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THE RELATION BETWEEN THE ABSTRACT AND GEOMETRIC DUALS OF A PLANAR GRAPH

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Abstract. The reciprocal development theories, of the graphs, respectively of the matroids, has also maintained by the manner in which the duality acts in the interior of the two combinatorics branchs. In [4], at the page 91, a wrong example is given, in the wish to prove, that for a graph G, its abstract dual $\tilde{G}$ is not always a geometric dual G^{*}. Our paper comes to explain the error of the example from [4], and to introduce another, by which we shall justify the anterior mentioned property, and also, the fact that the abstract duals, of a graph G, are not all isomorphic.

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We shall use unoriented graphs $G=(V(G),E(G))$ and we shall consider both the cycles and their cocycles, those the latters representing minimal sets $A\subset E(G)$, by whose removing, the connected components number of G increases.

Definition 1. The graph $\tilde{G}=(V(\tilde{G}),E(\tilde{G}))$ is the abstract dual for $G=(V(G),E(G))$, if the bijection $\varphi:E(G)\to E(\tilde{G})$ exists, such that X is a cycle in G, if and only if $\varphi(X)$ is a cocycle in $\tilde{G}$. The map φ is named the dual function from G to $\tilde{G}$.

Definition 2. The graph $G=(V(G),E(G))$ is planar, if it is isomorphic to $G'=(V(G'),E(G'))$, embedded in a plane Π, the edges of G' being Jordan curves in Π.

Definition 3. Let $G=(V(G),E(G))$ be a planar graph, and in the plane Π, in which G is embedded, let A be a set which does not contain points from $V(G)$ or $E(G)$. We say that x and y, from A, are in the relation "$\sim$", writing $x\sim y$, if and only if x can be connected to y, by a Jordan curve lying in Π, and which does not intersect any vertex and any edge of G.

Remark 1. Obviously, "$\sim$" is an equivalence relation.

Definition 4. Related to the relation "$\sim$", the set of equivalence classes designates the faces of $G=(V(G),E(G))$.

Definition 5. Let $G=(V(G),E(G))$ be a graph. Its geometric dual, marked $G^{*}=(V(G^{*}),E(G^{*}))$, is a graph obtained by the following rules:

i) at any face F_i of G, we associate only one of its interior points P_i;

ii) if two faces F_i and F_j, are adjacent in G, by the edges $e_1,e_2,\dots,e_k\in E(G)$, then, their corresponding points P_i, respectively P_j, will be joined through the edges $e_1', e_2', ..., e_k' \in E(G^*)$, such that the latters cross them, the firsts;

iii) if an edge e there is into an unique face F_i of G, being a pendant edge, we shall associate the loop $e' \in E(G^*)$, at the corresponding point $P_i \in F_i$.

Remark 2. The geometric dual G^*, of the *Definition 5*, is in conformity with [1], [3] and [4]. For the abstract dual introduced into the *Definition 1*, the description made in [3] is more difficult than that one of [4].

We know from [4], the:

Theorem 1. For the graphs G and $\tilde{G}$, the following assertions are equivalent:

i) $\tilde{G}$ is an abstract dual of G;

ii) the cocycle matroid $M^*(\tilde{G})$, of $\tilde{G}$, is isomorphic to the cycle matroid $M(G)$, of G;

iii) the cycle matroid $M(\tilde{G})$, of $\tilde{G}$, is isomorphic to the cocycle matroid $M^*(G)$, of G;

iv) G is an abstract dual of $\tilde{G}$,

whose proof rigorous, new in part, and corrected confronted by [4], there is in [2].

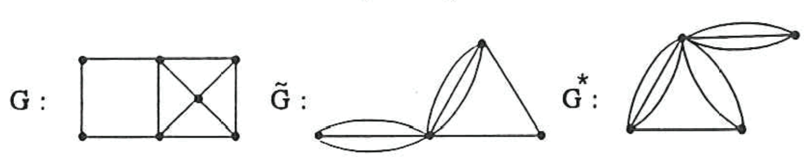
From [4], and in detail proved in [2], we know that the anterior theorem acknowledges the:

Corollary. Let G^* be the geometric dual of a graph G. Then, G^* is also the abstract dual of G.

For blocks, namely for 2 - connected graphs G, or equivalently for graphs which has connected cycles matroids $M(G)$, from [3] and [4], with a rigorous proof in [2], we know the:

Theorem 2. Let $G = (V(G), E(G))$ be a planar block. Then, a graph G^*, without isolated vertices, is an abstract dual of G, if and only if it is a geometric dual of G.

Therefore, the equivalent condition between the geometric dual G^* and the abstract dual $\tilde{G}$, of a graph G, is very restrictive. Moreover, when G has many abstract duals, it does not follow that all are obligatory isomorphic ! We must note, that no reference exists at some anterior problems, in [1]. Especially maintaining the fact that not every abstract dual comes from a geometric one, in [4], at the pages 91 - 92, we find an example, with the indication that it was given by D.R. Woodall.



a) b) c)

Figure 1

So, going to the graph G of the *Figure 1a)*, it is mentioned that G^*, from the

Figure 1c), should be its geometric dual. Obviously false, because the true G^* for G, drawn in the *Figure* 2, must contain 6 vertices and surely not 4, because G has 6 faces !

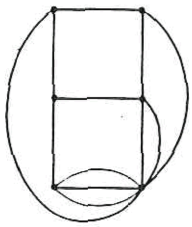


Figure 2

We tried to cure this error which impedes any continuation of the reasoning relative to G, $\tilde{G}$, G^*, searching for the references of [4], all the papers quoted here and written by D.R.Woodall. But, in none of them (see [5], [6], [7]) there are connexions with the duality ! This is the fact which stimulated us to find *the new example* of the graph G, drawn in the *Figure 3*.

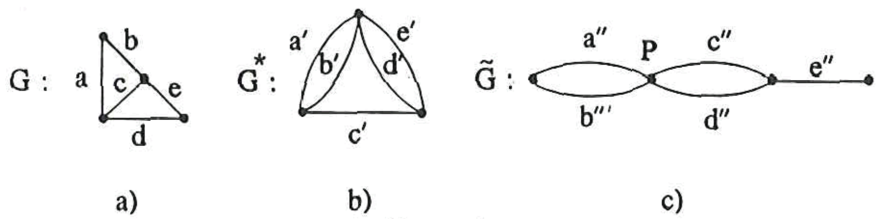


Figure 3

Obviously, $\tilde{G}$ from the *Figure 3c)* is not a geometric dual of G ! Really, in a such situation, the parallel edges a'',b'', respectively c'',d'' could not only be all incident in P, but also should be adjacent two by two, through an intermediate edge, corresponding to a crossing of c, in G. However, we observe that G^*, from the *Figure 3b)*, is a geometric dual, because it respects the *Definition 5*.

In the following, we prove that $\tilde{G}$ is an abstract dual of G. Therefor, we shall show that a bijection exists from $E(G)$ to $E(\tilde{G})$, leading the cycles of G, to the cocycles of $\tilde{G}$, and reciprocally.

We observe that in G, we have the following 3 cycles:

$$\{a,b,c\} \ ; \ \{c,d,e\} \ ; \ \{a,b,d,e\},$$

and in G^*, we have 3 cocycles:

$$\{a',b',c'\} \ ; \ \{c',d',e'\} \ ; \ \{a',b',d',e'\}.$$

In the graph $\tilde{G}$, there are also 3 cocycles:

$\{e''\}$; $\{a'',b''\}$; $\{c'',d''\}$.

We construct the bijection $\Phi : E(G) \to E(G^*)$ which leads the cycles of G, to the cocycles of G^*:

$$\Phi(\{a,b,c\}) = \{a',b',c'\} \ ; \ \Phi(\{c,d,e\}) = \{c',d',e'\} \ ; \ \Phi(\{a,b,d,e\}) = \{a',b',d',e'\}.$$

We emphasize the bijection $\psi : E(G^*) \to E(\tilde{G})$ which leads the cocycles of G^*, to the cocycles of $\tilde{G}$:

$$\psi(\{a',b',c'\}) = \{a'',b''\} \ ; \ \psi(\{c',d',e'\}) = \{c'',d''\} \ ; \ \psi(\{a',b',d',e'\}) = \{e''\}.$$

We now establish that the compound function $\psi \circ \Phi : E(G) \to E(\tilde{G})$ can be constructed, and it is obviously a bijection, because both, Φ and ψ, are bijections. The map $\psi \circ \Phi$ leads the cycles of G, to the cocycles of $\tilde{G}$, as it follows:

$$(\psi\circ\Phi)(\{a,b,c\}) = \{a'',b''\} \ ; (\psi\circ\Phi)(\{c,d,e\}) = \{c'',d''\} \ ; (\psi\circ\Phi)(\{a,b,d,e\}) = \{e''\}.$$

So, we can decide that $\tilde{G}$ is an abstract dual of G.

In conformity with the assertion of the *Corollary*, we more conclude that the geometric dual G^*, from the *Figure 3b*), is also an abstract dual. This fact can be directly deduced by the existence of the anterior bijection Φ, which led the cycles of G, to the cocycles of G^*, and reciprocally.

Although G^* and $\tilde{G}$ are some abstract duals for G, we observe that they are not isomorphic. Really, because $\tilde{G}$ has 2 vertices with degrees 1, respectively 2, fact which does not happen in G^* ! Hence, we conclude, that in general, the abstract duals of a graph G, are not isomorphic !

References

[1] **Capobianco M. Moluzzo J. C.** "Examples and counterexamples in graph theory", Elsevier North Holland Inc., 1978;

[2] **Modan L.** "Duality properties for graphs, proved by means of the matroids", (in Romanian), St. şi Cerc. de Calc. Ec. şi Cib. Ec., XXXI, 3, 1997 (will appear), also private communication at "The first National Conference of the Mathematical Society from Romania, June 1997";

[3] **Ore O.** "The four-color problem", Academic Press, 1967;

[4] **Welsh D. J. A.** "Matroids theory", Academic Press, 1976;

[5] **Woodall D. R.** "Applications of polymatroids and linear programming to transversals and graphs", Combinatorics, Cambridge University Press, 1973, pg. 195-201;

[6] **Woodall D. R.** "An exchange theorem for bases of matroids", J. of Comb. Th. (B), 16, 1974, pg. 227-8;

[7] **Woodall D. R.** "The induction of matroids by graphs", J. London Math. Soc., 2, 10, 1975, pg. 27-35.