

ON THE EXISTENCE AND UNIQUENESS OF INTEGRABLE SOLUTIONS FOR  
INTEGRAL EQUATIONS IN SEVERAL VARIABLES

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1. The aim of this note is to give a short proof of the existence and uniqueness of solution to the integral equation

$$(1) \quad u(x,y) = F\left(x,y, \int_0^x u(s,y)ds, \int_0^y u(x,t)dt, \int_0^x \int_0^y u(s,t)dtds, u(x,y)\right),$$

considered in the space of integrable functions.

This problem was discussed by M. B. Suryanarayana [4], [5] in the case when  $F$  does not depend on the last variable and by S. Walczak [6] under the additional assumption that  $F$  is linear with respect to the third, fourth and fifth variables. The proof provided in the present paper is very simple and it is based on the Banach contraction principle and A. Bielecki's [1] idea of weighted norms, (see C. Corduneanu [2] and M. Kwapisz [3]).

Observe that the Darboux problem for partial differential equation

$$(2) \quad \begin{cases} z_{xy}(x,y) = f(x,y, z_x(x,y), z_y(x,y), z(x,y), z_{xy}(x,y)), \\ z(x,0) = \varphi(x), \quad x \in I_a = [0,a], \\ z(0,y) = \psi(y), \quad y \in I_b = [0,b], \end{cases}$$

can be easily reduced to the equation (1). There is very extensive literature devoted to the problem (2) and equations of the form (1) - mostly under the assumptions that the given functions  $f$ ,  $F$ ,  $\varphi$ ,  $\psi$  are continuous. However, it is not the aim of the present paper to discuss this case, and to make any review of the problem.

2. Denote by  $L(I_a \times I_b, \mathbb{R})$  the space of all integrable real valued functions defined in the rectangle  $I_a \times I_b$ . Consider the family of norms in  $L(I_a \times I_b, \mathbb{R})$

$$(3) \quad \|u\|_\lambda = \sup_{(x,y) \in I_a \times I_b} \left\{ \left( \int_0^x \int_0^y |u(s,t)|dtds \right) \exp(-\lambda(x+y)) \right\}$$

with  $\lambda \geq 0$ . Clearly all these norms are equivalent with  $\|u\|_0$ .

Denote by  $\mathcal{F}(u)(x,y)$  the right hand side of the equation (1). We make the following

Assumption H. Assume that there is  $g \in L(I_a \times I_b, \mathbb{R})$ , such that

$$(4) \quad |F(x, y, 0, 0, 0, 0)| \leq g(x, y);$$

There are nonnegative constants A, B, C, and D;  $D < 1$ , such that

$$(5) \quad |F(x, y, p, q, r, s) - F(x, y, \bar{p}, \bar{q}, \bar{r}, \bar{s})| \leq \\ \leq A|p - \bar{p}| + B|q - \bar{q}| + C|r - \bar{r}| + D|s - \bar{s}|,$$

for every  $p, q, r, s, \bar{p}, \bar{q}, \bar{r}, \bar{s} \in \mathbb{R}$  and  $x \in I_a, y \in I_b$ .

3. Now we are in position to formulate the main result:

Theorem. If the Assumption H is satisfied, then there is in  $L(I_a \times I_b, \mathbb{R})$  a unique solution of equation (1), and it can be found as the limit of successive approximations  $u_n$ :

$$(6) \quad u_{n+1}(x,y) = F\left(x,y, \int_0^x u_n(s,y)ds, \int_0^y u_n(x,t)dt, \int_0^x \int_0^y u_n(s,t)dtds, u_n(x,y)\right),$$

$n = 0, 1, \dots, u_0(x,y) \equiv 0$ .

Proof. Observe first that the conditions (4) and (5) of Assumption H guarantee that  $\mathcal{F}u \in L(I_a \times I_b, \mathbb{R})$  for every  $u \in L(I_a \times I_b, \mathbb{R})$ . We will show that  $\mathcal{F}$  is a contraction in  $L(I_a \times I_b, \mathbb{R})$ , if  $\lambda$  is chosen conveniently.

Let  $u, v \in L(I_a \times I_b, \mathbb{R})$ . Put

$$d(u,v)(x,y) = \int_0^x \int_0^y |u(s,t) - v(s,t)| dtds.$$

We have

$$\begin{aligned}
d(\mathcal{F}u, \mathcal{F}v)(x, y) &= \int_0^x \int_0^y |F(\bar{x}, \bar{y}, \int_0^{\bar{x}} u(s, \bar{y}) ds, \int_0^{\bar{y}} u(\bar{x}, t) dt, \int_0^{\bar{x}} \int_0^{\bar{y}} u(s, t) dt ds, u(\bar{x}, \bar{y})) \\
&\quad - F(\bar{x}, \bar{y}, \int_0^{\bar{x}} v(s, \bar{y}) ds, \int_0^{\bar{y}} v(\bar{x}, t) dt, \int_0^{\bar{x}} \int_0^{\bar{y}} v(s, t) dt ds, v(\bar{x}, \bar{y}))| d\bar{x} d\bar{y} \leq \\
&\leq \int_0^x \int_0^y [A \int_0^{\bar{x}} |u(s, \bar{y}) - v(s, \bar{y})| ds + B \int_0^{\bar{y}} |u(\bar{x}, t) - v(\bar{x}, t)| dt + \\
&\quad + C \int_0^{\bar{x}} \int_0^{\bar{y}} |u(s, t) - v(s, t)| dt ds + D |u(\bar{x}, \bar{y}) - v(\bar{x}, \bar{y})|] d\bar{y} d\bar{x}.
\end{aligned}$$

Changing the order of integrations we get

$$\begin{aligned}
d(\mathcal{F}u, \mathcal{F}v)(x, y) &\leq A \int_0^x d(u, v)(\bar{x}, y) d\bar{x} + B \int_0^y d(u, v)(x, \bar{y}) d\bar{y} + \\
&\quad + C \int_0^x \int_0^y d(u, v)(\bar{x}, \bar{y}) d\bar{x} d\bar{y} + D d(u, v)(x, y) \leq \\
&\leq A \int_0^x d(u, v)(\bar{x}, y) \exp(-\lambda(\bar{x} + y)) \exp(\lambda(\bar{x} + y)) d\bar{x} + B \int_0^y d(u, v)(x, \bar{y}) \exp(-\lambda(x + \bar{y})) \exp(\lambda(x + \bar{y})) d\bar{y} \\
&\quad + C \int_0^x \int_0^y d(u, v)(\bar{x}, \bar{y}) \exp(-\lambda(\bar{x} + \bar{y})) \exp(\lambda(\bar{x} + \bar{y})) d\bar{x} d\bar{y} + D d(u, v)(x, y) \exp(-\lambda(x + y)) \cdot \exp(\lambda(x + y)).
\end{aligned}$$

Taking the corresponding supremums we find:

$$d(\mathcal{F}u, \mathcal{F}v)(x, y) \leq \left( \frac{A}{\lambda} + \frac{B}{\lambda} + \frac{C}{\lambda^2} + D \right) \|u - v\|_{\lambda} \exp(\lambda(x + y))$$

what implies the evaluation

$$\|\mathcal{F}u - \mathcal{F}v\|_{\lambda} \leq \left( \frac{A}{\lambda} + \frac{B}{\lambda} + \frac{C}{\lambda^2} + D \right) \|u - v\|_{\lambda}.$$

This means that  $\mathcal{F}$  is a contraction if  $\lambda$  is sufficiently large.

4. Note that a similar theorem can be formulated for system of equations. It means, we may assume that  $F$  is a vector-function and  $L(I_a \times I_b, \mathbb{R})$  is replaced by  $L(I_a \times I_b, \mathbb{R}^k)$ . Another observation is that the same idea can be used to prove existence and uniqueness result for more general equation of the form

$$u(x_1, x_2, \dots, x_m) = F\left(x_1, x_2, \dots, x_m, \int_0^{x_1} u(s_1, x_2, \dots, x_m) ds_1, \int_0^{x_2} u(x_1, s_2, \dots, x_m) ds_2, \dots, \int_0^{x_1} \int_0^{x_2} \dots \int_0^{x_m} u(s_1, s_2, \dots, s_m) ds_m ds_2 \dots ds_1, u(x_1, x_2, \dots, x_m)\right).$$

Here, it is assumed that in the right hand side of the equation the integrals are of any dimension, from zero to  $m$ .

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