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ON THE EXISTENCE OF A WEAK SOLUTION TO A VOLTERRA - SKOROHOD EQUATION

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In [6] we proved the existence of a weak solution for the equation

1) $$X_t = V_t + \int_0^t k(t, s, X_s)ds + \int_0^t g(t, s, X_s)dW_s +$$

$$+ \iint_{]o,t]\times\|v\|\leq 1} h(t, s, X_s, v)q(ds, dv) + \iint_{]o,t]\times\|v\|\geq 1} h(t, s, X_s, v)p(ds, dv)$$

in a real separable Hilbert space, where w is an abstract Wiener process and p(ds,dv) is the measure of "saltus" of a homogeneous R^d valued process Y(t) with independent increments.

We are dealing with the existence of a weak solution for a modified form of (1), equation which appears in the so-called mild solutions of stochastic evolution equation, that is, we consider the equation

2) $$X_t = U(t, o)X_o + \int_0^t U(t, s)k(s, X_s, \omega)ds + \int_0^t U(t, s)g(s, X_s, \omega)dW_s$$

$$+ \iint_{[0,t[\times\|v\|\leq 1} U(t, s)h(s, X_s, v, \omega)q(ds, dv) + \iint_{[0,t[\times\|v\|> 1} U(t, s)h(s, X_s, v, \omega)q(ds, dv)$$

We shall make the following notations:

- $\Delta = \{(t,s), \quad 0\leq s\leq t\leq T\}$
- $(\Omega, F, P, (F_t), t\in[o,T])$ is a filtering probability space satisfying the usual assumptions:
- (i,K,H) is an abstract Wiener space with P_t the Wiener measure, K,H being both, real, separable Hilbert spaces.

- we denote by $\| \|_H$ the norm in H and by $<\cdot, \cdot>$ its inner product; and by $\| \|$ the usual norm in R^d.

- W is the Wiener process corresponding to the Wiener abstract space (i,K,H), so its state space will be H.

- L(H) is the space of the bounded linear operators on H, and its norm is denoted by $\| \|_{L(H)}$.

- The Hilbert Schmidt operators on H are endowed with H.S. norm $\|\cdot\|_2$.

- Let $p(\omega,ds,dv)$ be a time homogeneous Poisson point process with Levy measure $\alpha(dv) = \frac{dv}{v^2}$ and $q(\omega,ds,dv) = p(\omega,ds,dv) - \frac{dt, dv}{v^2}$, $v \in R^d-\{0\}$, that is p(ds,dv) is the measure of "saltus" of a R^d valued process Y(t) with independent increments. Y(t) is supposed independent of W(t), and F_t adapted

-D([o,T],H) will be the space of functions on [o,T] with values in H, with discontinuities of at most the first kind, endowed with the Skorohod j_1, topology.

-U(t,s) is a mild evolution operator exponentially bounded on [o,T], [see[2],[3],[10]], that is, there is a $b > 0$, such that

3) $\| U(t,s) \|_{L(H)} \leq e^{b(t-s)}$, for all $0 \leq s \leq t \leq T$.

First we prove the following:

Lemma 3. The function U(t,s)x, **from** D **to** H **is Lipschitz in the first coordinate** t, **that is, for any** $x \in H$,

4) $\| U(t+h,s)x-U(t,s)x \|_H \leq Lh\|U(t,s)x\|_H$, L **a positive constant not depending on** t, s, $\| x \|_H$.

Proof Let $\{t_k\}$ be a monotonically decreasing sequence of positive numbers tending to 0 and let $L > 0$ be such that

5) $\| U(t_k+t,t)x-x \|_H \leq Lt_k\|x\|_H$ for any $x \in H$.

It is possible to get such a relation from the strongly continuity of U in Δ.

Let $h > 0$ and let n_k be a nonnegative integer such that $0 < h-n_k t_k < t_k$.

Then

$\| U(t+h,s)x-U(t,s)x\|_H = \| (U(t+h,t)-1)U(t,s)x \|_H$, $U(t,s)x = y$ is an element from H, so the previous relation becomes

6) $\|U(t+h)x-U(t,s)x \|_H = \| U(t+h,t)y-y\|_H =$

$= \| U(t+h,t+n_k t_k)U(t+n_k t_k,t)y-U(t+n_k t_k,t)y \|_H$

$+ \| U(t+n_k t_k,t)y-y \|_H$

Using the property of semigroup of evolution, and (5), we get

$\| U(t+n_k t_k,t)y-y\|_H = \| \sum_{j=1}^{n_k} U(t+jt_k,t+(j-1)t_k)U(t+(j-1)t_k,t)y-$

$-U(t+(j-1)t_k,t)y \|_H < Ln_k t_k \| y \|_H < Lh \| y \|_H$.

The first term of the right-hand side of (6) converges to 0 when

$n_k \to \infty$, so finally we get the relation (4).

We prove now the following:

Theorem. **Suppose, we are given** $(W_t)_{t\in[o,T]}$, **an abstract Wiener process, on K, a time-homogeneous Poisson point process, an exponentially bounded mild evolution satisfying** (3).

-X_0 **is a** F_o **measurable, H random variable**

-$k:[o.T] \times H \to H$ **measurable**

-$g:[o,T] \times H \to L^2(H)$ **strongly measurable**

Assume that:

I) **there exists a positive content** L **and a symmetric positive and compact operator** S **on** H **such that** S^{-1} **commutes with** U **and**

$E \| S^{-1} X_0 \|_H < \infty$

II) $\| S^{-1}k(s,X) \|_2 + Tr[S^{-2}g(t,x)Wg(t,x)] +$

$$+ \iint_{]0,t]\times\|v\|\leq 1} \frac{\left\|S^{-1}h(s,x,v)\right\|_H^2}{v^2} ds\, dv \leq L(1+\|x\|^2_H),$$

for all $s,x,v \in [0,T] \times H \times R^d - \{0\}$

III) **for every** $s \in [0,T]$, $k(s,x)$, $g(s,x)$, $h(s,x,v)$ **are continuous with respect to the norm** $\| \|_H$ **(respectively** $\| \|_2$, **and measurable in** s, **or** (s,v)**).**

IV) $$\lim_{\substack{s\to s_1\\ x\to x_1}} \int_{\|v\|\leq 1} \frac{\left\|h(s,x,v) - h(s_1,x_1,v)\right\|_H}{v^2} dv = 0 .$$

Then, the equation (2) possesses a weak Cadlag solution.

Remark: The form of the equation (2) and the properties of $U(t,s)$ allow to give up the Lipschitz condition of the coefficients of the equation (1).

Proof: For every n, define by induction the successive intervals $\left[k\frac{T}{n},(k+1)\frac{T}{n}\right]$, and the approximate process

$$7)\quad \begin{cases} x_t^n = x_0 & 0 \leq t \leq \frac{T}{n} \\ x_t^n = U(t,\frac{T}{n})x_0 + \int_0^{t-\frac{T}{n}} U(t,s)k(s,x_s^n)ds + \int_0^{t-\frac{T}{n}} U(t,s)g(s,x_s^n)dW_s + \end{cases}$$

$$+ \iint_{]0,t-\frac{T}{n}]\times\|v\|<1} U(t,s)h(s,x_s^n,v)q(ds,dv) +$$

$$+ \iint_{]0,t-\frac{T}{n}]\times\|v\|\geq 1} U(t,s)h(s,x_s^n,v)p(ds,dv) .$$

Since (8) $\sup_{t,s\in\Delta} \left\|U(t,s)\right\|_{L(H)} < \infty$, it follows by standard calculation that (9) $\sup_t E\left\|x_t^n\right\|_H^2 < \infty$.

For the simplicity of the notation, we write

$$x_t^n = A_n(t) + B_n(t) + C_n(t) + D_n(t) + E_n(t),$$

A,B,C,D,E representing the 5 terms of the right-hand side of (7).

In [9] it is proved that A(t), B(t), C(t) have a continuous version in the hypothesis of the theorem.

We intend to prove that D and E have Cadlag versions. Because a Hilbert space possesses the Radon-Nikodim property (see [7]), and the relation (9) takes place, it will be enough to prove that for any sequence $t_r \in [o,T]$, we have

$$10) \quad \sum_n E\left(\left\|\left(D_n(t_{r+1}) / F_{t_r}\right) - D_n(t_r)\right\|_H\right) \le KT.$$

We prove (10) by induction. Obviously, for $t_r \in \left[0, \frac{T}{n}\right]$, (10) happens, taking into account (7).

For any increasing sequence t_r belonging to $\left[k\frac{T}{n}, (k+1)\frac{T}{n}\right]$, we get

$$\sum_r E\left(\left\|E\left(D_n(t_{r+1}) / F_{t_r}\right) - D_n(t_r)\right\|_H\right) =$$

$$\sum_r E\Bigg\| E \iint_{]0,t_{r+1}-\frac{T}{n}]\times\|v\|\le 1} U(t_{r+1}, s)h\left(s, x_s^n, v\right)q(ds, dv) -$$

$$\iint_{]0,t_{r+1}-\frac{T}{n}]\times\|v\|\le 1} U(t_r, s)h\left(s, x_s^n, v\right)q(ds, dv) / F_{t_r}\Bigg\|_H$$

$$\le \sum_r E\Bigg\| E \iint_{]0,t_r]\times\|v\|\le 1} \Big[U(t_{r+1}, s)h\left(s, x_s^n v\right)$$

$$- U(t_r, s)h\left(s, x_s^n v\right)\Big]q(ds, dv) / F_{t_r}\Bigg\| +$$

$$+ \sum_r \Bigg\| E \int_{]t_r-\frac{T}{n}, t_{r+1}-\frac{T}{n}]\times\|v\|\le 1} U(t_{r+1}, s)h\left(s, x_s^n, v\right)q(ds, dv) / F_{t_r}\Bigg\|_H = D_1 + D_2$$

The first term of the last side is denoted by D_1, and second by D_2.

Using the result of the previous lemma and the fact that the measure $q(ds,dv) = p(ds,dv) - \frac{ds\,dv}{v^2}$, we obtain

$$D_1 \leq \sum_r E\left\| \left(\iint_{]0,t_r]\times\|v\|\leq 1} \left(U(t_{r+1}, s)h\left(s, x_s^n, v\right) - U(t_r, s)h\left(s, x_s^n, v\right)\right) p(ds, dv) / F_{t_r} \right)\right\|_H$$

$$+ \sum_r E\left\| E\left(\iint_{]0,t_r]\times\|v\|\leq 1} \left(U(t_{r+1}, s)h\left(s, x_s^n, v\right) - U(t_r, s)h\left(s, x_s^n, v\right)\right) \frac{ds\,dv}{v^2} / F_{t_r} \right)\right\|_H$$

$$\leq \sum_r E\left[E \iint_{]0,t_r]\times\|v\|\leq 1} \left\| U(t_{r+1}, s)h\left(s, x_s^n, v\right) - U(t_r, s)h\left(s, x_s^n, v\right)\right\|_H p(ds, dv) / F_{t_r} \right]$$

$$+ \sum_r E\left[E \iint_{]0,t_r]\times\|v\|\leq 1} \left\| U(t_{r+1}, s)h\left(s, x_s^n, v\right) - U(t_r, s)h\left(s, x_s^n, v\right)\right\|_H \frac{ds, dv}{v^2} / F_{t_r} \right]$$

$$\leq \sum L\left((t_{t+1} - t_r) \sup_{0\leq s\leq t_r - \frac{T}{n}} \|U(t_r, s)\|_{L(H)} \right) E \iint_{]0,t_r - \frac{T}{n}]\times\|v\|\leq 1} \left\| h\left(s, x_s^n, v\right)\right\|_H q(ds, dv) +$$

$$+ 2L \sum_r (t_{r+1} - t_r) \sup_{0\leq s\leq t_r - \frac{T}{n}} \|U(t_r, s)\|_{L(H)} \cdot \iint_{]0,t_r - \frac{T}{n}]\times\|v\|\leq 1} \left\| h\left(s, x_s^n, v\right)\right\|_H \frac{ds\,dv}{v^2}$$

Taking into account that $E \iint_{]0,t_r - \frac{T}{n}]\times\|v\|\leq 1} \left\| h\left(s, x_s^n, v\right)\right\|_H q(ds, dv) = 0$

and the hypothesis II) of the theorem, we get

$$D_1 \leq 2L_1 T \sup_{0\leq s\leq t\leq T} \|U(t, s)\|_{L(H)} \|X_0\|_H .$$

For t_{r+1} fixed, $D_2 = 0$, because the integral appearing in its expression is a martingale.

We conclude that (7) has a Cadlag version. In [9] it is proved that

11) $\lim_{r\to\infty} \sup_{n,t} P\left(\left\|S^{-1} Z_t^n\right\|_H \geq r\right) = 0$

12) $\lim_{h \to 0} \sup_{n} \sup_{|t-s| \le h} P\left(\left\| Z_t^n - Z_s^n \right\|_H \ge \varepsilon\right) = 0$

for every $\varepsilon > 0$, where $Z_n = X_0, A_n, B_n, C_n, D_n, E_n$ $(X_n = A_n+B_n+C_n+D_n+F_n)$.

We have in mind to prove 1) also for $Z_n = D_n, E_n$.

First, we sketch the proof of 12) for D_n.

$$13) \quad E\left\|D_n(t) - D_n(U)\right\|_H^2 = E\left(\left\| \iint_{]0,\, t-\frac{T}{n}] \times \|v\| \le 1} U(t,s)h\left(s, x_s^n, v\right)q(ds, dv) - \right.\right.$$

$$\left.\left. \iint_{]0,\, u-\frac{T}{n}] \times \|v\| \le 1} U(u,s)h\left(s, x_s^n, v\right)q(ds, dv\right\|_H\right)^2 \le$$

$$2 \int_{]0,\, t-\frac{T}{n}]} \int_{\|v\| \le 1} E\left\|U(t,s)h\left(s, x_s^n, v\right) - U(u,s)h\left(s, x_s^n, v\right)\right\|_H^2 \frac{ds\, dv}{v^2}.$$

$$+ 2 \int_{]0t-\frac{T}{n},\, u-\frac{T}{n}] \times} \int_{[\|v\| \le 1]} E\left\|U(u,s)h\left(s, x_s^n, v\right)\right\|_H^2 \frac{ds\, dv}{v^2} = K_1 + K_2.$$

Lemma (1) and the condition II and I of the theorem, the assumption that $\frac{T}{n} < 1$, the form (7) of the approximate process result in $K_1 \le (T-u)^2 C_1$, C_1 a constant depending on T, L, $\| S \|_{L(H)}$ $\left\|S^{-1} X_0\right\|_H^2$, but not on n.

The condition (8), (9), the assumption I and II of the theorem implies $K_2 \le (t-u) C_2$, C_2 a constant depending only on T, L, $\| S \|_{L(H)}$, $\| S^{-1} X_0 \|_H$.

So, finally, we get
$E\left\|C_n(t) - C_n(u)\right\|_H^2 \le (t-u)C$, from which we get 12) for $Z_n = D_n$.

For $Z_n = E_n$ we get (12) using Lemma 1, and the Lemma 1, page 56 of [8] which gives the existence of a random number γ and random points $z_1, z_2, \ldots, z_\gamma$ on $[0,T]$, which $v_1, v_2, \ldots v_\gamma \in R^d, \ldots, \|v_j\| > 1$, for $j=\overline{1,\gamma}$

$$F_n = \sum_{K=1}^{\gamma} U(t, z_k) h\left(z_K, x^n_{z_K}, v_k\right)$$

(8), (9) and the hypothesis of the theorem implies (11) for $Z_n = D_n$, $Z_n = E_n$.

Now we continue using the same argument as in [9]. Applying again a result of Skorohod [8], we obtain a system $\left(\tilde{X}^n, \tilde{A}_n, \tilde{B}_n, \tilde{C}_n, \tilde{D}_n, \tilde{F}_n, \tilde{X}_0, \tilde{Y}_n, \tilde{W}_n\right)$ on $D([0,T],H)$ and which converges in probability pointwise to a system $(\tilde{X}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{F}, \tilde{X}, \tilde{Y}, \tilde{W})$. The process $\tilde{W}$ is a Wiener process and $\tilde{Y}$ is a R^d-valued homogeneous process, with independent increments.

$\tilde{Y}_n$ had the same law as Y, so $\tilde{Y}_n$ are processes with independent increments having the same law as Y, hence $\tilde{Y}$ inherits the same properties. We can think about the measures of saltus of the process $\tilde{Y}, \tilde{p}, \tilde{q}$ having the same law (and properties) as p and q defined at the beginning of the paper. For details see [11].

The process $\tilde{W}$ is an abstract Wiener process having the same low as W. For details see [1] and [9].

The identity of finite dimensional distributions implies that $\left(\tilde{A}^n_t\right)_{t \le \frac{T}{n}}$ has a continuous version, hence, a.s.

$\tilde{A}^n_t = \tilde{X}^n_0$ for all $t < \frac{T}{n}$. If $t > \frac{T}{n}$ we deduce from Lemma 2.2 in [1] that for each t, $\tilde{A}^n_t = U(t, \frac{T}{n})\tilde{X}^n_0$ w.p.1.

Therefore, the process $\left(\tilde{X}^n_t\right)_{t<\frac{T}{n}}$ may be considered continuous.

Also, we have (as in [9])

$$\text{14)} \quad \begin{cases} \tilde{X}^n_t = \tilde{A}_n(t) & 0 \le t \le \frac{T}{n}. \\ \tilde{X}^n_t = \tilde{A}_n(t) + \tilde{B}_n(t) + \tilde{C}_n(t) + \tilde{D}_n(t) + \tilde{F}_n(t), & \frac{T}{n} < t \le T. \end{cases}$$

Now, we shall show by induction on the successive subintervals $I(K,n)=\left[K\frac{T}{n},(K+1)\frac{T}{n}\right]$ that $\left(\tilde{X}^n\right)$ has a Cadlag version and that $\tilde{A}_n(t), \tilde{B}_n(t), \tilde{C}_n(t), \tilde{D}_n(t), \tilde{E}_n(t)$ have the same expression as $A_n(t), B_n(t), C_n(t), D_n(t), F_n(t), \tilde{X}_t^n$ playing the role of x_t^n, $\tilde{W}_n(t)$ of $W(t)$, $\tilde{q}_n(ds,dv)$ of $q(ds,dv)$, $\tilde{p}_n(ds,dv)$ of $p(ds,dv)$.

Actually, the statement about $\tilde{A}_n(t), \tilde{B}_n(t), \tilde{C}_n(t)$, was proved in [9], we show only that

15) $\displaystyle \tilde{D}_n(t) = \iint_{]0,t-\frac{T}{n}]\times\|v\|\leq 1} U(t,s)h\left(s,\tilde{X}_s^n,v\right)\tilde{q}_n(ds,dv).$

16) $\displaystyle \tilde{F}_n(t) = \int_{]0,t-\frac{T}{n}\times\|v\|>1} U(t,s)h\left(s,\tilde{X}_s^n,v\right)\tilde{p}_n(ds,dv).$

Suppose that $\left(\tilde{X}_s^n\right)_{s\in I(K,n)}$ has a Cadlag version and that 15), 16) hold, for $t\in I(K,n)$ Fix $t\in I(K+1,n)$.

We have $\tilde{D}_n(t)=\alpha_t\left(\tilde{X}_s^n\right)_{s\in I(K,n)}$ with $\alpha_t : D(I(K,n),H)\to H$, the measurable function defined by

$$\alpha_t(X) = \int_{]0,t-\frac{T}{n}]\times\|v\|\leq 1} U(t,s)h(s,X_s,v)\tilde{q}^n(ds,dv).$$

The process $\left(x_s^n\right)_{s\in I(K,n)}$, $\left(\tilde{X}_s^n\right)_{s\in I(K,n)}$ have the same low in $D(I(k,n),H)$, hence by Lemma 2.2 from [1] 15) follows and we deduce that $\tilde{D}_n(t)$ has a Cadlag version on $I(K+1,n)$, hence also $\tilde{X}^n$.

From (7) and (14) we conclude

$$
\left\{
\begin{aligned}
&\tilde X^{n}_{t} = \tilde X^{n}_{0} \qquad 0 \le t \le \tfrac{T}{n} \\
&\tilde X^{n}_{t} = U\left(t, \tfrac{T}{n}\right)\tilde X^{n}_{0} + \int_{0}^{T-\frac{T}{n}} U(t,s)k\left(s, \tilde X^{n}_{s}, u\right)ds + \\
&+ \int_{0}^{T-\frac{T}{n}} U(t,s)g\left(s, \tilde X^{n}_{s}\right)d\tilde W^{n}_{s} + \iint_{]0,t-\frac{T}{n}]\times\|v\|\le 1} U(t,s)\,h\left(s, \tilde X^{n}_{s}, v\right)\tilde q_{n}(ds,dv) + \\
&+ \iint_{]0,t-\frac{T}{n}]\times\|v\|> 1} U(t,s)\,h\left(s, \tilde X^{n}_{s}, v\right)\tilde p_{n}(ds,dv)\,.
\end{aligned}
\right. \tag{17}
$$

Using the hypothesis of the theorem it follows from (17), by taking the limit in probability that $\left(\tilde X_t\right)$ is a solution of (2).

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