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THE SINGULAR LAGRANGE SPACES

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1. ***Introduction.*** *In the R. Miron and M. Anastasiei's monograph, [2], a geometrization of regular Lagrangians is provided and several applications of it are worked out. But there exists also many problems in Mechanics and Theoretical Physics involving nonregular Lagrangians. It is enough to recall that any variational problem subjects to constraints may lead to a nonregular Lagrangian. Unlike the regularity, the nonregularity may have more menings. In this paper it will mean that the metrical tensor field derived from Lagrangian has a constant rank strictly smaller than the dimension of the base manifold. Our purpose is to initiate a geometrization of such non-regular Lagrangians which is called singular in the following using the Gh. Atanasiu, M. Kirkovits, E. Stoica and H. Kawaguchi's result of the paper [1]. After we introduce the notion of singular Lagrangian in §2, some considerations on the externals of a singular Lagrangian are made in §3. Then the possibility to consider a cannonical metrical connection is pointed out in §4. Our results could be useful in the study of anholonomic mechanical systems.*

The terminology and the notations are those from [2]

2. ***The singular Lagrangian spaces.*** *Let M be a smooth, i.e. C^∞, manifold of dimension n and TM is tangent manifold. If (x^i), i= 1, 2, ..., n, denote the local coordinates on M, then (x^i, y^i) will denote the local coordinates on TM, adapted to the tangent fibration $\tau: TM \to M$.*

Definition 2.1 *A smooth function $L: TM \to R$ is called a singular Lagrangian if the Hessian of L*

$$g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j} \tag{2.1}$$

has a constant rank k· n on TM.

Definition 2.2 *A manifold M endowed with a singular Lagrangian will be called a singular Lagrange space and it will be denoted by $L^n = (M, L)$.*

Examples.

1. Any singular Finsler spaces, [1], is a singular Lagrange space. Indeed, if $K^n = (M, K)$ is a singular Finsler space, then $L(x, y) = K^2(x, y)$ is a singular Lagrangian. We notice that $K^2(x, y)$ is positively homogenous of degree two with respect to y.

2. Let $K^n = (M, K)$ be a singular Finsler space and $\gamma_{ij}(x, y)$ its fundamental metric tensor. Then the function

$$L(x, y) = \gamma_{ij}(x, y)y^i y^j + A^i(x)y^i + U(x),$$

where $A_i(x)$ is a d-connector field and U a real function on TM, is a singular Lagrangian.

Indeed, it is easy to see that $\frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j} = \gamma_{ij}(x, y)$

Let VTM be the vertical bundle over TM. Its fibre $V_u TM$, $u = (x, y) \in TM$, is the kernel of the Jacobian map of τ, $V_u TM = Ker\, D\tau$. it results that $V_u TM$ is spanned by $\left(\frac{\partial}{\partial y^i}\right)_u$.

We denote by $\chi(TM)$ the set of vector fields on TM and by $V(TM)$ the set of sections in the vertical bundle (vertical vector fields). An element $A \in V(TM)$ is of the form $A = A^i \frac{\partial}{\partial y^i}$ and the components (A^i) define a d-vector field or a Finsler vector field.

Techniques from the geometry of vector bundles of [2], [3] suggest the following considerations

The matrix (2.1) defines a symmetric bilinear mapping $g_u: V_uTM \times V_uTM \to \mathbb{R}$, $g_u(A,B) = g_{ij}(x,y) A^i B^j$, $A, B \in V_uTM$.

The mapping g may be regarded as a linear mapping

(2.2) $\tilde{g}_u : V_uTM \to (V_uTM)^*$, $A \to \tilde{g}_u(A)$, $\tilde{g}_u(A)B = g(A,B)$, *for* $A, B \in V_uTM$.

The condition on $g_{ij}(x, y)$ in the Definition 2.1 is equivalent to rank $\tilde{g}_u = k < n$, for every $u \in TM$. It follows that if we set $\overset{o}{V}_u = Ker\, \tilde{g}_u$, the mapping $\overset{o}{V} : u \to \overset{o}{V}_u$ defines a smooth and regular distribution of local dimension k which is a substitution of the substitution of the vertical distribution $V : u \to V_uTM$ of local dimension n. The vertical distribution is integrable. Generally, the distribution $\overset{o}{V}$ is not integrable. In order to find a condition for its integrability, let be the symmetric d-tensor field $C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$. This may be viewed as a trilinear mapping $C: V_uTM \times V_uTM \times V_uTM \to \mathbb{R}$, $C(X, Y, Z) = X^i Y^j Z^k C_{ijk}$, . When the first two arguments are fixed, one gets an element $C(X, Y)$ of $(V_uTM)^*$. We have

Proposition 2.1 *The distribution $\overset{o}{V}$ is integrable if it is a subdistribution of the distribution Ker $C(X, Y)$ for every $X, Y \in V_uTM$*

Proof. *First we notice that*

$$Ker\ \tilde{g}_u = \left\{ X = X^i \frac{\partial}{\partial y^i} \mid g_{ij}(x,y) X^j = 0 \right\}.$$

For $A = A^i \frac{\partial}{\partial y^i}$, $B = B^i \frac{\partial}{\partial y^i}$ the bracket $[A, B]$ is $[A, B] = Z^i \frac{\partial}{\partial y^i}$, where $Z^i = A^j \frac{\partial B^i}{\partial y^j} - B^j \frac{\partial A^i}{\partial y^j}$. Now, from $g_{kj} A^i = 0$ it results $2C_{ikj}A^i + g_{kj}\frac{\partial A^i}{\partial y^i} = 0$ and by hypothesis one gets $g_{kj}\frac{\partial A^i}{\partial y^i} = 0$. Similarly one gets $g_{kj}\frac{\partial B^j}{\partial y^i} = 0$. These equations imply $g_{ki} Z^i = 0$, i.e. $[A,B]$ belongs to the distribution $\overset{o}{V}$, q.e.d.

Let $\overset{l}{V}_u$ be a fixed supplement of $\overset{o}{V}_u$ in V_uTM for every $u \in TM$ in such a way that the mapping $u \to \overset{l}{V}_u$ to give a n regular smooth subdistribution of the vertical distribution. In the other words, we consider a decomposition

(2.3) $$V = \overset{o}{V} \oplus \overset{l}{V}$$

of the vertical distribution on TM.

*The decomposition (2.3) defines two supplementary projections **l** and **m** so that we have*

(2.4) $$\begin{cases} l(V) = \overset{o}{V}, \ m(V) = \overset{1}{V}, \ l + m = I(identity), \\ l^2 = l, \ m^2 = m, \ lm = ml = 0 \end{cases}$$

Proposition 2.2 *There exists a unique linear mapping* $\tilde{g}_u^{-1} : (V_u TM) \to V_u TM$, *so that*

(2.5) $$\tilde{g}_u^{-1} \circ g_u = m_n, \quad l_n \circ \tilde{g}_u^{-1} = 0.$$

Proof. The existence follow from the definition of $\tilde{g}_u$ *and the uniqueness results by contradiction.*

Remark 2.1 *The mapping* $\tilde{g}^{-1}$ *depends on the choice of* $\overset{1}{V}$.

Let (m^i_j) *and* (l^i_j) *the matrices of* **m** *and* **l** *with respect to the basis* $(\frac{\partial}{\partial y^k})$. *Regarding* $\tilde{g}^{-1}$ *as a symmetric bilinear mapping* $g^{-1} : (VTM)^* \times (VTM)^* \to \mathbb{R}$, $g^{-1}(\alpha,\beta) = \alpha(g^{-1}(\beta))$, $\alpha, \beta \in (VTM)^*$ *and denoting by* (g^{ij}) *its matrix with respect to the basis* (dy^k), *the condition (2.5) takes the form [1], [3]*

(2.6) $$g^{ik} g_{kj} = m^i_j, \quad g^{ik} l^j_k = 0.$$

We note also the following equations (see (2.4))

(2.7) (a) $$l^i_j l^j_k = l^i_k, \quad m^i_j m^j_k = m^i_k$$

(b) $$m^i_j + l^i_j = \delta^i_j, \quad l^i_j m^j_k = m^i_j l^j_k = 0.$$

3. External curves in a singular Lagrange space

The external curves for the singular Lagrangian **L** *are the curves* $t \to c(t) \in M$, $t \in [0,1]$ *along which the integral of action*

(3.1) $$I(c) = \int_0^1 L(x(t), \dot{x}(t))dt \quad , \dot{x}^i(t) = \frac{dx^i}{dt},$$

affords external values.

These external values are solutions of the Euler-Lagrange equation

(3.2) $$E_i =: \frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = 0.$$

These equations (3.2) expand to

(3.3) $$g_{ij} \ddot{x}^j + \frac{1}{2}\left[\frac{\partial^2 L}{\partial y^i \partial x^k} \dot{x}^k - \frac{\partial L}{\partial x^i}\right] = 0, \quad y^i = \dot{x}^i$$

Multiplying (3.3) by g^{si} *one gets*

$$E^s =: m^s_j \ddot{x}^j + \frac{1}{2} g^{si} \left[\frac{\partial^2 L}{\partial y^i \partial x^k} \dot{x}^k - \frac{\partial L}{\partial x^i} \right] = 0. \tag{3.4}$$

We notice that in sprite of the similarity of the functions

$$G^s = \frac{1}{4} g^{si} \left[\frac{\partial^2 L}{\partial y^i \partial x^k} \dot{x}^k - \frac{\partial L}{\partial x^i} \right]$$

with some well-known functions for regular Lagrangians, these functions do not provide a semi-spray as it happens when **L** *is regular. As in the case of regularity, a canonical nonlinear connection is derived from* G^s, *this possibility falls when L is singular. Then we need a different method for associating to L a nonlinear connection.*

Such a method will be printed out in the next section.

As E_i *is a d-covector field, it follows that* E^s *is a d-vector field. Using (2.6) and (2.7) we obtain* $l^h_s E^s = 0$, *equivalently* $m^h_s E^s = E^h$, *i.e.* E^s *belongs to the distribution* $\overset{1}{V}$.

This fact has to be regarded in conjuction with the use of the multipliers of Lagrange for solving the variational problems subject to constraints.

4. Metrical connections in a singular Langrange space

Let ∇ *be a linear connection in the vertical bundle, i.e.* $\nabla : \chi(TM) \times V(TM) \to V(TM)$, $(X, A) \to \nabla_X A$, *with the usual properties of a linear connection (Koszul's definition).*

Definition 4.1 *We say that* ∇ *is metrical or it is compatible with respect to g if*

$$(\nabla_X g)(A, B) =: Xg(A, B) - g(\nabla_X A, B) - g(A, \nabla_X B) = 0, \tag{4.1}$$

for every $X \in \chi(TM)$, $A, B \in V(TM)$.

Now we extend ∇_X *to 1-forms vertical valued and to tensor fields by usual procedures, keeping same symbol* ∇ *for these extensions. For instance, for 1-form* α *we set*

$$(\nabla_X \alpha)(A) = X(\alpha(A)) - \alpha(\nabla_X A). \tag{4.2}$$

Then the covariant derivative of $\tilde{g}$ *is given by*

$$(\nabla_X \tilde{g})(A) = \nabla_X \tilde{g}(A) - \tilde{g}(\nabla_X A). \tag{4.3}$$

by (4.2) and (4.3) we infer that the condition (4.1)is equivalent to

$$\nabla_X \tilde{g} = 0 \ , \ \forall X \in \chi(TM). \tag{4.4}$$

Using (4.3) and (4.4) one sees that if $A \in Ker\,\tilde{g}$ *then* $\nabla_X A \in Ker\,\tilde{g}$. *In other words, a metrical connection* ∇ *preserves by parallelism the distribution* $\overset{o}{V}$. *As* $\overset{1}{V}$ *was arbitrarily chosen, we may take it so that to be preserved by parallelism with respect to* ∇. *Consequently we have:*

$$\nabla_X l = 0 \ , \ \nabla_X m = 0, \quad \forall X \in \chi(TM). \tag{4.5}$$

Let us put

(4.6) $$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial y^j} = \Gamma_{ij}^k \frac{\partial}{\partial y^k} \ , \ \nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = C_{ij}^k \frac{\partial}{\partial y^k}$$

The functions Γ_{ij}^k and C_{ij}^k will be calles the local coefficients od the linear connection ∇.

We have

Proposition 4.1 *The linear connection ∇ is metrical if and only if*

(4.7) $$\nabla_i g_{jk} =: \frac{\partial g_{jk}}{\partial x^i} - \Gamma_{ji}^h g_{hk} - \Gamma_{ki}^h g_{hj} = 0 \ ,$$

$$\dot{\nabla}_i g_{jk} =: \frac{\partial g_{jk}}{\partial y^i} - C_{ji}^h g_{hk} - C_{ki}^h g_{hj} = 0 \ .$$

Proof. By a direct calculation one obtains that (4.1) is equivalent to (4.7).

The curvature of ∇ is given as follows:

(4.8) $$R(X,Y)A = \nabla_X \nabla_Y A - \nabla_Y \nabla_X A - \nabla_{[X,Y]} A, \ \forall X, Y \in \chi(TM), \ A \in V(TM).$$

As $\nabla_X l = 0$ is equivalent to $l \circ \nabla_X = \nabla_X \circ l$ one easily obtains

(4.9) $$l \circ R(X,Y) = R(X,Y) \circ l \ , \ m \circ R(X,Y) = R(X,Y) \circ m, \ \forall X, Y \in \chi(TM)$$

Let $C = y^i \frac{\partial}{\partial y^i}$ be the Lionville vector field on TM.

Definition 4.2 *A linear connection ∇ in the vertical bundle is said to be regular if the distribution $H = \{X \mid \nabla_X C = 0\}$ is supplementary to the vertical distribution.*

The distribution H, called the horizontal distribution defines a nonlinear connection on TM. Thus it appears as important to find regular linear connections compatible with respect to g.

First, we prove:

Theorem 4.1 *A linear connection $\nabla = (\Gamma^i_{jk} \ , \ C^i_{jk})$ is regular if and only if*

(4.10) $$\det\left(\delta^i_j + y^h C^i_{hj}\right) \neq 0.$$

Proof. Let us assume that H exists and it is locally spanned by the local vector fields $\frac{\delta}{\delta x^i} =: \frac{\delta}{\delta x^i} - N_i^k(x,y) \frac{\delta}{\delta y^k}$ chosen such that $D\left(\frac{\delta}{\delta x^i}\right) = \frac{\delta}{\delta x^i}$. The condition $\nabla_{\frac{\partial}{\partial x^i}} C = 0$ is equivalent to

(4.11) $$y^h \Gamma^i_{hj} - N_j^h\left(\delta^i_h + y^j C^i_{jh}\right) = 0.$$

By (4.10), the matrix $A^i_h - \delta^i_h + y^j C^i_{jh}$ is invertible. Let B^h_k its inverse, i.e. $A^i_h B^k_k = \delta^i_k$. from (4.11) we explicitly find N^i_s in the form

(4.13) $$T_u TM = H_u \oplus V_u TM.$$

A basis adapted to this decomposition is $\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\right)$ *and as is well known, [2], it is convenient to express the various geometrical objects in yhis basis.*

For instance, the regular linear connection ∇ *is given by*

(4.14) $$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial y^i} = L^k_{ji} \frac{\partial}{\partial y^k}\ ,\ \nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^i} = C^k_{ji} \frac{\partial}{\partial y^k},$$

where

(4.15) $$L^k_{ji} = \Gamma^k_{ji} N^h_i C^k_{hj}.$$

Writing (4.1) in the above adapted basis one gets

Proposition 4.2 *A regular linear connection* ∇ *is metrical if and only if*

(4.16) $$g_{jk\,|i} =: \frac{\partial g_{jk}}{\partial x^i} - L^h_{ji} g_{hk} - L^h_{ki} g_{hj} = 0,$$

$$g_{jk}|_i =: \frac{\partial g_{jk}}{\partial y^i} - C^h_{ji} g_{hk} - C^h_{ki} g_{hj} = 0.$$

Starting with the equations (4.16) and using the same method as that the framework of singular Finsler spaces, [1], one proves the existence and one find the arbitrariness of regular metrical connections.

REFERENCES

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