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Ray Optics On Surfaces

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Abstract. Variational techniques are used to find path of light rays on surfaces. Using Fermat's principle of least time the problem is treated as a constraint optimization to obtain a system of partial differential equations.

Introduction. The time required for a beam of light to traverse a path is called the optical length of the curve. Fermat's principle states that in an optical medium, the path of light from a point A to a point B has the least optical length of all paths joining A and B. Lyusternik [2, Chapter 6] gives an elementary treatment of Fermat's principle and its consequences. Goldstein [1] has a historical treatment of Fermat's principle. Weinstock [4, Chapter 5] discusses Fermat's principle and applications in geometric optics. In this article we use variational techniques to find path of light rays constraint on a given surface. By using Fermat's principle of least time we obtain a system of partial differential equations.

Discussion. First, as a review, we give differential equations for path of light rays in nonhomogeneous media. This is done in Whitham [5, pgs. 247-249]. For simplicity the proof is given in three dimensions.

Assume $c = c(x, y, z)$ is speed of light in the medium. Let σ denote the total time travelled. Then

$$\sigma = \int \frac{ds}{\frac{ds}{dt}} = \int \frac{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}{c(x, y, z)}\, dt\,,$$

where dot denotes differentiation with respect to the arc-length s. Let

$$F(x,y,z,\dot{x},\dot{y},\dot{z}) = \frac{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}{c(x,y,z)} .$$

Using Euler-Lagrange equations in parametric form, we obtain

(1) $$F_x - \frac{d}{dt}F_{\dot{x}} = 0$$

(2) $$F_y - \frac{d}{dt}F_{\dot{y}} = 0$$

(3) $$F_z - \frac{d}{dt}F_{\dot{z}} = 0$$

Then from (1) and expression for $F(x,y,z,\dot{x},\dot{y},\dot{z})$ we have

$$-\frac{\frac{\partial c}{\partial x}}{c^2}\cdot\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2} - \frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\cdot\frac{1}{c}\right) = 0$$

Now we use the chain rule $\frac{d}{dt} = \frac{d}{ds}\cdot\frac{ds}{dt} = \frac{d}{ds}\cdot\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}$.

$$-\frac{\frac{\partial c}{\partial x}}{c^2}\cdot\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2} - \frac{d}{ds}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\cdot\frac{1}{c}\right)\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2} = 0.$$

Note that $\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} = \frac{\frac{dx}{dt}}{\frac{ds}{dt}} = \frac{dx}{ds}$. Then we obtain

(4) $$\frac{\partial c}{\partial x}\cdot\frac{1}{c^2} + \frac{d}{ds}\left(\frac{1}{c}\frac{dx}{ds}\right) = 0.$$

Similarly for y and z we obtain

(5) $$\frac{\partial c}{\partial y}\cdot\frac{1}{c^2} + \frac{d}{ds}\left(\frac{1}{c}\frac{dy}{ds}\right) = 0,$$

(6) $$\frac{\partial c}{\partial z}\cdot\frac{1}{c^2} + \frac{d}{ds}\left(\frac{1}{c}\frac{dz}{ds}\right) = 0.$$

The general form in n dimension with $c = c(x_1,\cdots,x_n)$ is given by

(7) $$\frac{\partial c}{\partial x_i}\cdot\frac{1}{c^2} + \frac{d}{ds}\left(\frac{1}{c}\cdot\frac{dx_i}{ds}\right) = 0.$$

Note that if c is constant rays are straight lines as expected.

Now we use ray optics on surfaces as a constraint optimization by minimizing σ given by

$$\sigma = \int \frac{ds}{c(x,y,z)} = \int \frac{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}{c(x,y,z)}\, dt$$

subject to a surface $G(x,y,z) = 0$. We use Lagrange multiplier $\lambda(t)$. Let

$$F(x,y,z,\dot{x},\dot{y},\dot{z},\lambda) = \frac{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}{c(x,y,z)} + \lambda(t)G(x,y,z)\,.$$

See page 273 of Simmons [3] for this kind of Constraint optimization. We use Euler-Lagrange equation in parametric form. After simplification we obtain the following system of partial differential equations

$$\frac{1}{c^2}\frac{\partial c}{\partial x} + \frac{d}{ds}\left(\frac{1}{c}\frac{dx}{ds}\right) = \lambda(t)\frac{G_x}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\,, \tag{8}$$

$$\frac{1}{c^2}\frac{\partial c}{\partial x} + \frac{d}{ds}\left(\frac{1}{c}\frac{dy}{ds}\right) = \lambda(t)\frac{G_y}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\,, \tag{9}$$

$$\frac{1}{c^2}\frac{\partial c}{\partial z} + \frac{d}{ds}\left(\frac{1}{c}\frac{dz}{ds}\right) = \lambda(t)\frac{G_z}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\,. \tag{10}$$

If we assume c is constant we obtain the following equations for geodesics given in Simmons [3, p 374] as a special case

$$\frac{\frac{d^2x}{ds^2}}{G_x} = \frac{\frac{d^2y}{ds^2}}{G_y} = \frac{\frac{d^2z}{ds^2}}{G_z}\,.$$

Also, if we let $f = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$ we can rewrite the above equations as

$$\frac{\frac{d}{dt}\left(\frac{\dot{x}}{f}\right)}{G_x} = \frac{\left(\frac{d}{dt}\right)\left(\frac{\dot{y}}{f}\right)}{G_y} = \frac{\left(\frac{d}{dt}\right)\left(\frac{\dot{z}}{f}\right)}{G_z}\,.$$

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