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PROPERTIES OF WALLMAN TYPE SPACES

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1. INTRODUCTION. Let X be an abstract set and $\mathcal{L}$ a lattice of subsets of X. Associated with the pair X, $\mathcal{L}$ are the general Wallman spaces $I_R(\mathcal{L})$ and $I_R^{\sigma}(\mathcal{L})$ (see below for definitions). In section 3 we briefly review the fundamental properties of these spaces and interconnections between these spaces when a pair of lattices $\mathcal{L}_1$, $\mathcal{L}_2$ of subsets of X is given with $\mathcal{L}_1 \subset \mathcal{L}_2$. We then turn attention to the space $I_R^{\sigma}(\mathcal{L})$ and the lattice $W_{\sigma}(\mathcal{L})$. If $\mathcal{L}$ is disjunctive, then $W_{\sigma}(\mathcal{L})$ is replete as is well-known. We consider conditions equivalent to $W_{\sigma}(\mathcal{L})$ being prime complete as well as $W_{\sigma}(\mathcal{L})$ being Lindelöf.

Finally, we investigate the preservation of some of the properties introduced for the case of a pair of lattices $\mathcal{L}_1$, $\mathcal{L}_2$ with $\mathcal{L}_1 \subset \mathcal{L}_2$.

Throughout, we adhere to standard lattice and measure terminology (see [1],[3],[5],[7] and [9]). We summarize these for the reader's convenience in section 2. In section 3, we consider the space $I_R(\mathcal{L})$, while section 4 is devoted to the space $I_R^{\sigma}(\mathcal{L})$ and the associated lattices therein.

2. BACKGROUND AND NOTATIONS. Let X be an abstract set and $\mathcal{L}$ a lattice of subsets of X. It is assumed that $\emptyset, X \in \mathcal{L}$. We denote by $\mathcal{A}(\mathcal{L})$, the algebra generated by $\mathcal{L}$; $\sigma(\mathcal{L})$, the σ-algebra generated by $\mathcal{L}$; $\delta(\mathcal{L})$, the lattice of all countable intersections of sets from $\mathcal{L}$; $t(\mathcal{L})$, the lattice of arbitrary intersections of sets of $\mathcal{L}$; $\rho(\mathcal{L})$, the smallest class closed under countable in-

tersections and unions which contains $\mathcal{L}$; $s(\mathcal{L})$, the lattice derived Souslin sets. The lattice $\mathcal{L}$ is called:

(2.1) <u>delta lattice</u> (δ-lattice) if $\mathcal{L}$ is closed under countable intersections.

(2.2) <u>complemented</u> if $L\in\mathcal{L}$ implies $L'\in\mathcal{L}$ (where prime denotes complement).

(2.3) <u>complement generated</u> if $L\in\mathcal{L}$ implies $L=\bigcap_{n=1}^{\infty} L_n'$, $L_n\in\mathcal{L}$.

(2.4) <u>separating</u> (or T_1) if $x,y\in X$ and $x\neq y$ implies there exists $L\in\mathcal{L}$ such that $x\in L$, $y\notin L$.

(2.5) <u>disjunctive</u> if for $x\in X$ and $L_1\in\mathcal{L}$ such that $x\notin L_1$ there exists $L_2\in\mathcal{L}$ with $x\in L_2$ and $L_1\cap L_2=\emptyset$.

(2.6) <u>T_2-lattice</u> if for $x,y\in X$, $x\neq y$, there exist $L_1,L_2\in\mathcal{L}$ such that $x\in L_1'$, $y\in L_2'$ and $L_1'\cap L_2'=\emptyset$.

(2.7) <u>regular</u> if for $x\in X$ and $L_1\in\mathcal{L}$ such that $x\notin L_1$ there exist L_2, $L_3\in\mathcal{L}$ with $x\in L_2'$, $L_1\subset L_3'$ and $L_2'\cap L_3'=\emptyset$.

(2.8) <u>normal</u> if for any $L_1,L_2\in\mathcal{L}$ with $L_1\cap L_2=\emptyset$ there exist L_3, $L_4\in\mathcal{L}$ with $L_1\subset L_3'$, $L_2\subset L_4'$ and $L_3'\cap L_4'=\emptyset$.

(2.9) <u>compact</u> if for any collection $\{L_\alpha\}$ of sets of $\mathcal{L}$ with $\bigcap L_\alpha=\emptyset$, there exists a finite subcollection with empty intersection.

(2.10) <u>countably compact</u> if for any countable collection $\{L_\alpha\}$ of sets of $\mathcal{L}$ with $\bigcap L_\alpha=\emptyset$, there exists a finite subcollection with empty intersection.

(2.11) <u>Lindelöf</u> if for any collection $\{L_\alpha\}$ of sets of $\mathcal{L}$ with $\bigcap L_\alpha=\emptyset$, there exists a countable subcollection with empty intersection.

We give now some measure terminology which will be used throughout. $M(\mathcal{L})$ denotes the set of finite valued bounded finitely additive measures on $\mathcal{A}(\mathcal{L})$.Without loss of generality may assume throughout that all measures are non-negative. A measure $\mu\in M(\mathcal{L})$ is called:

(2.12) <u>σ-smooth on $\mathcal{L}$</u> if for all sequences $\{L_n\}$ of sets of $\mathcal{L}$ with $L_n\downarrow\emptyset$, $\mu(L_n)\dashrightarrow 0$.

(2.13) <u>σ-smooth on $\mathcal{A}(\mathcal{L})$</u> if for all sequences $\{A_n\}$ of sets of $\mathcal{A}(\mathcal{L})$ with $A_n\downarrow\emptyset$, $\mu(A_n)\dashrightarrow 0$.

(2.14) <u>τ-smooth on $\mathcal{L}$</u> if for every net $\{L_\alpha\}$ of sets from $\mathcal{L}$ with $L_\alpha\downarrow\emptyset$, $\mu(L_\alpha)\dashrightarrow 0$.

(2.15) <u>$\mathcal{L}$-regular</u> if for any $A\in\mathcal{A}(\mathcal{L})$,

$$\mu(A)=\sup\{\mu(L)/L\subset A,\ L\in\mathcal{L}\}.$$

In addition we denote by:

$M_R(\mathcal{L})$, the set of $\mathcal{L}$-regular measures of $M(\mathcal{L})$;

$M_\sigma(\mathcal{L})$, the set of σ- smooth measures on $\mathcal{L}$ of $M(\mathcal{L})$;

$M_\tau(\mathcal{L})$, the set of τ-smooth measures on $\mathcal{L}$ of $M(\mathcal{L})$;

$M^\sigma(\mathcal{L})$, the set of σ-smooth measures on $\mathcal{A}(\mathcal{L})$ of $M(\mathcal{L})$;

$M_R^\sigma(\mathcal{L})$, the set of $\mathcal{L}$-regular measures of $M^\sigma(\mathcal{L})$;

$M_R^\tau(\mathcal{L})$, the set of $\mathcal{L}$-regular measures of $M(\mathcal{L})$ which are τ- smooth on $\mathcal{L}$.

$I(\mathcal{L})$, $I_R(\mathcal{L})$, $I_\sigma(\mathcal{L})$, $I_R^\sigma(\mathcal{L})$, $I_R^\tau(\mathcal{L})$ are the subsets of the corresponding M's which consist of the non-trivial zero-one valued measures.

For $\mu\in M(\mathcal{L})$, the <u>support of μ</u> is

(2.16) $S(\mu)=\bigcap\{L\in\mathcal{L}/\mu(L)=\mu(X)\}$

(2.17) <u>$\mathcal{L}$ is replete</u> iff for any $\mu\in I_R^\sigma(\mathcal{L})$, $S(\mu)\neq\emptyset$.

(2.18) <u>$\mathcal{L}$ is fully-replete</u> iff for any $\mu\in I^\sigma(\mathcal{L})$, $S(\mu)\neq\emptyset$.

(2.19) <u>$\mathcal{L}$ is prime complete</u> iff for any $\mu\in I_\sigma(\mathcal{L})$, $S(\mu)\neq\emptyset$.

The following results are not difficult to prove:

(2.20) $\mathcal{L}$ is compact iff $S(\mu)\neq\emptyset$ for every $\mu\in I_R(\mathcal{L})$.

(2.21) If $\mathcal{L}$ is countably compact, then $I_R(\mathcal{L})=I_R^\sigma(\mathcal{L})$.

(2.22) $\mathcal{L}$ is normal iff for each $\mu\in I(\mathcal{L})$, there exists a unique $\nu\in I_R(\mathcal{L})$ such that $\mu\leq\nu$ on $\mathcal{L}$.

(2.23) $\mathcal{L}$ is regular iff whenever $\mu_1,\mu_2\in I(\mathcal{L})$ and $\mu_1\leq\mu_2$ on $\mathcal{L}$, then $S(\mu_1)=S(\mu_2)$.

(2.24) $\mathcal{L}$ is strongly normal iff whenever $\mu_1,\mu_2,\mu\in I(\mathcal{L})$ and $\mu\leq\mu_2$, $\mu\leq\mu_1$ on $\mathcal{L}$ then $\mu_1\leq\mu_2$ on $\mathcal{L}$ or $\mu_2\leq\mu_1$ on $\mathcal{L}$

Our final set of definitions concerns relationships between two lattices $\mathcal{L}_1$, $\mathcal{L}_2$ of subsets of X, such that $\mathcal{L}_1\subset\mathcal{L}_2$.

(2.25) <u>$\mathcal{L}_1$ semiseparates $\mathcal{L}_2$</u> if $A\in\mathcal{L}_1$, $B\in\mathcal{L}_2$ and $A\cap B=\emptyset$ implies there exists $C\in\mathcal{L}_1$ such that $B\subset C$ and $A\cap C=\emptyset$.

(2.26) <u>$\mathcal{L}_1$ separates $\mathcal{L}_2$</u> if $A,B\in\mathcal{L}_2$ and $A\cap B=\emptyset$, implies there exist $C,D\in\mathcal{L}_1$ such that $A\subset C$, $B\subset D$ and $C\cap D=\emptyset$.

(2.27) <u>$\mathcal{L}_2$ is $\mathcal{L}_1$-countably paracompact</u> if for any $A_n\in\mathcal{L}_2$, $n=1,2,\dots$ with $A_n\downarrow\emptyset$, there exists $B_n\in\mathcal{L}_1$, $n=1,2,\dots$ such that $A_n\subset B'_n$ and $B'_n\downarrow\emptyset$.

(2.28) $\mathcal{L}_2$ is countably bounded $\mathcal{L}_1$ lattice if for $A_n \in \mathcal{L}_2$, $n=1,2,\ldots$ and $A_n \downarrow \emptyset$, there exists $B_n \in \mathcal{L}_1$, $n=1,2,\ldots$ with $A_n \subset B_n$ and $B_n \downarrow \emptyset$.

3. ON LATTICE SEPARATION AND THE WALLMAN SPACE $I_R(\mathcal{L})$. In this section we briefly summarize some known facts (see e.g. [2], and [7]) about the Wallman space $I_R(\mathcal{L})$.

Let X be an abstract set and let $\mathcal{L}$ be a lattice of subsets of X. If $x \in X$, then μ_x is the measure concentrated at x:

$$\mu_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \qquad \text{where } A \in \mathcal{A}(\mathcal{L})$$

and $\mu_x \in I_R(\mathcal{L})$ iff $\mathcal{L}$ is disjunctive.

Therefore there exists a mapping $\phi: X \dashrightarrow I_R(\mathcal{L})$, where $\phi(x) = \mu_x$ for $x \in X$, if $\mathcal{L}$ is disjunctive. If $\mathcal{L}$ is separating, ϕ is one-to-one. If $\mathcal{L}$ is separating and disjunctive and if X is given the topology $t(\mathcal{L})$ of closed sets and if $I_R(\mathcal{L})$ is given the Wallman topology, then ϕ is a homeomorphism of X into $I_R(\mathcal{L})$.

The Wallman topology is obtained by taking all

$$W(L) = \{\mu \in I_R(\mathcal{L}) \;/\; \mu(L) = 1\}, \quad L \in \mathcal{L}$$

as a base for the closed sets. $I_R(\mathcal{L})$ is called the general Wallman space associated with X and $\mathcal{L}$. Since $\phi(X)$ is also dense in $I_R(\mathcal{L})$, $I_R(\mathcal{L})$ is a compactification of X, the general Wallman compactification of X.

We assume that $\mathcal{L}$ is disjunctive. Then if $A \in \mathcal{A}(\mathcal{L})$, let $W(A) = \{\mu \in I_R(\mathcal{L}) \;/\; \mu(A) = 1\}$. The following statements are true:

a) $W(A \cup B) = W(A) \cup W(B)$

b) $W(A \cap B) = W(A) \cap W(B)$

c) $W(A') = W(A)'$

d) $A \supset B$ iff $W(A) \supset W(B)$

e) $\mathcal{A}(W(\mathcal{L})) = W(\mathcal{A}(\mathcal{L}))$

It is known that $W(\mathcal{L})$ is disjunctive and that the topological space $(I_R(\mathcal{L}), tW(\mathcal{L}))$ is compact and T_1; it is T_2 iff $\mathcal{L}$ is normal (assuming as we have that $\mathcal{L}$ is disjunctive).

Next consider that X is an abstract set and $\mathcal{L}_1, \mathcal{L}_2$ are two lattices of subsets of X with $\mathcal{L}_1 \subset \mathcal{L}_2$. We note here some results on extensions and restrictions of zero-one regular measures

<u>Theorem 3.1</u> Let $\mathcal{L}_1 \subset \mathcal{L}_2$ be two lattices of subsets of X such that $\mathcal{L}_1$ semiseparates $\mathcal{L}_2$. The restriction map $\Psi : I_R(\mathcal{L}_2) \dashrightarrow I_R(\mathcal{L}_1)$ defined by $\Psi(\nu) = \mu = \nu / a(\mathcal{L}_1)$, $\nu \in I_R(\mathcal{L}_2)$ is onto and is one-to-one iff $\mathcal{L}_1$ separates $\mathcal{L}_2$.
(Note that semiseparation implies that μ is $\mathcal{L}_1$-regular and that the result is equally true if $\nu \in M_R(\mathcal{L}_2)$).

<u>Theorem 3.2</u> Let $\mathcal{L}_1 \subset \mathcal{L}_2$ be two lattices of subsets of X. Then the following are true:

a) Any $\mu \in I_R(\mathcal{L}_1)$ can be extended to a $\nu \in I_R(\mathcal{L}_2)$ and if $\mathcal{L}_2$ is $\mathcal{L}_1$ countably paracompact or $\mathcal{L}_2$ is countably compact, or $\mathcal{L}_2$ is a countably bounded $\mathcal{L}_1$-lattice then any $\mu \in I_R^{\sigma}(\mathcal{L}_1)$ can be extended to a $\nu \in I_R^{\sigma}(\mathcal{L}_2)$.

b) If $\mathcal{L}_1$ semiseparates $\mathcal{L}_2$ and $\mathcal{L}_2$ is $\mathcal{L}_1$ countably paracompact or a countably bounded $\mathcal{L}_1$ lattice, the restriction map $\Psi : I_R^{\sigma}(\mathcal{L}_2) \dashrightarrow I_R^{\sigma}(\mathcal{L}_1)$ is onto.

c) If $\mathcal{L}_1$ separates $\mathcal{L}_2$ and $\mathcal{L}_2$ is $\mathcal{L}_1$-countably paracompact or $\mathcal{L}_2$ is a countably bounded $\mathcal{L}_1$ lattice, then the restriction map $\Psi : I_R^{\sigma}(\mathcal{L}_2) \dashrightarrow I_R^{\sigma}(\mathcal{L}_1)$ is one-to-one.

<u>Theorem 3.3</u> Let $\mathcal{L}_1 \subset \mathcal{L}_2$ be two lattices of subsets of X. If $\mathcal{L}_1$ semiseparates $\mathcal{L}_2$, the restriction map $\Psi : I_R(\mathcal{L}_2) \dashrightarrow I_R(\mathcal{L}_1)$ is onto and continuous with respect to the Wallman topologies. Ψ is one-to-one (and therefore a homeomorphism) iff $\mathcal{L}_1$ separates $\mathcal{L}_2$.

4. THE WALLMAN SPACE $I_R^{\sigma}(\mathcal{L})$ AND THE LATTICES $W_\sigma(\mathcal{L})$ AND $tW_\sigma(\mathcal{L})$.

Here we turn attention first to the general Wallman space $I_R^{\sigma}(\mathcal{L})$ in the case where $\mathcal{L}$ is again a disjunctive lattice; this space is the generalization of the usual realcompactification and the N-compactification. We investigate when $I_R^{\sigma}(\mathcal{L})$, $W_\sigma(\mathcal{L})$ is prime complete. In particular we give necessary and sufficient conditions for $I_R^{\sigma}(\mathcal{L})$, $tW_\sigma(\mathcal{L})$ to be Lindelöf generalizing thereby results of [6] for the realcompactification.
First, we summarize some known results about zero-one measures and

filter correspondence:

a) There exists a one-to-one correspondence between all elements of $I_R(\mathcal{L})$ and all $\mathcal{L}$-ultrafilters.
There exists a one-to-one correspondence between all elements of $I_R^\sigma(\mathcal{L})$ and all $\mathcal{L}$-ultrafilters with the countable intersection property. The correspondence is given by the following rule: with each $\mathcal{L}$-ultrafilter F we associate the zero-one measure defined on $\mathcal{A}(\mathcal{L})$ by:

$$\mu_F(E) = \begin{cases} 1 & \text{if there exists } A \in F,\ A \subset E \\ 0 & \text{if there exists } A \in F,\ A \subset E' \end{cases}$$

b) There exists a one-to-one correspondence between all elements of $I(\mathcal{L})$ and all prime $\mathcal{L}$-filters, given by the following rule: with each $\mu \in I(\mathcal{L})$ we associate the prime $\mathcal{L}$-filter given by $F = \{A \in \mathcal{L} / \mu(A) = 1\}$. This correspondence induces a one-to-one correspondence. between prime $\mathcal{L}$-filters with the countable intersection property and $I_\sigma(\mathcal{L})$.

Consider an abstract set X and a disjunctive lattice of subsets of X, $\mathcal{L}$. Let $\mu \in I_R(\mathcal{L})$ and define μ' on $\mathcal{A}(W_\sigma(\mathcal{L})) = W_\sigma(\mathcal{A}(\mathcal{L}))$ by $\mu'(W_\sigma(A)) = \mu(A)$, $A \in \mathcal{A}(\mathcal{L})$ where $W_\sigma(A) = \{\mu \in I_R^\sigma(\mathcal{L}) / \mu(A) = 1\}$ and $W_\sigma(\mathcal{L}) = \{W_\sigma(A) / A \in \mathcal{A}(\mathcal{L})\}$
Clearly for $A, B \in \mathcal{A}(\mathcal{L})$ the properties a)-e) that we stated in section 3 are still valid. Note also that $W_\sigma(\mathcal{L})$ is a disjunctive lattice.

<u>Theorem 4.1</u> If $\mu \in I_R(\mathcal{L})$ then $\mu \in I_R^\sigma(\mathcal{L})$ iff $\mu' \in I_R^\sigma(W_\sigma(\mathcal{L}))$. (More generally: if $\mu \in I(\mathcal{L})$ then $\mu' \in I(W_\sigma(\mathcal{L}))$ and $\mu \in I_R(\mathcal{L})$ iff $\mu' \in I_R(W_\sigma(\mathcal{L}))$).

Proof. See[2].

<u>Theorem 4.2</u> Let $\mathcal{L}$ be a disjunctive lattice of subsets of an abstract set X. Then $W_\sigma(\mathcal{L})$ is replete.

Proof. Let $\mu' \in I_R^\sigma(W_\sigma(\mathcal{L}))$, $\mu' \neq 0$, defined by

$\mu'(W_\sigma(L)) = \mu(L)$, $L \in \mathcal{L}$, $\mu \in I_R^\sigma(\mathcal{L})$.

$S(\mu') = \bigcap \{W_\sigma(L) / \mu'(W_\sigma(L)) = 1,\ L \in \mathcal{L}\}$

But $\mu'(W_\sigma(L)) = \mu(L) = 1$ implies $\mu \in S(\mu') \neq \emptyset$ and therefore $W_\sigma(\mathcal{L})$ is replete.

A lattice satisfies condition:

(1) if for any $\mu \in I_\sigma(\mathcal{L})$ there exists $\nu \in I_R^\sigma(\mathcal{L})$ such that $\mu \le \nu\ (\mathcal{L})$ (i.e. $\mu(L) \le \nu(L)$ for all $L \in \mathcal{L}$).

(2) if for every $\mathcal{L}$-filter with the countable intersection property $\pi \in \Pi_\sigma(\mathcal{L})$, there exists $\mu \in I_R^\sigma(\mathcal{L})$ such that $\pi \le \mu\ (\mathcal{L})$ We note that F is an $\mathcal{L}$-ultrafilter with the countable intersection property if $A_i \in F$ for all $i=1,2,\dots$ implies $\bigcap_{i=1}^\infty A_i \ne \emptyset$.

(3) if for every $\mu \in I_\sigma(\mathcal{L}')$ there exists $\nu \in I_R^\sigma(\mathcal{L})$ such that $\mu \le \nu\ (\mathcal{L})$.

<u>Theorem 4.3</u> Let $\mathcal{L}$ be a disjunctive lattice. Then $\mathcal{L}$ satisfies condition (1) iff $W_\sigma(\mathcal{L})$ is prime complete.

Proof. Sufficiency: Let $\mu \in I_\sigma(\mathcal{L})$ and the associated $\mu' \in I_\sigma(W_\sigma(\mathcal{L}))$ defined by $\mu'(W_\sigma(L)) = \mu(L)$, $L \in \mathcal{L}$. Since $W_\sigma(\mathcal{L})$ is prime complete, $S(\mu') \ne \emptyset$ and then there exists $\nu \in S(\mu')$ and $\mu \le \nu\ (\mathcal{L})$

Necessity: Let $\mu' \in I_\sigma(W_\sigma(\mathcal{L}))$ and the associated $\mu \in I_\sigma(\mathcal{L})$ such that $\mu'(W_\sigma(L)) = \mu(L)$, $L \in \mathcal{L}$. Since $\mathcal{L}$ satisfies condition (1), there exists $\nu \in I_R^\sigma(\mathcal{L})$ such that $\mu \le \nu\ (\mathcal{L})$. But then $\nu' \in I_R^\sigma(W_\sigma(\mathcal{L}))$ and $\mu' \le \nu''\ (W_\sigma(\mathcal{L}))$. Therefore $S(\nu') \subset S(\mu')$ and since $W_\sigma(\mathcal{L})$ is replete $S(\nu') \ne \emptyset$. Then $S(\mu') \ne \emptyset$ which proves that $W_\sigma(\mathcal{L})$ is prime complete.

<u>Theorem 4.4</u> If $\mathcal{L}$ satisfies condition (2) and if $\mathcal{L}$ is replete then $\mathcal{L}$ is Lindelöf.

Proof. Let $\pi \in \Pi_\sigma(\mathcal{L})$. Then there exists $\mu \in I_R^\sigma(\mathcal{L})$ and $\pi \le \mu(\mathcal{L})$ $\mathcal{L}$ replete implies $S(\mu) = \bigcap_{\alpha \in \Lambda} \{L_\alpha \in \mathcal{L} / \mu(L_\alpha) = 1\} \ne \emptyset$ and since $\pi \le \mu$ $S(\mu) \subset S(\pi) \ne \emptyset$. But π has the countable intersection property and then $\bigcap_{\alpha=1}^\infty \{L_\alpha \in \mathcal{L} / \pi(L_\alpha) = 1\} \ne \emptyset$, i.e. $\mathcal{L}$ is Lindelöf.

<u>Theorem 4.5</u> If $\mathcal{L}$ is a countably compact lattice then $\mathcal{L}$ satisfies condition (2).

Proof. Let $\{L_\alpha\}_{\alpha=1}^{\infty}$ be a collection of subsets of X such that $\bigcap_{\alpha=1}^{\infty}\{L_\alpha\}\neq\emptyset$. Since $\mathcal{L}$ is countably compact, $\bigcap_{\alpha=1}^{\infty}\{L_\alpha\in\mathcal{L}\}\neq\emptyset$ and then $\{L_\alpha\}_\alpha$ is a filter base which generates an $\mathcal{L}$-filter with the countable intersection property, $\pi\in\Pi_\sigma(\mathcal{L})$. We enlarge it to F, an $\mathcal{L}$-ultrafilter with the countable intersection property. To F it corresponds uniquely $\mu\in I_R^\sigma(\mathcal{L})$. Now from a filter- ultrafilter argument it follows that $\pi\leq\mu(\mathcal{L})$.

Theorem 4.6 If $\mathcal{L}$ is disjunctive and Lindelöf then $\mathcal{L}$ satisfies condition (2).

Proof. Let $\pi\in\Pi_\sigma(\mathcal{L})$ and let $\{L_\alpha\}_{\alpha\in\Lambda}$ be a family of subsets of X. Then $\bigcap_{\alpha=1}^{\infty}\{L_\alpha/\pi(L_\alpha)=1\}\neq\emptyset$ and since $\mathcal{L}$ is Lindelöf $\bigcap_{\alpha\in\Lambda}\{L_\alpha/\pi(L_\alpha)=1\}=S(\pi)\neq\emptyset$.
Let $x\in S(\pi)$, and consider μ_x. Clearly $\pi\leq\mu_x(\mathcal{L})$. From the countable intersection property of π it follows that $\mu_x\in I_\sigma(\mathcal{L})$ and since $\mathcal{L}$ is disjunctive $\mu\in I_R(\mathcal{L})$. Therefore $\mu_x\in I_R^\sigma(\mathcal{L})$ and $\mathcal{L}$ satisfies condition (2).

Theorem 4.7 If $\mathcal{L}$ is disjunctive then $\mathcal{L}$ satisfies condition (2) iff $(I_R^\sigma(\mathcal{L}),tW_\sigma(\mathcal{L}))$ is Lindelöf.

Proof. Necessity: First we show that $W_\sigma(\mathcal{L})$ satisfies condition (2). Let $\pi\in\Pi_\sigma(\mathcal{L})$. There exists $\mu\in I_R^\sigma(\mathcal{L})$ and $\pi\leq\mu(\mathcal{L})$. Corresponding to π and μ we have (by Theorem 4.1) $\pi'\in\Pi_\sigma(W_\sigma(\mathcal{L}))$ and $\mu'\in I_R^\sigma(W_\sigma(\mathcal{L}))$ with $\pi'\leq\mu'(W_\sigma(\mathcal{L}))$, therefore $W_\sigma(\mathcal{L})$ satisfies condition (2). Since $\mathcal{L}$ is disjunctive, $W_\sigma(\mathcal{L})$ is replete. Then by Theorem 4.4 it follows that $W_\sigma(\mathcal{L})$ is Lindelöf and $W_\sigma(\mathcal{L})\subset tW_\sigma(\mathcal{L})$ implies that $tW_\sigma(\mathcal{L})$ is Lindelöf.

Sufficiency: $tW_\sigma(\mathcal{L})$ Lindelöf implies that $W_\sigma(\mathcal{L})$ is Lindelöf and since $W_\sigma(\mathcal{L})$ is also disjunctive, by Theorem 4.6 it follows that $W_\sigma(\mathcal{L})$ satisfies condition (2). Therefore for $\pi'\in\Pi_\sigma(W_\sigma(\mathcal{L}))$ there exists $\mu'\in I_R^\sigma(W_\sigma(\mathcal{L}))$ such that $\pi'\leq\mu'(W_\sigma(\mathcal{L}))$. To π' and μ' correspond $\pi\in\Pi_\sigma(\mathcal{L})$ and $\mu\in I_R^\sigma(\mathcal{L})$ such that $\pi\leq\mu(\mathcal{L})$.

Theorem 4.8 If $\mathcal{L}$ is regular and Lindelöf then $\mathcal{L}$ satisfies condition (1).

Proof. Let $\mu\in I_\sigma(\mathcal{L})$. Since $\mathcal{L}$ is Lindelöf, $S(\mu)\neq\emptyset$. Let then $x\in S(\mu)$ and consider μ_x which is $\mathcal{L}$-regular since $\mathcal{L}$ is regular and therefore disjunctive.

μ_x is also σ-smooth, hence $\mu_x \in I_R^{\sigma}(\mathcal{L})$ and $\mu \le \mu_x$ $(\mathcal{L})$.

Theorem 4.9 If $\mathcal{L}$ satisfies condition (1) and is replete then $\mathcal{L}$ is prime complete.

Proof. If $\mu \in I_{\sigma}(\mathcal{L})$, there exists $\nu \in I_R^{\sigma}(\mathcal{L})$ such that $\mu \le \nu$ $(\mathcal{L})$. Therefore $S(\mu) \supset S(\nu)$ and since $\mathcal{L}$ is replete $S(\nu) \neq \emptyset$.

Remark If $\mathcal{L}$ satisfies condition (3) and is replete $\mathcal{L}'$ is prime complete. Note that if $\mathcal{L}$ satisfies condition (1) and is replete then $\mathcal{L}$ is fully replete. Therefore in this case repleteness is equivalent to fully-repleteness.

Lemma Let $\mathcal{L}$ be a lattice of subsets of X. Then:

a) If $\mathcal{L}$ is complement generated then $\mathcal{L}$ is countably paracompact.

b) If $\mathcal{L}$ is countably paracompact then $I_{\sigma}(\mathcal{L}') \subset I_{\sigma}(\mathcal{L})$

Proof. a) Let $A_n \in \mathcal{L}$, $A_n \downarrow \emptyset$, $A_n = \bigcap_{j=1}^{\infty} B'_{nj}$, $B_{nj} \in \mathcal{L}$. Then $A_n \subset B'_{nj} \downarrow \emptyset$.

b) Let $\mu \in I_{\sigma}(\mathcal{L}')$ and $B_n \in \mathcal{L}$, $B_n \downarrow \emptyset$. There exists $A_n \in \mathcal{L}$ such that $B_n \subset A'_n \downarrow \emptyset$ and $\mu(B_n) \le \mu(A'_n) \to 0$.

Theorem 4.10 If $\mathcal{L}$ is complement generated then $\mathcal{L}$ satisfies condition (3).

Proof. Let $A \in \mathcal{L}, A = \bigcap A'_n$, $A_n \in \mathcal{L}$ and let $\mu \in I_{\sigma}(\mathcal{L}')$. Then $\mu \le \nu$ $(\mathcal{L})$ for any $\nu \in I_R(\mathcal{L})$. Suppose that $\mu(A)=0$ and $\nu(A)=1$ and that $A'_n \downarrow$. By the $\mathcal{L}$-regularity of ν it follows that $\mu = \nu(\mathcal{L})$ therefore $I_{\sigma}(\mathcal{L}') \subset I_R(\mathcal{L})$. By statement b) of Lemma we have that $I_{\sigma}(\mathcal{L}') \subset I_{\sigma}(\mathcal{L})$ and therefore $\mu \in I_R^{\sigma}(\mathcal{L})$.

Theorem 4.11 Let $\mathcal{L}$ be a disjunctive lattice of subsets of X

a) If $\mathcal{L}$ satisfies condition (3) then $W_{\sigma}(\mathcal{L})'$ is prime complete.

b) If $W_{\sigma}(\mathcal{L})'$ is prime complete then $\mathcal{L}'$ satisfies condition (1).

Proof. a) Let $\mu' \in I_{\sigma}(W_{\sigma}(\mathcal{L})')$ with the associated $\mu \in I_{\sigma}(\mathcal{L}')$. There exists $\nu \in I_R^{\sigma}(\mathcal{L})$ such that $\mu \le \nu$ $(\mathcal{L})$. Then $\mu' \le \nu'$ $(W_{\sigma}(\mathcal{L})')$ and since $W_{\sigma}(\mathcal{L})$ is replete, $S(\nu') \neq \emptyset$ which implies $S(\mu') \neq \emptyset$.

b) Let $\mu \in I_\sigma(\mathcal{L}')$ and consider the associated $\mu' \in I_\sigma(W_\sigma(\mathcal{L})')$ $S(\mu') \neq \emptyset$, since $W_\sigma(\mathcal{L})'$ is prime complete. Therefore there exists $\nu \in S(\mu')$, $\nu \in I_R^\sigma(\mathcal{L}')$ and $\mu \leq \nu$ $(\mathcal{L}')$.

Theorem 4.12 Let $\mathcal{L}_1 \subset \mathcal{L}_2$ be two lattices of subsets of X. Suppose that $\mathcal{L}_1$ satisfies condition (1). Then the following are true:

a) If $\mathcal{L}_1$ is strongly normal and if $\mathcal{L}_2$ is $\mathcal{L}_1$ countably bounded then $\mathcal{L}_2$ satisfies condition (1);

b) If $\mathcal{L}_1$ is normal and semiseparates $\mathcal{L}_2$ and if $\mathcal{L}_2$ is $\mathcal{L}_1$ countably bounded then $\mathcal{L}_2$ satisfies condition (1);

c) If $\mathcal{L}_1$ is normal and semiseparates $\mathcal{L}_2$ and if $\mathcal{L}_2$ is $\mathcal{L}_1$ countably paracompact then $\mathcal{L}_2$ satisfies condition (1).

Proof. Let $\nu \in I_\sigma(\mathcal{L}_2)$. Then $\nu|_{a(\mathcal{L}_1)} = \mu \in I_\sigma(\mathcal{L}_1)$ and since $\mathcal{L}_1$ satisfies condition (1) there exists $\lambda \in I_R^\sigma(\mathcal{L}_1)$ such that $\mu \leq \lambda$ $(\mathcal{L}_1)$. Let $\tau \in I_R(\mathcal{L}_2)$. Then $\nu \leq \tau(\mathcal{L}_2)$ and $\mu \leq \tau|_{a(\mathcal{L}_1)}(\mathcal{L}_1)$.

a) Since $\mathcal{L}_1$ is strongly normal $\tau|_{a(\mathcal{L}_1)} \leq \lambda(\mathcal{L}_1)$ and then $\tau|_{a(\mathcal{L}_1)} \in I_\sigma(\mathcal{L}_1)$. Since $\mathcal{L}_2$ is $\mathcal{L}_1$ countably bounded $\tau \in I_\sigma(\mathcal{L}_2)$. Hence $\tau \in I_R^\sigma(\mathcal{L}_2)$.

b) Since $\mathcal{L}_1$ semiseparates $\mathcal{L}_2$ $\tau|_{a(\mathcal{L}_1)} \in I_R(\mathcal{L}_1)$. But since $\mathcal{L}_1$ is normal we must have $\tau|_{a(\mathcal{L}_1)} = \lambda$ and therefore $\tau|_{a(\mathcal{L}_1)} \in I_\sigma(\mathcal{L}_1)$. As in a) it follows that $\tau \in I_R^\sigma(\mathcal{L}_2)$.

c) As in b) we get that $\tau|_{a(\mathcal{L}_1)} \in I_\sigma(\mathcal{L}_1)$. Since $\mathcal{L}_2$ is $\mathcal{L}_1$ countably paracompact it follows then that $\tau \in I_\sigma(\mathcal{L}_2)$, hence $\tau \in I_R^\sigma(\mathcal{L}_2)$.

Theorem 4.13 Let $\mathcal{L}_1 \subset \mathcal{L}_2$ be two lattices of subsets of X and suppose that $\mathcal{L}_2$ satisfies condition (1). Then the following are true:

a) If $\mathcal{L}_1$ semiseparates $\mathcal{L}_2$ and if $\mathcal{L}_2$ is $\mathcal{L}_1$ countably bounded then $\mathcal{L}_1$ satisfies condition (1);

b) If $\mathcal{L}_1$ is δ and $\sigma(\mathcal{L}_1) \subset s(\mathcal{L}_1)$ and if $\mathcal{L}_2$ is $\mathcal{L}_1$ countably bounded then $\mathcal{L}_1$ satisfies condition (1).

Proof. Let $\mu \in I_\sigma(\mathcal{L}_1)$ and extend it to $\tau \in I_\sigma(\mathcal{L}_2)$ since

$\mathcal{L}_2$ is $\mathcal{L}_1$ countably bounded.

$\mathcal{L}_2$ satisfies condition (1), therefore there exists $\nu \in I_R^{\sigma}(\mathcal{L}_2)$ such that $\tau \leq \nu(\mathcal{L}_2)$. Consider the restriction $\nu|_{a(\mathcal{L}_1)}$.

If $\mathcal{L}_1$ semiseparates $\mathcal{L}_2$ (or if $\mathcal{L}_1$ is δ and $\sigma(\mathcal{L}_1) \subset s(\mathcal{L}_1)$)

then $\nu|_{a(\mathcal{L}_1)} \in I_R^{\sigma}(\mathcal{L}_1)$ and $\mu \leq \nu|_{a(\mathcal{L}_1)}(\mathcal{L}_1)$.

Therefore $\mathcal{L}_1$ satisfies condition (1).

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