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Existence of Bounded Solutions to Nonlinear Discrete Equations

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The aim of this paper is to obtain some existence results for discrete equations of the form

(1)
$$x_{n+1} = f(n, x_n),$$

in which $n \in Z_+ = \{0, 1, 2, ...\}$ or $n \in Z$, and f is a map from $Z_+ \times R$ into R, or from $Z \times R$ into R ($R =$ the real line). We restrict first our considerations to the scalar case, and then we will indicate how similar results can be obtained in the vector case($x_n \in R^m$).

Our main concern is to provide conditions on $f(n, x)$, such that (1) has bounded solutions on Z_+ or on Z. Let us point out that in the papers [1], [3], the existence of bounded solutions has been assumed in order to prove other qualitative properties, such as the convergence to a limit as $n \to \infty$ (in case of equations defined on Z_+), or the almost periodicity (in case of equations defined on Z). Of course, adequate assumptions must be made on $f(n, x)$, in order to assume such kind of behavior.

1 Case of Linear Equations on Z_+ and Z

Before we consider the general equation (1), we need to get some information on the very simple linear equation

(2)
$$x_{n+1} = Mx_n + b_n,$$

in which $M > 1$ is a constant, and $\{b_n\}$ is a bounded sequence:$| b_n |\leq B$, for some $B > 0$, and $n \in Z_+$ or $n \in Z$, according to the case under discussion.

First, let us deal with equation (2) when $n \in Z_+$, $M > 1$, and $\{b_n\}$ is bounded. It is easily seen that the following formula holds true(by induction) for any $n \geq 1$:

(3)
$$x_n = M^n(x_0 + \frac{b_0}{M} + \frac{b_1}{M^2} + ... + \frac{b_{n-1}}{M^n}).$$

If we wish $\{x_n\}$ to be bounded, the only possibility is to choose the initial value x_0 in such a way that

(4)
$$x_0 + \frac{b_0}{M} + \frac{b_1}{M^2} + ... + \frac{b_{n-1}}{M^n} \to 0 \qquad \text{as } n \to \infty.$$

Since $\{b_n\}$ is bounded and $M > 1$, it is obvious that the series

(5)
$$\frac{b_0}{M} + \frac{b_1}{M^2} + ... + \frac{b_{n-1}}{M^n} + ...$$

converges absolutely. Therefore, the only possible choice for x_0 to obtain a bounded solution of (1), i.e. , to assure the boundedness of the sequence (3), is

$$x_0 = -\left(\frac{b_0}{M} + \frac{b_1}{M^2} + \ldots + \frac{b_{n-1}}{M^n} + \ldots\right). \tag{6}$$

If x_0 from (6) is substituted in (3), after elementary operations one obtains

$$x_n = -\frac{1}{M}\left(b_n + \frac{b_{n+1}}{M} + \frac{b_{n+2}}{M^2} + \ldots\right). \tag{7}$$

Again, the convergence of the series (5) gives legitimacy to the formula (7). It is obvious that (7) leads immediately to $Mx_n + b_n = x_{n+1}$, i.e., the equation (2).

It is important for what follows to obtain an estimate for the solution $\{x_n\}$ given by (7). That this solution is bounded one can easily see from the fact that (5) converges absolutely. One obtains from (7) and our assumptions:

$$\mid x_n \mid \leq \frac{1}{M}\left(B + \frac{B}{M} + \frac{B}{M^2} + \ldots\right), \qquad n \in Z_+ - \{0\}, \tag{8}$$

which leads to

$$\mid x_n \mid \leq \frac{1}{M}\left(\frac{B}{1 - M^{-1}}\right) = \frac{B}{M-1}, \qquad n \in Z_+. \tag{9}$$

If we take $B = \sup \mid b_n \mid$, then (9) can be also written as

$$\sup_{n \in Z_+} \mid x_n \mid \leq \frac{1}{M-1} \sup_{n \in Z_+} \mid b_n \mid . \tag{10}$$

This estimate provides the necessary tool in order to carry out the discussion from the linear case to the nonlinear one represented by the equation (1).

If instead of $n \in Z_+$ in the equation (2) we consider the case $n \in Z$, still assuming that $\{b_n\}$ is bounded on Z and $M > 1$, then a similar discussion can be conducted and we obtain instead of (10) the inequality

$$\sup_{n \in Z} \mid x_n \mid \leq \frac{1}{M-1} \sup_{n \in Z} \mid b_n \mid . \tag{11}$$

It is important to notice that in either case ($n \in Z_+$ *or* $n \in Z$), the solution given by (7) is the only bounded solution of equation (2).

Indeed, formula (3) provides an arbitrary solution of (2) on Z_+, where x_0 is the (arbitrary) initial value. It shows that for any choice of x_0, excepting that value given by (6), the solution of (2) is unbounded on Z_+ (because $M^n \to \infty$ as $n \to \infty$, while the parenthesis tends to something different of zero). This discussion obviously covers the case of equation (2) on Z.

2 Case of Nonlinear Equations on Z_+

We consider now the nonlinear discrete equation (1), under the following assumptions on $f(n,x)$:

a) The sequence $\{f(n,0)\}$, $n \in Z_+$, is bounded.

b) For every $n \in Z_+$, the derivative $f_x(n,x)$ does exist for any $x \in R$, and satisfies the inequalities (12), where m and M are constants.

$$1 < m \leq f_x(n,x) \leq M \tag{12}$$

The following theorem can be proven by using the Banach contraction mapping theorem.

Theorem 1 *If the map $f : Z_+ \times R \to R$ satisfies conditions a) and b) stated above, then equation (1) has a unique bounded solution on Z_+.*

Proof. We will choose as underlying space for our proof the space of all bounded maps (sequences) from Z_+ into R. This space, say X, is a Banach space if the norm is chosen as the supremum norm, i.e., for $x = \{x_0, x_1, x_2, ..., x_n, ...\} \in X$, one denotes

$$| x |= \sup\{| x_k |; k \in Z_+\}. \tag{13}$$

On the space X, the following operator U is defined as follows: for any $x \in X$, let $y \in X$ be the unique solution of the discrete equation

$$y_{n+1} = My_n + f(n,x_n) - Mx_n, \quad n \in Z_+. \tag{14}$$

We agree to set $y_0 = x_0$. Then we say that $y = Ux$.

Of course, it is necessary to show that $b_n = f(n,x_n) - Mx_n$ is a bounded sequence, for any $x \in X$ (i.e., for any bounded x). In order to show that $b = \{b_0, b_1, ..., b_n, ...\} \in X$, we notice the fact

$$| f(n,x_n) |\leq| f(n,x_n) - f(n,0) | + | f(n,0) | .$$

Hence,

$$| b_n |\leq M | x_n | + | f(n,0) |,$$

which shows that $b \in X$. Consequently, the discrete equation (14) has a unique solution $y \in X$, for every $x \in X$, which means that the operator U is defined on the whole space X.

It remains to be shown that U is a contraction mapping on X. Indeed, let x, ξ be arbitrary in X, and let $y = Ux$, $\quad \eta = U\xi$. We easily obtain from (14) and

$$\eta_{n+1} = M\eta_n + f(n,\xi_n) - M\xi_n, \quad \eta_0 = \xi_0, \tag{15}$$

the equation

$$y_{n+1} - \eta_{n+1} = M(y_n - \eta_n) + f(n,x_n) - f(n,\xi_n) - M(x_n - \xi_n). \tag{16}$$

This means that $y - \eta$ satisfies an equation of the form (2). As seen above, the following estimate, derived from (10), holds true:

$$\sup_{n \in Z_+} | y_n - \eta_n | \leq \frac{1}{M-1} \sup_{n \in Z_+} | f(n, x_n) - f(n, \xi_n) - M(x_n - \xi_n) | . \tag{17}$$

If we take into account condition b), (17) leads to the inequality

$$\sup_{n \in Z_+} | y_n - \eta_n | \leq \frac{M-m}{M-1} \sup_{n \in Z_+} | x_n - \xi_n | . \tag{18}$$

In other words, we can write

$$| Ux - U\xi | \leq \frac{M-m}{M-1} | x - \xi | . \tag{19}$$

The inequality (19) holds true for any $x, \xi \in X$, and condition (12) in b) implies $\frac{M-m}{M-1} < 1$. Therefore, the operator U is a contraction mapping of X into itself.

The unique fixed point of the operator U obviously satisfies the equations (1).

This ends the proof of Theorem 1.

Corollary 1 *If we assume, besides a) and b), that*

$$\lim_{n \to \infty} f(n, x) = f_\infty(x) \tag{20}$$

does exist uniformly with respect to x in any bounded set R, then as shown in [3], the solution of (1) whose existence has been proved is such that $\lim_{n \to \infty} x_n$ exists.

Remark 1. The condition b), particularly the inequality (12) imposed on the derivative $f_x(n, x)$, can be somewhat relaxed. If we assume only $f_x(n, x) \geq m > 1$, as well as the boundedness of $f_x(n, x)$ in any strip $Z_+ \times I$, where I is any finite interval of R, then the existence and uniqueness of the bounded solution of (2) on Z_+ is still assured. The proof goes on the same lines, with the difference that Banach fixed point theorem must be applied into a sufficiently large ball $| x | \leq r$, instead of the whole space of bounded sequences. There is also a price to pay for this, namely one must assume $\sup | f(n, 0) |$ small enough, to have a solution in $| x | \leq r$.

Remark 2. A similar situation to that encountered in Theorem 1 occurs when the inequality (12) in condition b) is substituted by

$$-M \leq f_x(n, x) \leq -m < -1, \tag{21}$$

for all $n \in Z_+$ and $x \in R$. The discussion of this case can be conducted exactly in the same way, first showing that the linear equation $x_{n+1} = -Mx_n + b_n$ has a unique bounded solution on Z_+, for b_n bounded there.

Remark 3. We also notice the fact that a condition of the form

$$-1 < -m \leq f_x(n, x) \leq m < 1 \tag{22}$$

on the derivative, implies the Lipschitz type condition for $f(n,x)$:

$$| f(n,x) - f(n,y) | \leq m \mid x - y \mid, \tag{23}$$

for all $n \in Z_+$ and $x, y \in R$.

It can be easily seen in this case that the usual iterative method is convergent to the unique bounded solution of equation (1). In other words, the fixed point method has to be applied to the mapping $y = Ux$ defined by

$$y_{n+1} = f(n, x_n), \qquad n \geq 0, y_0 = x_0.$$

3 Case of Nonlinear Equations on Z

We have remarked at the end of §1 that the equation (2), considered for $n \in Z$, has also a unique bounded solution on Z, provided $\{b_n\}_{n \in Z}$ is bounded. This unique bounded solution satisfies the estimate (11).

We shall now consider the equation (1) on Z, under the following hypotheses:

a_1) The sequence $f(n, 0)$, $n \in Z$, is bounded.

b_1) For every $n \in Z$, the derivative $f_x(n,x)$ exists for all $x \in R$, and satisfies the inequality

$$1 < m \leq f_x(n,x) \leq M, \tag{24}$$

for some constants m and M.

The following existence result can be proven for the equation (1).

Theorem 2 *Assume that the map $f : Z \times R \to R$ satisfies the hypotheses a_1) and b_1). Then the equation (1) has a unique bounded solution on Z.*

Proof. This time, the underlying space X consists of all bounded sequences on Z, with values in R, and with the supremum norm. The operator U is formally defined by means of equation (14), this time without initial condition, and for all $n \in Z$.

The remaining part of the proof can be conducted on the same lines as the proof of Theorem 1.

Corollary 2 *If besides a_1) and b_1) one assumes that the map $n \to f(n,x)$ from Z into R, is almost periodic, uniformly with respect to x in any bounded set of R, then the unique bounded solution (on Z) of (1) is almost periodic.*

The proof of almost periodicity has been given in [3].

4 Case of Vector Valued Functions

In concluding this paper, we want to indicate the procedure to follow in order to cover the case when $f(n,x)$ is vector valued(say, $f \in R^m$).

The only real difference which appears, in comparison with the scalar case discussed above, is the condition on the derivative f_x. Actually, we do not need the existence of the derivative and a monotonicity condition of the form

$$m \mid x - y \mid^2 \leq \langle f(n,x) - f(n,y), x - y \rangle \leq M \mid x - y \mid^2,$$

in which $m > 1$, will suffice to obtain the estimates that show the convergence. This procedure has been amply used in [3], where various estimates have been obtained for proving the almost periodicity of a bounded solution.

Finally, let us mention that existence of bounded solutions (on Z_+ or Z) can be obtained for the second order equation

$$x_{n+2} = f(n, x_n, x_{n+1}). \tag{25}$$

In [3], we have shown that bounded solutions of (25), if they exist, satisfy also various qualitative properties, in accordance with the properties enjoyed by (25).

In a forthcoming paper we shall discuss the second order case.

It should be mentioned that various topics related to discrete processes are treated in [2].

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