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THREE-PERSONS GAMES WITH A CROSS-ENTROPIC CRITERION

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Abstract

In this paper we use a cross-entropic criterion to find an optimal strategy with respect to an apriori given probability distribution of a player having a countably infinite number of pure strategies in a three-persons game. A dual approach on a linearly constrained entropy optimization problem is used here. It extends others entropic criteria approaches to the problem of optimal strategies of a player with only a finite number of pure strategies.

1 Introduction

Entropy optimization problems with explicit linear restrictions have received much attention in recent years because they have applications in many fields including information theory and game theory. We follow here the idea from [3],[4] of finding the maximum of one player's gain function by approaching a constrained entropy optimization problem from its dual side according to Ben-Tal [1],[2], Fang and Tsao [6],[9]. It will permit us to determine, by using a cross-entropic criterion in the gain function, the optimal strategy of a player, having a countably infinite number of pure strategies, with respect to an apriori given probability distribution. Most results on optimal strategies in three-persons games with entropic criteria were centering around probability distributions with a finite number of pure strategies of the players [7],[3],[4].

We shall present our framework and explicitly define the entropy optimization problem in Section 2. In Section 3 we shall associate its dual to our entropy optimization problem and use a dual approach based on some results from [6],[9]. It will permit us to obtain some results on the optimal strategy of a player in the given game. In Section 4 we shall establish the corresponding results in three–persons games with a cross-entropic criterion in which the players have a finite number of pure strategies. We shall conclude our findings in Section 5.

2 Problem statement

Let us consider a nonzero sum game with the players J_1, J_2, J_3 having a countably infinite number of pure independent strategies, $X = \{x_1, x_2, ...\}$, $Y = \{y_1, y_2, ...\}$, $Z = \{z_1, z_2, ...\}$. Let

$$\begin{aligned} &\xi = (\xi_1, \xi_2, ...);\ \eta = (\eta_1, \eta_2, ...);\ \zeta = (\zeta_1, \zeta_2, ...); \\ &\xi_i \geq 0\ (i = 1, 2, ...);\ \eta_j \geq 0\ (j = 1, 2, ...);\ \zeta_k \geq 0\ (k = 1, 2, ...); \\ &\sum_{i=1}^{\infty} \xi_i = \sum_{j=1}^{\infty} \eta_j = \sum_{k=1}^{\infty} \zeta_k = 1, \end{aligned} \tag{2.1}$$

be random strategies on a countably infinite sample space adopted by the three players. Let u^1_{ijk} be the utility for the player J_1 of the variant of the game composed by the strategies x_i, y_j, z_k of the players. Then the expected utility of the player J_ℓ, $(\ell = 1, 2, 3)$ will be

$$\mathcal{U}^\ell(\xi, \eta, \zeta) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} u^\ell_{ijk} \xi_i \eta_j \zeta_k, \tag{2.2}$$

where the sum is taken over all the pure strategies of the players.

In what follows, let us suppose the random strategies η and ζ of the players J_2 and J_3 are fixed. Let $p = (p_1, p_2, ...)$, with $p_i > 0$, $i = 1, 2, ...$, $\sum_{i=1}^{\infty} p_i = 1$ be an apriori given distribution that estimates $\{\xi_i\}$. The expected utility of the player J_1 depends explicitely in this case only on his random strategy

$$\mathcal{U}^1(\xi) = \sum_{i=1}^{\infty} u_i \xi_i, \text{ where } u_i = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} u^1_{ijk} \eta_j \zeta_k,\ i = 1, 2, ... \tag{2.3}$$

if all u_i, $i = 1, 2, ...$, are finite and the series $\sum_{i=1}^{\infty} u_i \xi_i$ is convergent.

The random strategy ξ contains a quantity of uncertainty, in the given hypothesis, measured by the cross-entropy between the probability distributions ξ and p, which is defined by

$$(2.4) \qquad CE(\xi, p) = \sum_{i=1}^{\infty} \xi_i \ln\left(\frac{\xi_i}{p_i}\right), \text{ if } \sum_{i=1}^{\infty} \xi_i \ln\left(\frac{\xi_i}{p_i}\right) \text{ is convergent.}$$

Since the expected utility and the cross-entropy are defined as infinite sums and the infinite sums do not always exist, we naturally confine our attention to only those probability distributions $\xi = (\xi_1, \xi_2, ...)$ such that these infinite sums exists. That is why we shall suppose that the random strategies possibly used by the player J_1 satisfy the condition

$$(2.5) \qquad \text{the series } \sum_{i=1}^{\infty} \xi_i \ln\left(\frac{\xi_i}{p_i}\right) \text{ and } \sum_{i=1}^{\infty} u_i \xi_i \text{ are convergent.}$$

Let us suppose that the aim of the player J_1 in the considered three-persons game is the maximization of his gain defined as a weighted combination of his expected utility and of the cross-entropy $CE(\xi, p)$.

Let us suppose that the random strategies ξ possibly used by the player J_1 satisfy, besides the condition (2.5), a set of linear restrictions of the form

$$(2.6) \qquad \sum_{i=1}^{\infty} w_{ji} \xi_i = d_j, \ (j = 1, ..., m).$$

We may suppose, without restrain the generality, that the linear restrictions $\sum_{i=1}^{\infty} w_{ji} \xi_i = d_j$, $(j = 1, ..., m)$ imply $\sum_{i=1}^{\infty} \xi_i = 1$.

Let us denote by H the set of the random strategies $\xi > 0$, that means completely mixt, of the player J_1 which satisfy the conditions (2.5) and (2.6). This leads us to define the UCER Pattern (expected Utility-Cross-Entropy-Restrictions).

The UCER Pattern. The gain function of the player J_1 has the form

$$(2.7) \qquad \mathcal{F}_1(\xi) = \lambda_1 \sum_{i=1}^{\infty} u_i \xi_i - \lambda_2 \sum_{i=1}^{\infty} \xi_i \ln\left(\frac{\xi_i}{p_i}\right), \ \lambda_1, \lambda_2 > 0,$$

with apriori given $p = (p_1, p_2, ...)$, $p_i > 0$, $(i = 1, 2, ...)$ and $\xi \in H$.

The gain functions of the players J_2 and J_3 have similar forms. We are interested in finding a probability distribution, among all admissible ones, attaining the maximum gain function of the player J_1 with respect to the given apriori, the random strategies η, ζ being supposed known. Such a strategy will be referred as an optimal strategy.

We note by $\mathbb{N}$ the set of the natural numbers and by $\mathbb{N}^* = \mathbb{N} - \{0\}$. For every pair $\lambda_1, \lambda_2 > 0$ we associate to the UCER Pattern the following optimization problem

$$\text{(P)} \qquad \sup_{\xi \in H} \left(\lambda_1 \sum_{i=1}^{\infty} u_i \xi_i - \lambda_2 \sum_{i=1}^{\infty} \xi_i \ln \left(\frac{\xi_i}{p_i} \right) \right).$$

Problem (P) gives the optimal strategy of the player J_1 in the game according with the UCER Pattern, for $\lambda_1, \lambda_2 > 0$ fixed, that means the optimal strategy to pass from the apriori distribution $p = (p_1, p_2, ...)$ estimating $\{\xi_i\}$ to the probability distribution $\xi = (\xi_1, \xi_2, ...,) \in H$.

The dual approach to determine the optimal strategy of the player J_1 compatible with the apriori known probability distribution p is based on the following two lemmas we shall give without proof.

3 Main results on optimal strategies

Lemma 3.1. (Corollary 1 in [9]) *Let $p = (p_1, p_2, ...)$ with $p_i > 0$, $i \in \mathbb{N}^*$ and $\xi = (\xi_1, \xi_2, ...)$ with $\xi_i > 0$ for any $i \in \mathbb{N}^*$ be two probability distributions. Then, for any real numbers y_i, $i \in \mathbb{N}^*$, we have the inequality*

$$(3.1) \qquad \sum_{i=1}^{\infty} \xi_i \ln \left(\frac{\xi_i}{p_i} \right) \geq -\ln \left(\sum_{i=1}^{\infty} p_i e^{y_i} \right) + \sum_{i=1}^{\infty} \xi_i y_i,$$

whenever the three infinite sums exist.

Lemma 3.2. (Corollary 2 in [9]) *Let $p = (p_1, p_2, ...)$ be with $p_i > 0$, $i \in \mathbb{N}^*$ and $\xi = (\xi_1, \xi_2, ...)$ with $\xi_i > 0$, $i \in \mathbb{N}^*$, two probability distributions. Then the inequality 3.1 from Lemma 3.1 becomes an equation if and only if*

$$(3.2) \qquad \xi_i = \frac{p_i e^{y_i}}{\sum_{j=1}^{\infty} p_j e^{y_j}}, \ i \in \mathbb{N}^*.$$

Moreover, in this case,

$$-\sum_{i=1}^{\infty}\xi_i \ln\left(\frac{\xi_i}{p_i}\right)+\sum_{i=1}^{\infty}\xi_i y_i=\ln\sum_{i=1}^{\infty}p_i e^{y_i}.$$

Lemma 3.3. *Let $\lambda_1,\lambda_2>0$, $\xi=(\xi_1,\xi_2,...)$ and $p=(p_1,p_2,...)$ with $p_i>0$, $\xi_i>0$ for any $i\in\mathbb{N}^*$ be two probability distributions. If $\xi\in H$, then, for any numbers $u_i>0$, $i\in\mathbb{N}^*$ and any vector with real components $y=(y_1,...,y_m)$ we have the inequality*

$$\begin{aligned}&\lambda_1\sum_{i=1}^{\infty}u_i\xi_i-\lambda_2\sum_{i=1}^{\infty}\xi_i\ln\left(\frac{\xi_i}{p_i}\right)\leq\\ &\leq\lambda_2\ln\left(\sum_{i=1}^{\infty}p_i\exp\left(\lambda_2^{-1}\left(\lambda_1u_i+\sum_{j=1}^{m}w_{ji}y_j\right)\right)\right)-\sum_{j=1}^{m}d_jy_j,\end{aligned}\tag{3.3}$$

whenever the infinite sum of the right side exists and has finite value.

Proof. For $\xi\in H$ the infinite sums of the left side of the inequality (3.3) exist and have finite values. For $i\in\mathbb{N}^*$ let us consider the real numbers z_i defined by

$$z_i=\frac{\lambda_1u_i+\sum_{j=1}^{m}w_{ji}y_j}{\lambda_2}=\lambda_2^{-1}\left(\lambda_1u_i+\sum_{j=1}^{m}w_{ji}y_j\right).\tag{3.4}$$

For the real numbers z_i, $i\in\mathbb{N}^*$ and the probability distributions ξ and p we apply Lemma 3.1. One obtains

$$\sum_{i=1}^{\infty}\xi_i\ln\left(\frac{\xi_i}{p_i}\right)\geq-\ln\left(\sum_{i=1}^{\infty}p_ie^{z_i}\right)+\sum_{i=1}^{\infty}\xi_iz_i,\tag{3.5}$$

if the two infinite sums of the right side exist.

Multiplying the inequality (3.5) by $-\lambda_2,\lambda_2>0$ it follows

$$-\lambda_2\sum_{i=1}^{\infty}\xi_i\ln\left(\frac{\xi_i}{p_i}\right)\leq\lambda_2\ln\sum_{i=1}^{\infty}p_ie^{z_i}-\lambda_2\sum_{i=1}^{\infty}\xi_iz_i.\tag{3.6}$$

On the other hand, because $\xi\in H$ we have the following relations

$$\sum_{i=1}^{\infty}w_{ji}\xi_i=d_j,\ (j=1,...,m)\tag{3.7}$$

and from (3.4) we have

$$\lambda_1 u_i - \lambda_2 z_i + \sum_{j=1}^{m} w_{ji} y_j = 0, \; i \in \mathbb{N}^*. \tag{3.8}$$

For each j, $(j = 1, ..., m)$ we multiply the relation (3.7) by y_j and we sum by $j = 1, ..., m$. It follows that

$$\sum_{j=1}^{m} \sum_{i=1}^{\infty} y_j w_{ji} \xi_i - \sum_{j=1}^{m} d_j y_j = 0. \tag{3.9}$$

Now we multiply the relation (3.8) by ξ_i and then we sum by $i \in \mathbb{N}^*$ obtaining

$$\lambda_1 \sum_{i=1}^{\infty} u_i \xi_i - \lambda_2 \sum_{i=1}^{\infty} z_i \xi_i + \sum_{i=1}^{\infty} \xi_i \sum_{j=1}^{m} w_{ji} y_j = 0 \tag{3.10}$$

equivalent, according to the relation (3.9), to the relation

$$\lambda_1 \sum_{i=1}^{\infty} u_i \xi_i - \lambda_2 \sum_{i=1}^{\infty} z_i \xi_i + \sum_{j=1}^{m} d_j y_j = 0. \tag{3.11}$$

Adding (3.11) to (3.6) one obtains

$$\lambda_1 \sum_{i=1}^{\infty} u_i \xi_i - \lambda_2 \sum_{i=1}^{\infty} \xi_i \ln\left(\frac{\xi_i}{p_i}\right) \le \lambda_2 \ln \sum_{i=1}^{\infty} p_i e^{z_i} - \sum_{j=1}^{m} d_j y_j. \tag{3.12}$$

Replacing z_i given by the relation (3.4) in (3.12) one obtains the inequality (3.3) from Lemma 3.3 if the infinite sum of the right side exists. □

According to Lemma 3.2, the inequality (3.3) becomes an equation if

$$\xi_i = \frac{p_i \exp\left(\lambda_2^{-1}\left(\lambda_1 u_i + \sum_{j=1}^{m} y_j w_{ji}\right)\right)}{\sum_{k=1}^{\infty} p_k \exp\left(\lambda_2^{-1}\left(\lambda_1 u_k + \sum_{j=1}^{m} y_j w_{jk}\right)\right)}, \; i \in \mathbb{N}^*, \tag{3.13}$$

where $\sum_{k=1}^{\infty} p_k \exp\left(\lambda_2^{-1}\left(\lambda_1 u_k + \sum_{j=1}^{m} y_j w_{jk}\right)\right) < \infty$.

Let be $\Omega = \left\{ z = (z_1, z_2, ...) \mid \text{the series } \sum_{i=1}^{\infty} p_i e^{z_i} \text{ and } \sum_{i=1}^{\infty} \xi_i z_i \text{ are convergent} \right\}$ and

$$\Omega' = \{ y = (y_1, ..., y_m) \mid \lambda_2 z_i = \lambda_1 u_i + \sum_{j=1}^{m} y_j w_{ji},\ z = (z_1, z_2, ...) \in \Omega \}.$$

We remark that for $\xi \in H$ and $y \in \Omega'$ all the infinite sums from the inequality (3.3) exist and have finite values. The inequality (3.3) allows us to associate to Problem (P) its dual

$$\text{(D)} \qquad \inf_{y \in \Omega'} \left(\lambda_2 \ln \left(\sum_{i=1}^{\infty} p_i \exp \left(\lambda_2^{-1} \left(\lambda_1 u_i + \sum_{j=1}^{m} w_{ji} y_j \right) \right) \right) - \sum_{j=1}^{m} d_j y_j \right).$$

Problem (D) is an optimization problem without explicit restrictions.

We note by $F(\xi)$ and $G(y)$ the objective functions of the dual problems (P), respectively (D). For $\xi \in H$ and $y \in \Omega'$ all the infinite sums appearing in their expression exist and have finite value

$$F(\xi) = \lambda_1 \sum_{i=1}^{\infty} u_i \xi_i - \lambda_2 \sum_{i=1}^{\infty} \xi_i \ln \left(\frac{\xi_i}{p_i} \right),$$

$$G(y) = \lambda_2 \ln \left(\sum_{i=1}^{\infty} p_i \exp \left(\lambda_2^{-1} \left(\lambda_1 u_i + \sum_{j=1}^{m} y_j w_{ji} \right) \right) \right) - \sum_{j=1}^{m} d_j y_j.$$

Regarding the pair of problems (P) and (D) we have the following important results:

Theorem 3.1. (weak duality) *If $\xi \in H$ is an admissible solution of the problem* (P) *and $y \in \Omega'$, then*

$$F(\xi) \leq G(y).$$

Proof. It follows from Lemma 3.3, the form of the dual and the definition of Ω'. ◻

Theorem 3.2. (strong duality) *Let $y^0 \in \mathrm{int}(\Omega')$ be an optimal solution of the problem (D) and*

$$(3.14) \qquad \xi_i{}^0 = \frac{p_i \exp\left(\lambda_2^{-1}\left(\lambda_1 u_i + \sum_{j=1}^{m} y_j^0 w_{ji}\right)\right)}{\sum_{k=1}^{\infty} p_k \exp\left(\lambda_2^{-1}\left(\lambda_1 u_k + \sum_{j=1}^{m} y_j^0 w_{jk}\right)\right)}, \; i \in \mathbb{N}^*.$$

If $\xi^0 \in H$, then ξ^0 is an optimal solution of the problem (P) and

$$F(\xi^0) = G(y^0) = \left(\sum_{k=1}^{\infty} p_k \exp\left(\lambda_2^{-1}\left(\lambda_1 u_k + \sum_{j=1}^{m} y_j^0 w_{ji}\right)\right)\right)^{-1} \cdot$$
$$\cdot \sum_{i=1}^{\infty} p_i \left[\lambda_2 \ln\left(\sum_{k=1}^{\infty} p_k \exp\left(\lambda_2^{-1}\left(\lambda_1 u_k + \sum_{j=1}^{m} y_j^0 w_{ji}\right)\right)\right) - \sum_{j=1}^{m} y_j^0 w_{ji}\right] \cdot$$
$$\cdot \exp\left(\lambda_2^{-1}\left(\lambda_1 u_i + \sum_{j=1}^{m} y_j^0 w_{ji}\right)\right).$$

Proof. Since $y^0 \in \Omega'$, the sum $\sum_{k=1}^{\infty} p_k \exp\left(\lambda_2{}^{-1}\left(\lambda_1 u_k + \sum_{j=1}^{m} y_j^0 w_{jk}\right)\right)$ from (3.14) exists and has finite value. If y^0 is an interior point of the set Ω' and y^0 is an optimal solution of Problem (D), it must satisfy the condition of stationary point, that means

$$\left(\frac{\partial G(y)}{\partial y_k}\right)_{y=y^0} = 0, \; (k = 1, ..., m).$$

But

$$\left(\frac{\partial G(y)}{\partial y_k}\right)_{y=y^0} = \lambda_2 \frac{\sum_{i=1}^{\infty} p_i \frac{w_{ki}}{\lambda_2} \exp\left(\lambda_2^{-1}\left(\lambda_2 u_i + \sum_{j=1}^{m} y_j^0 w_{ji}\right)\right)}{\sum_{k=1}^{\infty} p_k \exp\left(\lambda_2^{-1}\left(\lambda_1 u_k + \sum_{j=1}^{m} y_j^0 w_{jk}\right)\right)} - d_k =$$

$$= \sum_{i=1}^{\infty} \left(\frac{p_i \exp\left(\lambda_2^{-1}\left(\lambda_1 u_i + \sum_{j=1}^{m} y_j^0 w_{ji}\right)\right)}{\sum_{k=1}^{\infty} p_k \exp\left(\lambda_2^{-1}\left(\lambda_1 u_k + \sum_{j=1}^{m} y_j^0 w_{jk}\right)\right)} \right) w_{ki} - d_k.$$

The optimality conditions of the first order are

$$\sum_{i=1}^{\infty} \left(\frac{p_i \exp\left(\lambda_2^{-1}\left(\lambda_1 u_i + \sum_{j=1}^{m} y_j^0 w_{ji}\right)\right)}{\sum_{k=1}^{\infty} p_k \exp\left(\lambda_2^{-1}\left(\lambda_1 u_k + \sum_{j=1}^{m} y_j^0 w_{jk}\right)\right)} \right) w_{ki} = d_k, \ (k = 1, ..., m)$$

which, according to the relation (3.14) are reducing to

$$\sum_{i=1}^{\infty} \xi_i{}^0 w_{ki} = d_k, \ (k = 1, , ..., m)$$

that means that ξ^0 is an admissible solution to Problem (P) in the hypothesis that the corresponding cross-entropy is well defined and the series $\sum_{i=1}^{\infty} u_i \xi_i$ is convergent. As a direct consequence of Theorem 3.1, we have that ξ^0 is an optimal solution of Problem (P).

Moreover, in this case, we have that the optimal values of the two dual problems coincide, that is

$$F(\xi^0) = G(y^0).$$

The optimal value of the objective functions corresponding to the pair of dual problems (P) and (D) is obtained by a direct calculus.

Remark 3.1. The optimal solution of Problem (P) is the optimal strategy of the player J_1 with respect to the apriori given probability distribution in the three-persons game according to the UCER Pattern.

4 Considerations on the finite number of pure strategies case

All our results obtained in Section 3 for a countably infinite number of pure strategies can be reformulated for three-persons games with a finite number of pure strategies for each player and an apriori given distribution.

Let $p = (p_1, ..., p_r)$ be the apriori given distribution. We shall consider J_1's random strategies $\xi = (\xi_1, ..., \xi_r)$ satisfying linear constraints

$$(4.1) \qquad \sum_{i=1}^{r} w_{ji}\xi_i = d_j, \ (j = 1, ..., m).$$

We also assume, without loss of generality, that (4.1) implies $\sum_{i=1}^{r} \xi_i = 1$. Denote $\{\xi = (\xi_1, ..., \xi_r \mid \xi_i > 0, (i = 1, ..., r) \text{ verifying } (4.1)\}$ by $\widetilde{H}$.

The UCERF Pattern (UCER Pattern in the Finite case). The gain function of the player J_1 is

$$(4.2) \qquad \mathcal{F}_1(\xi) = \lambda_1 \sum_{i=1}^{r} u_i\xi_i - \lambda_2 \sum_{i=1}^{r} \xi_i \ln\left(\frac{\xi_i}{p_i}\right), \lambda_1, \lambda_2 > 0, \xi \in \widetilde{H}.$$

We consider the following pair of dual problems:

$$(\text{PF}) \qquad \sup_{\xi \in \widetilde{H}} \left(\lambda_1 \sum_{i=1}^{r} u_i\xi_i - \lambda_2 \sum_{i=1}^{r} \xi_i \ln\left(\frac{\xi_i}{p_i}\right)\right),$$

$$(\text{DF}) \qquad \inf_{y \in \widetilde{\Omega}} \left(\lambda_2 \ln \sum_{i=1}^{r} p_i \exp\left({\lambda_2}^{-1}\left(\lambda_1 u_i + \sum_{j=1}^{m} w_{ji}y_j\right)\right)\right) - \sum_{j=1}^{m} d_j y_j,$$

unde $\widetilde{\Omega} = \{y = (y_1, ..., y_m) \mid \lambda_2 z_i = \lambda_1 u_i + \sum_{j=1}^{m} y_j w_{ji}, z_i \in \mathbb{R}, i = 1, ..., r\}$.

Denote the two objective functions by $F(\xi)$ and, respectively, $G(y)$.

Regarding the pair of problems (PF) and (DF) we have the following duality theorem.

Theorem 4.1.

(i) (weak duality) *If* $\xi \in \widetilde{H}$ *is a admissible solution of Problem* (PF) *and* $y \in \widetilde{\Omega}$, *then* $F(\xi) \leq G(y)$.

(ii) (strong duality) *Let* $y^0 \in \text{int}(\widetilde{\Omega})$ *the optimal solution of the problem* (DF) *and*

$$\xi_i^0 = \frac{p_i \exp\left({\lambda_2}^{-1}\left(\lambda_1 u_i + \sum_{j=1}^{m} y_j^0 w_{ji}\right)\right)}{\sum_{k=1}^{r} p_k \exp\left({\lambda_2}^{-1}\left(\lambda_1 u_k + \sum_{j=1}^{m} y_j^0 w_{jk}\right)\right)}, (i = 1, ..., r).$$

If $\xi^0 \in \widetilde{H}$, then ξ^0 is an optimal solution of Problem (PF) *and the optimal values of the two dual problems coincide.*

Remark 4.1. The optimal solution of Problem (PF) is the optimal strategy of the player J_1 with respect to the apriori given distribution in the three–persons game according to UCERF Pattern. Taking the uniform repartition as the apriori given distribution we recover our results on optimal strategies from [3],[4], which generalize some results obtained by Guiaşu in [7].

5 Conclusions

In this paper we obtained some results on optimal strategy of a player with respect to an apriori given distribution in three–persons games with a cross–entropic criteria. We treated the countably infinite number of pure strategies case and formulated the corresponding results for the finite number case. We used a dual approach generalizing the dual problems considered in [9], so that this paper extends the application of optimal entropy analysis in game theory.

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