LIBERTAS MATHEMATICA, VOL. XVIII (1998)

Asymptotic Behavior of Differential Equations

Raúl Naulin and Jaime Urbina

Key words and phrases: Scalar dichotomies, asymptotic integration.

AMS (MOS) subject classification: Primary 39A11, Secondary 39A10.

Abstract

In this paper, a notion of a scalar dichotomy for an ordinary differential equations of order n is introduced. Using this concept, a general theory of asymptotic integration of the nonautonomous equation $x^{(n)} + (a_{n-1}(t) + b_{n-11}(t))x^{(n-1)} + \cdots + (a_0(t) + b_0(t))x = 0$ is given, provided the estimates of the basic solutions of the linear equation $x^{(n)} + a_{n-1}(t)x^{(n-1)} + \cdots + a_0(t)x = 0$, up to the p-derivative, $0 \leq p \leq n-1$, are known.

1 Introduction

Let us consider the differential operators

$$M[x](t) = x^{(n)}(t) + a_{n-1}(t)x^{(n-1)}(t) + \cdots + a_0(t)x(t)$$

and

$$N[x](t) = b_p(t)x^{(p)}(t) + b_{p-1}(t)x^{(p-1)}(t) + \cdots + b_0(t)x(t), \quad 0 \leq p \leq n-1,$$

where the coefficients a_i and b_i are continuous functions on the interval $J = [0, \infty)$. This paper treats the problem of asymptotic integration of the equation

$$M[y] = N[y], \tag{1}$$

provided we have some information about

$$\mathcal{X} = \{x_1, x_2, ..., x_n\}, \tag{2}$$

a basis of the space of solutions of the nonperturbed equation

$$M[x] = 0. \tag{3}$$

Our main hypothesis is the assumption of existence of positive functions $m_i^r(t)$, such that

$$|x_i^{(r)}(t)| \le m_i^r(t),\ t \in I(t_0),\ r \in \mathcal{N}_p, \tag{4}$$

where $I(t_0) = [t_0, \infty)$ and

$$\mathcal{N} = \{1, 2, ..., n\}, \mathcal{N}_p = \{0, 1, ..., p\},\quad 0 \le p \le n-1.$$

The desired result is the obtaining of n linearly independent solutions y_i of Eq. (3) having the asymptotic formulas

$$y_i^{(r)} = x_i^{(r)} + o(m_i^r),\ r \in \mathcal{N}_p,$$

where $o(m_i^r)$ denotes a function safisfying

$$\lim_{t\to\infty} \frac{o(m_i^r)(t)}{m_i^r(t)} = 0.$$

This problem is classical [2, 3]. In the scope of this paper we refer to the research accomplished in [5, 6, 7], where a notion of a scalar dichotomy is introduced as an analog to the Levinson dichotomy for diagonal systems [1, 4].

In this paper we introduce a new definition of a scalar Levinson dichotomy, different from that given in [5, 6]. We show that this concept can be used in the description of the solutions of Eq. (1). In an example displayed in the section 4 we show how the asymptotic integration of Eq. (1) can be obtained for equations where the Levinson asymptotic theorem [1, 2] cannot be applied. The notion of a Levinson scalar dichotomy is motivated by the paper of Kusano [3].

2 Preliminaries

For a positive function h, $L^1(h)$ will denote the space of h-integrable functions, that is $f \in L^1(h)$ iff $h^{-1}f$ is integrable.

Let us assume that Eq. (3) has the basis (2). By W we denote the Wronskian of these solutions; W_i is the determinant obtained from the Wronskii matrix by replacing the i-column by the n-basic vector $e_n = \mathrm{col}(0,0,...,1)$. The following identities are known

$$\sum_{i=1}^{n} x_i^{(r)} W_i(t) = 0,\ 0 \le r \le n-2, \quad \sum_{i=1}^{n} x_i^{(n-1)}(t) W_i(t) = W(t). \tag{5}$$

Further, besides the functions m_i^r satisfying (4) we suppose that there exists a set of positive and continuous functions, R_i, $1 \le i \le n$, satisfying

$$\left|\frac{W_i(t)}{W(t)}\right| \le R_i(t), \quad t \in I(t_0). \tag{6}$$

Let us introduce the following notations

$$G_{i,j,r}(t) = \frac{m_i^r(t)}{m_i^r(s)} \frac{m_j^r(s)}{m_j^r(t)}, \quad 1 \le i,j \le n, \quad r \in \mathcal{N}_p,$$

and

$$h_r(t) = \max\{m_i^r(t) R_i(t) :\ 1 \le i \le n\},\ r \in \mathcal{N}_p, \tag{7}$$

where p represent an integer number, such that $p \in \mathcal{N}_p$. In the future, the ordered collection of positive functions

$$\Omega = \{h_0, h_1, ..., h_p\}$$

will be called the weight of the scalar dichotomy of precision p.

Definition 1 *We shall say that Eq. (3) allows a Levinson scalar dichotomy of precision p iff:*

D_1 *: There exist a basis (2) of the space of solutions of Eq. (3) and functions* m_i^r, R_i *satisfying (4) and (6).*

D_2 *: For every pair of integers* $(i,j) \in \mathcal{N} \times \mathcal{N}$ *one satisfies*

$$\lim_{t\to\infty} \int_{t_0}^{\infty} G_{i,j,r}(t,\tau)|f(\tau)|d\tau = 0, \quad \forall f \in L^1(h_r), \quad 0 \le r \le p, \tag{8}$$

either

$$\lim_{t\to\infty} \int_{t}^{\infty} G_{i,j,r}(t,\tau)|f(\tau)|d\tau = 0, \quad \forall f \in L^1(h_r), \quad 0 \le r \le p. \tag{9}$$

3 Asymptotic integration

Suppose that the estimates (4) and (6) are valid. To each solution $x_i \in X$ and $0 \leq p \leq n-1$, $C^{(p)}(x_i)$ denote the functional space of p-times continuously differentiable funtions f, such that $(m_i^r)^{-1}f^{(r)}$ is bounded for any $r \in \mathcal{N}_p$. The space $C^{(p)}(x_i)$ can be endowed with the norm

$$|f|_{(p)} = \max\{|(m_i^r)^{-1}f^{(r)}|_\infty : \quad 0 \leq r \leq p\},$$

where

$$|f|_\infty = \sup\{|f(t)| : t \in I(t_0)\}.$$

Clearly, the pair $(C^{(p)}(x_i), \ |\cdot|_{(p)})$ is a complete normed space. For each fixed $i \in N$, let us define the following sets of integers numbers:

$$I_i = \{j \in \mathcal{N} : (j,i) \text{ satisfies } (14)\},$$

$$F_i = \{j \in \mathcal{N} : (j,i) \text{ satisfies } (15)\}.$$

Lemma 1 *If $y \in C^{(p)}(x_i)$, then*

$$|N[y](t)| \leq \sum_{r=0}^{p} m_i^r(t)|b_r(t)||y|_{(p)}.$$

Proof. If follows from

$$|N[y](t)| \leq \sum_{r=0}^{p} |b_r(t)|m_i^r(t)|(m_i^r)^{-1}(t)y^{(r)}(t)|$$

and the definition of the norm $|\cdot|_{(p)}$. □

With the notation

$$U_j(t,s) = x_j(t)\frac{W_j(s)}{W(s)}; \quad 1 \leq j \leq n,$$

the identities (5) can be written in the form

$$\sum_{i=1}^{n} \frac{\partial^r}{\partial t^r}U_j(t,t) = 0, \ r \in \mathcal{N}_{n-2}, \ \sum_{i=1}^{n} \frac{\partial^n}{\partial t^n}U_j(t,t) = 1.$$

For each i, $i \in \mathcal{N}$, let us consider the operator

$$\mathcal{T}(y)(t) = x_i(t) + \sum_{k \in I_i} \int_{t_0}^{t} U_k(t,s)N[y](s)ds - \sum_{k \in F_i} \int_{t}^{\infty} U_k(t,s)N[y](s)ds \quad (10)$$

By differentiating (10) we obtain for $0 \leq r \leq p$

$$\begin{aligned} \mathcal{T}^{(r)}(y)(t) &= x_i^{(r)}(t) + \sum_{k \in I_i} \int_{t_0}^{t} \frac{\partial^r}{\partial t^r} U_k(t,s) N[y](s) ds \\ &- \sum_{k \in F_i} \int_{t}^{\infty} \frac{\partial^r}{\partial t^r} U_k(t,s) N[y](s) ds, \end{aligned} \tag{11}$$

and

$$\begin{aligned} \mathcal{T}^{(n)}(y)(t) &= x_i^{(n)}(t) + \sum_{k \in I_i} \int_{t_0}^{t} \frac{\partial^n}{\partial t^n} U_k(t,s) N[y](s) ds \\ &- \sum_{k \in F_i} \int_{t}^{\infty} \frac{\partial^n}{\partial t^n} U_k(t,s) N[y](s) ds + N[y](t). \end{aligned} \tag{12}$$

Although the last formulas (11) and (12) have a formal deduction, they can be justified by means of the following

Theorem 1 *Let us assume that Eq. (1) allows the Levinson scalar dichotomy of precision p (8)-(9). If $b_r \in L^1(h_r)$, $0 \leq r \leq p$, then the Eq. (1) has a set of n linearly independent solutions $y_1, y_2, \ldots, y_n$, such that*

$$y_i^{(r)} = x_i^{(r)} + o(m_i^r), \quad 0 \leq r \leq p. \tag{13}$$

Proof. We will prove that

$$\mathcal{T} : C^{(p)}(x_i) \to C^{(p)}(x_i), \tag{14}$$

and that $\mathcal{T}$ is a contraction. From (11) we have

$$\begin{aligned} |m_i^r(t)^{-1}\mathcal{T}^{(r)}(y)(t)| &\leq |m_i^r(t)^{-1} x_i^{(r)}(t)| \\ &+ \sum_{k \in I_i} \int_{t_0}^{t} m_i^r(t)^{-1} |\frac{\partial^r}{\partial t^r} U_k(t,s)||N[y](s)| ds \\ &+ \sum_{k \in F_i} \int_{t}^{\infty} m_i^r(t)^{-1} |\frac{\partial^r}{\partial t^r} U_k(t,s)||N[y](s)| ds. \end{aligned}$$

Using Lemma 1, we can write the estimate

$$
\begin{aligned}
|m_i^r(t)^{-1}\mathcal{T}^{(r)}(y)(t)| &\leq |x_i|_{(p)} \\
&+ \sum_{k\in I_i}\sum_{r=0}^{p}\int_{t_0}^{t} G_{k,i,r}(t,s)R_k(s)m_k^r(s)|b_r(s)ds|y|_{(p)} \\
&+ \sum_{k\in F_i}\sum_{r=0}^{p}\int_{t}^{\infty} G_{k,i,r}(t,s)R_k(s)m_k^r(s)|b_r(s)ds|y|_{(p)}.
\end{aligned}
$$

Since $b_r \in L^1(h_r)$, the right hand side of this last estimate is bounded. Therefore (14) is satisfied. In analogous form we obtain

$$
\begin{aligned}
|m_i^r(t)^{-1}\mathcal{T}^{(r)}(y-z)(t)| &\leq \sum_{k\in I_i}\sum_{r=0}^{p}\int_{t_0}^{t} G_{k,i,r}(t,s)h_r(s)|b_r(s)|ds|y-z|_{(p)} \\
&+ \sum_{k\in F_i}\sum_{r=0}^{p}\int_{t}^{\infty} G_{k,i,r}(t,s)h_r(s)|b_r(s)|ds|y-z|_{(p)}.
\end{aligned}
$$

Using again (8) and (9), there exist a t_0 such that for $t \geq t_0$ we have

$$
\sum_{r=0}^{p}\left(\sum_{k\in I_i}\int_{t_0}^{t} + \sum_{k\in F_i}\int_{t}^{\infty}\right) G_{k,i,r}(t,s)h_r(s)|b_r(s)|ds < 1.
$$

From this we obtain that the operator $\mathcal{T}$ is contractive. Henceforth $\mathcal{T}$ has a fixed point y_i satisfying

$$
y_i(t) = x_i(t) + \sum_{k\in I_i}\int_{t_0}^{t} U_k(t,s)N[y](s)ds - \sum_{k\in F_i}\int_{t}^{\infty} U_k(t,s)N[y](s)ds.
$$

From this identity, we obtain that y_i is a function of class C^n, and from (11) and (12), we may prove that y_i satisfies Eq.(1). Now, for each $r \in \mathcal{N}_p$, Definition 1 implies that the function

$$
\sum_{k\in I_i}\int_{t_0}^{t} G_{k,i,r}(t,s)h_r(s)|b_r(s)|ds + \sum_{k\in F_I}\int_{t}^{\infty} G_{k,i,r}(t,s)h_r(s)|b_r(s)|ds
$$

tends to zero if t tends to infinity. Thus, formula (13) is valid. □

4 An example

Let us contemplate the Euler equation

$$x'' + t^{-1}x' - t^{-2}x = 0. \tag{15}$$

We are interested in the asymptotic integration of the equation

$$y'' + (t^{-1} - b_1(t))y' - (t^{-2} - b_0(t))y = 0. \tag{16}$$

A right step to solve this problem is in the reducing Eq. (16) to the two dimensional system:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \left[t^{-2}\begin{pmatrix} 0 & t^2 \\ 1 & -t \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ b_0 & b_1 \end{pmatrix} \right] \begin{pmatrix} x \\ y \end{pmatrix}, \quad t \geq 1, \tag{17}$$

a system that can be studied starting from the linear one

$$\begin{pmatrix} x \\ y \end{pmatrix}' = t^{-2}\begin{pmatrix} 0 & t^2 \\ 1 & -t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad t \geq 1.$$

We observe that the function

$$A(t) = \begin{pmatrix} 0 & 1 \\ t^{-2} & -t^{-1} \end{pmatrix},$$

satisfies

$$\lim_{t\to\infty} A(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The last says that system (17) cannot be integrated from the ideas of Levinson asymptotic theorem [2].

Nevertheless, Eq. (16) can be asymptotic integrated from Theorem 1 of this paper. We point out that the set $\{t^{-1}, t\}$ constitutes a basis of Eq. (15). Condition (4) is accomplished with $m_1^0(t) = t^{-1}$, $m_2^0(t) = t$, $m_1^1(t) = t^{-2}$, $m_2^1(t) = 1$, $R_1(t) = t^2$ and $R_2(t) = 1$. From definition (7) we obtain $h_0(t) = t$, $h_1(t) = 1$. The pairs of indexes $(1,1)$, $(2,2)$ and $(2,1)$ satisfy condition (9), while $(1,2)$ satisfies (8). Thus Eq. (15) has a Levinson scalar dichotomy of precision 1. From Theorem 1, the Eq. (16) has two linearly independent solutions satisfying

$$y_i^{(r)}(t) = (1 + o(1))x_i^{(r)}(t); \quad i = 1,2, \quad r = 0,1,$$

if $b_1 \in L^1(1)$ and $b_0 \in L^1(t^{-1})$.

The reader may observe that Eq. (16) can be reduced to the form

$$\ddot{z}(s) - b_1(e^s)\dot{z}(s) - (1 - b_0(e^s))z(s) = 0,$$

where $\dot{z} = \frac{dz}{ds}$. The asymptotic integration of this equation can be accomplished by Levinson asymptotic theorem [2] and the results of the asymptotic integration coincide with those obtained from Theorem 1 of this paper.

The reader may observe that the analysis of Eq. (15) could be adapted to the non Euler equation

$$x''(t) + \frac{2\sin^2 t}{\sin t \cos t - t}x'(t) + \frac{\sin t \cos t + t}{t - \sin t \cos t}x(t) = 0, \ t > 1,$$

with linearly independent solutions $x_1(t) = \sin t$, $x_2(t) = t\cos t$. It is not clear the possibility of applying the Levinson asymptotic theorem to the equation

$$x''(t) + \left(\frac{2\sin^2 t}{\sin t \cos t - t} + a(t)\right) x'(t) + \left(\frac{\sin t \cos t + t}{t - \sin t \cos t} + b(t)\right) x(t) = 0, \ t > 1.$$

Acknowledgements. This research was supported by the grant Proyecto CI-1-0101-0782/95.

References

[1] Coddington, E. A., Levinson, N., *Theory of Ordinary Differential Equations*, McGraw-Hill, New York (1965).

[2] Eastham, M. S. P., *The Asymptotic Solution of Linear Differential Systems (Applications of the Levinson Theorem)*, Clarendon Press, Oxford (1989).

[3] Kusano, T., Asymptotic relationships between two higher order ordinary differential equations, *Internat. J. Math. And. Math. Sci.*, 6, 559-566 (1983).

[4] Levinson, N., Asymptotic nature of solutions of linear systems of differential equations, *Duke Math. J.*, 15, 129-139 (1948).

[5] Naulin, R., Dichotomies and asymptotic integration of linear ordinary differential equations of order n, *Journal of Nonlinear Analysis, TMA*, Vol. 30, 2, 1125-1132 (1997).

[6] Naulin, R., Urbina, J., Asymptotic integration of linear ordinary differential equations of order n, *Acta Math. Hungar.*, 80 (1-2), 111-123 (1998).

[7] Urbina, J., *Integración Asintótica de Ecuaciones Diferenciales Ordinarias de Orden n*, Trabajo de Ascenso, Universidad de Oriente (1996).