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A linear delay-differential equation of the second order

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Abstract. The asymptotic behaviour of the solutions of certain 2nd order delay-differential equations with variable coefficients is studied. Some upper bounds for solutions and their derivatives are obtained, which are sometimes involved in getting stability results of exponential type. The method here employed is based on some appropriate integral inequalities.

1. Firstly, we consider the equation

$$\ddot{x}(t) + 2a(t)\dot{x}(t) + b(t)x(t) = c(t)x(t-T), \quad t \geq 0 \tag{1}$$

where $a \in C^2[-T,\infty)$, $b \in C^1[0,\infty)$, $c \in C[0,\infty)$ and $T > 0$ is a constant. In the case when the coefficients are positive constants, this equation, appearing in some problems of mechanics, was earlier studied by many authors; see, for example, the paper [1] and the author's paper [2]. Here, we consider more general equations under more general hypotheses. The solution $x = x(t)$ is subjected to the initial conditions

$$x(0_+) = x_0, \quad \dot{x}(0_+) = v_0, \quad x(t) = \varphi(t) \text{ for } t \in (-T,0) \tag{2}$$

where $\varphi = \varphi(t)$ is a given continuous function, having finite lateral limits at the points $t = -T$ and $t = 0$. Putting

$$x(t) = y(t)e^{-\int_0^t a(s)ds}, \quad m(t) = b(t) - a^2(t) - \dot{a}(t), \quad \gamma(t) = c(t)e^{\int_{t-T}^t a(s)ds} \tag{3}$$

we obtain the equation

$$\ddot{y}(t) + m(t)y(t) = \gamma(t)y(t-T), \quad t \geq 0 \tag{4}$$

with the initial conditions for the new unknown function

$$y(0_+) = x_0, \quad \dot{y}(0_+) = a(0)x_0 + v_0, \quad y(t) = \varphi(t)e^{\int_0^t a(s)ds} \text{ for } t \in (-T,0). \tag{5}$$

It follows by an elementary argument, that the problems (1)-(2), respectively (4)-(5) have a unique solution, continuous with its first derivative for $t \geq 0$, while the second derivatives have a discontinuity with finite lateral limits at the point $t = T$ only. In what follows, we shall search upper bounds for $y = y(t)$ and its derivatives, writing $y(0)$ and $\dot{y}(0)$ instead of $y(0_+)$ and $\dot{y}(0_+)$. Of course, the same convention will be also adopted for the solution $x = x(t)$ of the problem (1)-(2).

Multiplying (4) by $2\dot{y}(t)$ and integrating on $[0, t]$, we obtain

$$\dot{y}^2(t) + \int_0^t m(s)(y^2(s))' ds = \dot{y}^2(0) + 2\int_0^t \gamma(s)\dot{y}(s)y(s-T)ds \tag{6}$$

Imposing the supplementary restriction $\dot{m}(t) \leq 0$ and $m(t) \geq m_\infty > 0$ for $t \geq 0$, we obtain from the preceding equality that

$$\dot{y}^2(t) + m_\infty y^2(t) \leq \dot{y}^2(0) + m(0)y^2(0) + 2\int_0^t |\gamma(s)| \cdot |\dot{y}(s)y(s-T)|\, ds, \quad t \geq 0 \tag{7}$$

We shall also use the inequalities

$$2\int_0^t |\gamma(s)| \cdot |\dot{y}(s)y(s-T)|\, ds \leq \frac{1}{\sqrt{m_\infty}} \int_0^t |\gamma(s)|\, (\dot{y}^2(s) + m_\infty y^2(s-T))ds, \quad t \geq 0 \tag{8}$$

and

$$\int_0^t |\gamma(s)|\, y^2(s-T)ds \leq C + \int_0^t |\gamma(s+T)|\, y^2(s)ds, \quad t \geq 0 \tag{9}$$

where

$$C = \int_{-T}^0 |\gamma(s+T)|\, \varphi^2(s) e^{2\int_0^s a(\tau)d\tau} ds = const. \tag{10}$$

Denoting $g(t) = \max\{|\gamma(t)|, |\gamma(t+T)|\}$ for $t \geq 0$ and taking into account (8) and (9), we get for $t \geq 0$

$$2\int_0^t |\gamma(s)| \cdot |\dot{y}(s)(s-T)|\, ds \leq \sqrt{m_\infty} C + (1/\sqrt{m_\infty}) \int_0^t g(s)(\dot{y}^2(s) + m_\infty y^2(s))ds \tag{11}$$

Denoting

$$u(t) = \dot{y}^2(t) + m_\infty y^2(t), \quad t \geq 0 \tag{12}$$

we obtain, from (7) and (11), the following inequality

$$u(t) \leq A^2 + (1/\sqrt{m_\infty}) \int_0^t g(s)u(s)ds, \quad t \geq 0 \tag{13}$$

where $A \geq 0$ is a constant, given by

$$A = (\dot{y}^2(0) + m(0)y^2(0) + \sqrt{m_\infty} C)^{1/2} \tag{14}$$

The Gronwall-Bellman lemma implies

$$u(t) \leq A^2 e^{(1/\sqrt{m_\infty})\int_0^t g(s)ds}, \quad t \geq 0 \tag{15}$$

from which we derive

$$|y(t)| \leq (A/\sqrt{m_\infty})e^{(1/2\sqrt{m_\infty})\int_0^t g(s)ds}, \quad |\dot{y}(t)| \leq Ae^{(1/2\sqrt{m_\infty})\int_0^t g(s)ds} \tag{16}$$

for $t \geq 0$. The constant A depends on the arbitrary initial values x_0, v_0 and on the function φ, if we remember the conditions (5). Taking into account the above obtained bounds and the substitution $x = y\, exp(-\int_0^t a(s)ds)$, we can formulate the following

Proposition 1. *Under the hypotheses*

(h_1) $a \in C^2[-T, \infty)$, $b \in C^1[0, \infty)$, $c \in C[0, \infty)$, $T > 0$

(h_2) $\dot{m}(t) \leq 0$, $m(t) \geq m_\infty > 0$ for $t \geq 0$

we have, for the solution $x = x(t)$ of the problem (1) – (2), the bounds

$$|x(t)| \leq (A/\sqrt{m_\infty})e^{-\int_0^t h(s)ds}, \quad |\dot{x}(t)| \leq B(t)e^{-\int_0^t h(s)ds}, \quad t \geq 0 \tag{17}$$

where $B = B(t) \geq 0$ and $h = h(t)$ are given by

$$B(t) = A(1 + (|a(t)| / \sqrt{m_\infty})), \quad h(t) = a(t) - (1/2\sqrt{m_\infty})g(t), \quad t \geq 0 \tag{18}$$

Of course, a corresponding bound for $|\ddot{x}(t)|$ may be obtained from equation (1).

Corollary 1.1. *Assume that, in (1), the coefficients are constant: $a > 0$, $b > a^2$ and $|c| < 2a\sqrt{b - a^2}e^{-aT}$. Then, we have for the solution $x = x(t)$ of the problem (1)-(2), the bounds*

$$|x(t)| \leq (A_1/\sqrt{b - a^2})e^{-\alpha t}, \quad |\dot{x}(t)| \leq B_1 e^{-\alpha t}, \quad t \geq 0 \tag{19}$$

where the constants $A_1 \geq 0$, $B_1 \geq 0$ and $\alpha > 0$ are given by

$$A_1 = \left[(ax_0 + v_0)^2 + (b - a^2)x_0^2 + |c|\, e^{aT}\sqrt{b - a^2}\int_{-T}^0 \varphi^2(s)e^{2as}ds\right]^{1/2}, \tag{20}$$

$$B_1 = A_1(1 + (a/\sqrt{b - a^2})), \quad \alpha = a - (|c|\, e^{aT}/2\sqrt{b - a^2}).$$

Now, let us consider the more general equation

$$\ddot{x}(t) + 2a(t)\dot{x}(t) + b(t)x(t) = \sum_{i=1}^{n} c_i(t)x(t - T_i), \quad t \geq 0 \tag{21}$$

where $a = a(t)$, $b = b(t)$ satisfy the conditions of Proposition 1 and the functions $c_i = c_i(t)$, $i = \overline{1, n}$ are continuous for $t \geq 0$. It is assumed that $0 < T_1 < T_2 < \ldots < T_n < \infty$ and the initial conditions are

$$x(0) = x_0, \quad \dot{x}(0) = v_0, \quad x(t) = \varphi(t) \text{ for } t \in (-T_n, 0) \tag{22}$$

where $\varphi = \varphi(t)$ is continuous on $(-T_n, 0)$ and has finite lateral limits at the points $t = -T_n$ and $t = 0$. As it is easy to prove, the unique solution $x = x(t)$ of the problem (21)-(22), which may be successively constructed on the intervals $(0, T_1), (T_1, T_2), \ldots, (T_n, \infty)$, belongs to the class $C^1[0, \infty)$ and its second derivative $\ddot{x}(t)$ may have discontinuities with finite lateral limits at the point T_i, $i = \overline{1,n}$ only. Using the substitution and the notations introduced in the preceding case, we obtain the equation

$$\ddot{y}(t) + m(t)y(t) = \sum_{i=1}^{n} \gamma_i(t)y(t - T_i), \quad t \geq 0 \tag{23}$$

with the initial conditions

$$y(0) = x_0, \ \dot{y}(0) = a(0)x_0 + v_0, \ y(t) = \varphi(t)e^{\int_0^t a(s)ds} \ \text{for} \ t \in (-T_n, 0) \tag{24}$$

We have also used the notations

$$\gamma_i(t) = c_i(t)e^{\int_{t-T_i}^t a(s)ds}, \quad i = \overline{1,n}, \quad t \geq 0 \tag{25}$$

Proceeding as in the case of the equation (1), we obtain

$$\dot{y}^2(t) + m_\infty y^2(t) \leq \dot{y}^2(0) + m(0)y^2(0) + 2\sum_{i=1}^{n} \int_0^t |\gamma_i(s)| \cdot |\dot{y}(s)y(s - T_i)| ds, \ t \geq 0 \tag{26}$$

and

$$2\int_0^t |\gamma_i(s)| \cdot |\dot{y}(s)y(s - T_i)| \, ds \leq \sqrt{m_\infty}C_i + (1/\sqrt{m_\infty})\int_0^t g_i(s)(\dot{y}^2(s) + m_\infty y^2(s))ds \tag{27}$$

where

$$C_i = \int_{-T_i}^{0} |\gamma_i(s + T_i)| \, \varphi^2(s)e^{2\int_0^s a(r)dr}ds, \quad g_i(t) = \max\{|\gamma_i(t)|, \ |\gamma_i(t + T_i)|\} \tag{28}$$

for $i = \overline{1,n}$. Then, for $u = u(t)$ defined by (12), it follows a similar bound with the one of the r.h.s. of (15). Denoting

$$\tilde{A} = (\dot{y}^2(0) + m(0)y^2(0) + \sqrt{m_\infty}\sum_{i=1}^{n} C_i)^{1/2}, \quad \tilde{g}(t) = \sum_{i=1}^{n} g_i(t) \tag{29}$$

we can state the following

Corollary 1.2. *Under the hypotheses*
(h_3) $a \in C^2[-T, \infty)$, $b \in C^1[0, \infty)$, $c_i \in C[0, \infty)$ *and* $T_i > 0$ *for* $i = \overline{1,n}$
(h_4) $\dot{m}(t) \leq 0$, $m(t) \geq m_\infty > 0$ *for* $t \geq 0$
we have, for the solution $x = x(t)$ *of the problem* (21) − (22), *the bounds*

$$|x(t)| \leq (\tilde{A}/\sqrt{m_\infty})e^{-\int_0^t \tilde{h}(s)ds}, \quad |\dot{x}(t)| \leq \tilde{B}(t)e^{-\int_0^t \tilde{h}(s)ds}, \quad t \geq 0 \tag{30}$$

where $\tilde{B}(t) \geq 0$ *and* $\tilde{h}(t)$ *are respectively defined by*

(31) $$\tilde{B}(t) = \tilde{A}(1 + (|a(t)|/\sqrt{m_\infty})), \quad \tilde{h}(t) = a(t) - (1/2\sqrt{m_\infty})\tilde{g}(t), \quad t \geq 0$$

Another method to obtain bounds for the solution of the problem (1)-(2) is based on the direct twice repeated integration of (4) on the interval $[0,t]$. Of course, $m(t)$ and $\gamma(t)$ reamain those defined by (3).

Assuming that $|m(t)| \leq M < \infty$, $|\gamma(t)| \leq G < \infty$ for $t \geq 0$ and denoting

(32) $$K = M + G, \quad L = \int_{-T}^{0} |\gamma(s+T)| \cdot |\varphi(s)| \, e^{\int_0^s a(\tau)d\tau} ds$$

we obtain, after some elementary computations, the inequality

(33) $$|y(t)| \leq |y(0)| + (|\dot{y}(0)| + GL)t + K\int_0^t (t-s)\,|y(s)|\, ds, \quad t \geq 0$$

Putting

(34) $$f(t) = |y(0)| + (|\dot{y}(0)| + GL)t, \quad k(t,s) = K(t-s)$$

we can rewrite (33) under the form

(35) $$|y(t)| \leq f(t) + \int_0^t k(t,s)\,|y(s)|\, ds, \quad t \geq 0$$

which implies

(36) $$|y(t)| \leq f(t) + \int_0^t r(t,s) f(s) ds, \quad t \geq 0$$

where $r(t,s)$ is the resolvent kernel of $k(t,s)$. Since $f = f(t)$ is nondecreasing for $t \geq 0$, and a simple computation shows that

(37) $$r(t,s) = \sqrt{K} sh \sqrt{K}(t-s), \quad t \geq s \geq 0,$$

we get

(38) $$|y(t)| \leq f(t)\left(1 + \sqrt{K}\int_0^t sh\sqrt{K}(t-s)ds\right) = f(t) ch\sqrt{K}t \leq f(t)e^{\sqrt{K}t}$$

for $t \geq 0$. By a simple integration of (4) on $[0,t]$, we have

(39) $$|\dot{y}(t)| \leq |\dot{y}(0)| + GL + K\int_0^t |y(s)|\, ds, \quad t \geq 0$$

from which we easily obtain

(40) $$|\dot{y}(t)| \leq |\dot{y}(0)| + GL + (\sqrt{K}/2) f(t) e^{\sqrt{K}t}, \quad t \geq 0$$

Using the inequality

$$(41)\qquad |x(t)|\le|y(t)|\,e^{-\int_0^t a(s)ds},\ \ |\dot{x}(t)|\le(|a(t)y(t)|+|\dot{y}(t)|)e^{-\int_0^t a(s)ds},\ \ t\ge 0$$

and the notation $\alpha(t)=a(t)-\sqrt{K}$, we obtain

$$(42)\qquad \begin{array}{c}|x(t)|\le|f(t)|\,e^{-\int_0^t \alpha(s)ds},\ \ |\dot{x}(t)|\le(|\dot{y}(0)|+GL)e^{-\int_0^t a(s)ds}+\\ (|a(t)|+(\sqrt{K}/2))f(t)e^{-\int_0^t \alpha(s)ds},\ \ t\ge 0\end{array}$$

Recalling that the constants K and L are given by (32), we conclude these considerations by stating the following

Proposition 2. *Under the hypotheses*
(h_5) $a\in C^1[-T,\infty)$, b *and* $c\in C[0,\infty)$, $T>0$
(h_6) $m=m(t)$ *and* $\gamma=\gamma(t)$ *defined by* (3) *are bounded functions for* $t\ge 0$,
the bounds (42) *hold true for the solution* $x=x(t)$ *of the problem* (1)-(2).

Example 1. Let us consider the problem

$$(P)\qquad \begin{cases}\ddot{x}(t)+2a\dot{x}(t)+bx(t)=cx(t-T),\ \ t\ge 0\\ x(0_+)=x_0,\ \ \dot{x}(0_+)=v_0,\ \ x(t)=0\ \text{ for }\ t\in(-T,0)\end{cases}$$

where, a,b are positive constants and $c=const.$ The initial conditions (values) x_0 and v_0 are arbitrary constants. Here, we have

$$\begin{array}{l}M=|b-a^2|,\ \ G=|c|\,e^{aT},\ \ L=0,\ \ K=|b-a^2|+|c|\,e^{aT},\\ y(0)=x_0,\ \ \dot{y}(0)=ax_0+v_0,\ \ f(t)=|x_0|+|ax_0+v_0|\,t\ \text{ for }\ t\ge 0.\end{array}$$

The case $b\le a^2$ and $|c|<be^{-aT}$. We have $a>\sqrt{K}$ and we denote $\alpha=a-\sqrt{K}>0$. According to (42), for the solution of the problem (P) it follows

$$\begin{array}{c}|x(t)|\le(|x_0|+|ax_0+v_0|\,t)e^{-\alpha t},\ \ |\dot{x}(t)|\le|ax_0+v_0|\,e^{-at}+(a+(\sqrt{K}/2))\cdot\\ \cdot(|x_0|+|ax_0+v_0|\,t)e^{-\alpha t},\ \ t\ge 0\end{array}$$

The case $a^2<b<2a^2$ and $|c|<(2a^2-b)e^{-aT}$. We again have $\alpha=a-\sqrt{K}>0$ and the preceding bounds remain true. The solution $x=x(t)$ and its derivatives $\dot{x}(t)$, $\ddot{x}(t)$ tend exponentially to zero as $t\to\infty$.

Let us again consider the problem (21)-(22), maintaining the nonrestrictive condition $0<T_1<T_2<\ldots<T_n<\infty$. Using the substitution (3), we obtain the equation

$$(43)\qquad \ddot{y}(t)+m(t)y(t)=\sum_{i=1}^{n}\gamma_i(t)y(t-T_i),\ \ \text{where}\ \ \gamma_i(t)=c_i(t)e^{\int_{t-T_i}^{t}a(s)ds}$$

for $i=\overline{1,n}$ and $m=m(t)$ is the same as given in (3). Assuming $|m(t)|\le M<\infty$ and $|\gamma_i(t)|\le G_i$ for $t\ge 0$, $i=\overline{1,n}$ and also denoting

$$(44)\qquad \tilde{K}=M+\sum_{i=1}^{n}G_i,\ \ L_i=\int_{-T_i}^{0}|\gamma_i(s+T_i)|\cdot|\varphi(s)|\,e^{\int_0^s a(\tau)d\tau}ds,\ \ i=\overline{1,n}$$

we obtain the inequality

$$|\, y(t)\,| \le \tilde{f}(t) + \int_0^t \tilde{k}(t,s)\,|y(s)|\,ds, \quad t \ge 0 \tag{45}$$

where

$$\tilde{f}(t) = |y(0)| + \left(|\dot{y}(0)| + \sum_{i=1}^{n} G_i L_i\right) t, \quad \tilde{k}(t,s) = \tilde{K}(t-s). \tag{46}$$

From (45) and (46), it follows that

$$|y(t)| \le \tilde{f}(t)\, ch\sqrt{\tilde{K}}t \le \tilde{f}(t) e^{\sqrt{\tilde{K}}t}, \quad t \ge 0. \tag{47}$$

Acting as in the proof of Proposition 2 and putting $\tilde{\alpha}(t) = a(t) - \sqrt{\tilde{K}}$, we finally obtain

$$\begin{aligned} &|x(t)| \le \tilde{f}(t) e^{-\int_0^t \tilde{\alpha}(s)ds}, \quad |\dot{x}(t)| \le \left(|\dot{y}(0)| + \sum_{i=1}^{n} G_i L_i\right) e^{-\int_0^t a(s)ds} + \\ &+ \left(|\, a(t)\,| + \left(\sqrt{\tilde{K}}/2\right)\right) \tilde{f}(t) e^{-\int_0^t \tilde{\alpha}(s)ds}, \quad t \ge 0 \end{aligned} \tag{48}$$

Consequently, we can state the following

Corollary 2.1. *Under the hypotheses*

(h_7) $a \in C^1[-T,\infty)$, $b \in C[0,\infty)$, $c_i \in C[0,\infty)$, $i = \overline{1,n}$

(h_8) $m = m(t)$ *defined by* (3) *and* $\gamma_i = \gamma_i(t)$ *defined by* (43) *are bounded functions on* $[0,\infty)$,

the bounds (48) *hold true for the solution* $x = x(t)$ *of the problem* (21)-(22).

2. Next, we pass to the study of the behaviour of solutions for the equation

$$\ddot{x}(t) + 2a(t)\dot{x}(t) + b(t)x(t) = c(t)x(t-T) + d(t)\dot{x}(t-T), \quad t \ge 0 \tag{1'}$$

where a, b, c satisfy the hypotheses imposed in Proposition 1 and $d = d(t)$ is continuous for $t \ge 0$. The initial conditions are

$$x(0_+) = x_0, \quad \dot{x}(0_+) = v_0, \quad x(t) = \varphi(t) \ \text{ for } \ t \in (-T,0) \tag{2'}$$

where x_0 and v_0 are arbitrarily taken, $\varphi(t)$ and $\dot{\varphi}(t)$ are continuous on $(-T,0)$ and have finite lateral limits at the points $t = -T$ and $t = 0$. Making use of the substitution $x = y\, exp(-\int_0^t a(s)ds)$ and of the notations

$$\gamma(t) = (c(t) - d(t)a(t-T))e^{\int_{t-T}^t a(s)ds}, \quad \delta(t) = d(t)e^{\int_{t-T}^t a(s)ds} \tag{3'}$$

we obtain the equation

$$\ddot{y}(t) + m(t)y(t) = \gamma(t)y(t-T) + \delta(t)\dot{y}(t-T), \quad t \ge 0 \tag{4'}$$

where $m = m(t)$ is defined in (3). Of course, the initial conditions for $y = y(t)$ are those given by (5), $y(t)$ and $\dot{y}(t)$ being known functions on $(-T, 0)$. We omit to write here their explicit expressions.

Preceeding as in the proof of Proposition 1, we obtain an inequality similar to (7), namely

$$(5') \qquad \begin{aligned} \dot{y}^2(t) + m_\infty y^2(t) \leq \dot{y}^2(0) + m(0)y^2(0) + 2\int_0^t |\gamma(s)| \cdot |\dot{y}(s)y(s-T)|\, ds + \\ + 2\int_0^t |\delta(s)| \cdot |\dot{y}(s)\dot{y}(s-T)|\, ds, \quad t \geq 0 \end{aligned}$$

Using the inequality, valid for every $\alpha_1 > 0$ and $\alpha_2 > 0$,

$$(6') \qquad 2\,|\dot{y}(s)y(s-T)| \leq (1/\sqrt{\alpha_1\alpha_2})(\alpha_1\dot{y}^2(s) + \alpha_2 y^2(s-T)), \quad s \geq 0$$

and the notations

$$(7') \qquad g(t) = \max\{|\gamma(t)|,\ |\gamma(t+T)|,\ |\delta(t)|,\ |\delta(t+T)|\}, \quad t \geq 0$$

respectively

$$(8') \qquad C = \int_{-T}^{0} |\gamma(s+T)|\, y^2(s)ds, \quad D = \int_{-T}^{0} |\delta(s+T)|\, \dot{y}^2(s)ds$$

we obtain, after some elementary calculations, that

$$(9') \qquad \begin{aligned} \dot{y}^2(t) + m_\infty y^2(t) \leq (A^*)^2 + (2 + \sqrt{\alpha_1/\alpha_2})\int_0^t g(s)\dot{y}^2(s)ds + \\ + (1/m_\infty)\sqrt{\alpha_2/\alpha_1}\int_0^t g(s)(m_\infty y^2(s))ds, \quad t \geq 0 \end{aligned}$$

where the constant $A^* \geq 0$ is given by

$$(10') \qquad A^* = (\dot{y}^2(0) + m(0)y^2(0) + \sqrt{\alpha_2/\alpha_1}C + D)^{1/2}$$

Imposing the condition $2 + \sqrt{\alpha_2/\alpha_1} = (1/m_\infty)\sqrt{\alpha_2/\alpha_1}$, we get $\sqrt{\alpha_2/\alpha_1} = m_\infty\lambda$, where

$$(11') \qquad \lambda = 1 + \sqrt{1 + (1/m_\infty)}$$

Again using the notation (12) in the preceding section of this paper, we can rewrite (9') under the form

$$(12') \qquad u(t) \leq (A^*)^2 + \lambda\int_0^t g(s)u(s)ds, \quad t \geq 0$$

which implies

$$(13') \qquad u(t) \leq (A^*)^2 e^{\lambda\int_0^t g(s)ds}, \quad t \geq 0$$

where, obviously, we have taken $\sqrt{\alpha_2/\alpha_1} = m_\infty\lambda$, $y(0) = x_0$, $\dot{y}(0) = a(0)x_0 + v_0$ in (10′). Regarding the constants C and D, defined by (8′), we recall that y and $\dot{y}$ are known functions on the interval $(-T, 0)$.

From the inequality (13′), we deduce

$$(14') \qquad |y(t)| \leq (A^*/\sqrt{m_\infty})e^{(\lambda/2)\int_0^t g(s)ds}, \quad |\dot{y}(t)| \leq A^* e^{(\lambda/2)\int_0^t g(s)ds}, \quad t \geq 0.$$

Then, we can state the following

Proposition 3. *Under the hypotheses*

(H_1) $a \in C^2[-T,\infty)$, $b \in C^1[0,\infty)$, c *and* $d \in C[0,\infty)$, $T > 0$

(H_2) $m(t) = b(t) - a^2(t) - \dot{a}(t)$ *satisfies* $\dot{m}(t) \leq 0$, $m(t) \geq m_\infty > 0$ *for* $t \geq 0$

we have, for the solution $x = x(t)$ *of the problem* (1′) − (2′), *the bounds*

$$(15') \qquad |x(t)| \leq (A^*/\sqrt{m_\infty})e^{-\int_0^t h(s)ds}, \quad |\dot{x}(t)| \leq B^*(t)e^{-\int_0^t h(s)ds}, \quad t \geq 0$$

where we used the notations

$$(16') \qquad h(t) = a(t) - (\lambda/2)g(t), \quad B^*(t) = A^*(1 + (|a(t)|/\sqrt{m_\infty})), \quad t \geq 0$$

Example 2. Let us consider the problem with constant coefficients

$$(P') \qquad \begin{cases} \ddot{x}(t) + 2a\dot{x}(t) + bx(t) = cx(t-T) + d\dot{x}(t-T), & t \geq 0 \\ x(0_+) = x_0, \ \dot{x}(0_+) = v_0; \ x(t) = 0 \text{ for } t \in (-T, 0) \end{cases}$$

where $a > 0$, $b > a^2$, c and d are real constants. Here, we have

$$m_\infty = b - a^2 > 0, \quad \lambda = 1 + \sqrt{1 + (b - a^2)^{-1}}, \quad C = D = 0,$$

$$A^* = ((ax_0 + v_0)^2 + (b - a^2)x_0^2)^{1/2}, \quad B^* = A^*(1 + (|a|/\sqrt{b - a^2})) = const.,$$

$$g(t) = \max\{|c - ad|\, e^{aT}, \ |d|\, e^{aT}\} = \beta = const., \quad h(t) = a - (\lambda/2)\beta = \alpha = const.$$

Assuming $\alpha > 0 \Leftrightarrow \beta < 2a/\lambda$, we have for the solution of the problem (P')

$$|x(t)| \leq (A^*/\sqrt{b - a^2})e^{-\alpha t}, \quad |\dot{x}(t)| \leq A^*(1 + (|a|/\sqrt{b - a^2}))e^{-\alpha t}, \quad t \geq 0.$$

In the particular case $d = 0$, we also have the bounds given by Corollary 1.1 and the Example 1.

Let us again consider the problem (1′) − (2′), under other assumptions than above. Maintaining the hypothesis (H_1) of Proposition 3, we shall impose new hypotheses, namely

(H_3) $\delta \in C^1[0,\infty)$ *and* m *given by* (3) *satisfies* $|m(t)| \leq M < \infty$, $t \geq 0$

(H_4) γ *and* δ *defined by* (3′) *satisfy the conditions*

$$|\gamma(t)| \leq G_1 < \infty, \quad |\dot{\delta}(t)| \leq G_2 < \infty \ \ for \ \ t \geq 0, \quad I = \int_T^\infty |\delta(t)|\, dt < 1$$

The repeated integration of (4′) leads to

$$(17') \quad \begin{aligned} & y(t) - y(0) - \dot{y}(0)t + \int_0^t (t-s)m(s)y(s)ds = \\ & = \int_0^t ds \int_0^s \gamma(\tau)y(\tau - T)d\tau + \int_0^t ds \int_0^s \delta(\tau)\dot{y}(\tau - T)d\tau \end{aligned}$$

for $t \geq 0$. Because we have, for $s \geq 0$,

$$(18') \quad \begin{aligned} & |\int_0^s \gamma(\tau)y(\tau - T)d\tau| \leq G_1(C_1 + \int_0^s |y(t)|\, d\tau), \text{ where} \\ & C_1 = \int_{-T}^0 |y(s)|\, ds = const. \end{aligned}$$

and respectively

$$(19') \quad |\int_0^s \delta(\tau)\dot{y}(\tau - T)d\tau| \leq C_2 + |\delta(s)| \cdot |y(s-T)| + |\delta(T)y(0)| + G_2 \int_0^s |y(\tau)|\, d\tau, \quad s \geq 0$$

where

$$(20') \quad C_2 = \int_{-T}^0 |\delta(s+T)| \cdot |\dot{y}(s)|\, ds = const.$$

we easily obtain from (17′) that

$$(21') \quad |y(t)| \leq |y(0)| + (|\dot{y}(0)| + L)t + K_1 \int_0^t (t-s)\, |y(s)|\, ds + \int_0^t |\delta(s)| \cdot |y(s-T)|\, ds$$

for $t \geq 0$, where we have denoted

$$(22') \quad L = G_1C_1 + C_2 + |\delta(T)y(0)|, \quad K_1 = M + G_1 + G_2$$

Taking into account that

$$(23') \quad \int_0^t |\delta(s)| \cdot |y(s-T)|\, ds \leq C_3 + \int_0^t |\delta(s+T)| \cdot |y(s)|\, ds, \quad t \geq 0$$

where

$$(24') \quad C_3 = \int_{-T}^0 |\delta(s+T)| \cdot |y(s)|\, ds = const.$$

and using the notation

$$(25') \quad f_1(t) = |y(0)| + C_3 + (|\dot{y}(0)| + L)t, \quad t \geq 0$$

we can rewrite (21′) under the form

$$(26') \quad |y(t)| \leq f_1(t) + K_1 \int_0^t (t-s)\, |y(s)|\, ds + \int_0^t |\delta(s+T)| \cdot |y(s)|\, ds, \quad t \geq 0.$$

Denoting $z(t) = \sup |y(s)|$ for $0 \leq s \leq t$, and remarking that $|y(t) \leq z(t)$ for $t \geq 0$ and fixing $t_0 > 0$, we get

$$(27') \quad |y(t)| \leq f_1(t_0) + K_1 \int_0^{t_0} (t_0 - s)z(s)ds + z(t_0)I,$$

for every $t \in [0, t_0]$.

Taking here the supremum for $t \in [0, t_0]$, we have

$$z(t_0) \le f_1(t_0) + K_1 \int_0^{t_0} (t_0 - s) z(s) ds + z(t_0) I \tag{28'}$$

and, since $t_0 > 0$ was arbitrarily taken, the just obtained inequality holds true for every $t_0 > 0$, i.e. we have

$$(1 - I) z(t) \le f_1(t) + K_1 \int_0^t (t - s) z(s) ds, \quad t \ge 0 \tag{29'}$$

Putting

$$K = (1 - I)^{-1} K_1, \quad f(t) = (1 - I)^{-1} f_1(t) \text{ for } t \ge 0 \tag{30'}$$

it follows that

$$z(t) \le f(t) + K \int_0^t (t - s) z(s) ds, \quad t \ge 0 \tag{31'}$$

which is formally identical to (35) and implies

$$| y(t) | \le z(t) \le f(t) ch \sqrt{K} t \le f(t) e^{\sqrt{K} t}, \quad t \ge 0. \tag{32'}$$

On the other hand, integrating (4′) on $[0, t]$ and taking into account (18′) and (19′), we obtain, for the derivative $\dot{y}(t)$, the bound

$$| \dot{y}(t) | \le | \dot{y}(0) | + L + K_1 \int_0^t | y(s) | ds + | \delta(t) | \cdot | y(t - T) |, \quad t \ge 0 \tag{33'}$$

which implies that

$$| \dot{y}(t) | \le | \dot{y}(0) | + L + (K_1 / 2\sqrt{K}) f(t) e^{\sqrt{K} t} + | \delta(t) | f(t - T) e^{\sqrt{K}(t - T)} \tag{34'}$$

for $t \ge T$. Recalling (41) of Section 1, using (32′) and (34′) and also denoting $\alpha(t) = a(t) - \sqrt{K}$ for $t \ge 0$, we finally obtain (for the solution of the problem (1′) – (2′))

$$\begin{gathered} | x(t) | \le f(t) e^{-\int_0^t \alpha(s) ds}, \quad | \dot{x}(t) | \le (| \dot{y}(0) | + L) e^{-\int_0^t a(s) ds} + \\ + \left[(| a(t) | + (K_1 / 2\sqrt{K})) f(t) + e^{-\sqrt{K} T} | \delta(t) | f(t - T) \right] e^{-\int_0^t \alpha(s) ds} \end{gathered} \tag{35'}$$

for $t \ge 0$ and respectively for $t \ge T$. We conclude by stating

Proposition 4. *Under the hypotheses* (H_1), (H_3) *and* (H_4), *the upper bounds* (35′) *hold true, for the solution* $x = x(t)$ *of the problem* (1′) – (2′)).

Remark. Similar results with those given in Proposition 3 and 4 may also be obtained for a more general equation than (1′), namely

$$\ddot{x}(t) + 2a(t)\dot{x}(t) + b(t)x(t) = \sum_{i=1}^{n} (c_i(t) x(t - T_i) + d_i(t) \dot{x}(t - T_i)), \quad t \ge 0 \tag{36'}$$

under similar appropriate assumptions imposed to the coefficients a, b, c_i and d_i, $i = \overline{1, n}$.

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