THE MULTICOALITIONAL BARGAINING SET M_o VERSUS THE ONECOALITIONAL BARGAINING SET M.

Irinel Dragan

Some combinatorial aspects of the theory of the bargaining sets for cooperative n person games, initiated by R.J.Aumann and M.Maschler in [1] , have been considered in two previous papers [2,3] . In fact, we have tried to obtain a characterization of the payoffs belonging to a bargaining set defined by means of multicoalitional objections and counter objections, starting from the characterization of the payoffs belonging to $M_1^{(i)}$ given in [4]. However, we have been led to very complex combinatorial structures and we have remarked that few changes in the basic definitions may simplify very much the corresponding combinatorial structures. To emphasize the fact that we did not use simply extensions of the concepts of objection and counter objection, we have also introduced new terms like: bargaining proposal, bargaining distribution, bargaining counter proposal, bargaining counter distribution. In [2], a characterization of the payoffs belonging to the bargaining set M_o has been obtained. The bargaining set M_o has been defined as the set of stable payoffs with respect to multicoalitional bargaining proposals and counter proposals. The case of convex games has been considered in [3], where it has been shown that for convex games the bargaining set M_o coincides with the bargaining set M; the bargaining set M has been defined as the set of stable payoffs with respect to onecoalitional bargaining proposals and counter proposals. This was an expected result, taking into account the results given in [5]. In general, we have $C(G) \subseteq M_o \subseteq M$, where $C(G)$ is the core of G. The fact that $M_o = M$ for convex games is rising the question whether we may have $C(G) \subset M_o$ and/or $M_o \subset M$ for some class of non convex games. The present paper is showing such a class of games. Obviously, This result justifies the study of the multicoalitional bargaining sets. Some definitions and previous results from [2] and [3] will be given in the first section. In the second section we shall show a class of games for which $C(G) \subset M_o \subset M$.

1. Definitions and previous results.

Consider $G = (I,v,F)$ a cooperative n person game with coalition structures. We have $|I| = n$ and $v(\emptyset) = o$ and F is the union of all $F_{\mathcal{S}} = \{ x \mid x \in R^n, x(S) = v(S), \forall S \in \mathcal{S} \}$, for all coalition structures $\mathcal{S}$. The core of G is $C(G) = \{ x \mid x \in F, e(x,S) \leq o, \forall S \in P(I) \}$. Consider any $x \in F_{\mathcal{S}}$ for some coalition structure $\mathcal{S}$. As in [3], a generic coalition structure will be denoted by $\mathcal{T} = (P_1,..,P_{\pi};N_1,..,N_{\nu};O_1,..,O_{\omega})$, where P_i, $i=\overline{1,\pi}$, N_j, $j=\overline{1,\nu}$, O_k, $k=\overline{1,\omega}$ stand for the coalitions with positive, negative and zero excesses, respectively. Denote by P the union of all P_i, $i=\overline{1,\pi}$. Suppose that $x \notin C(G)$. Then, there exist coalition structures $\mathcal{S}$ with $P \neq \emptyset$.

Definition 1.1: Any coalition structure $\mathcal{T}$ with $P \neq \emptyset$ will be called a bargaining proposal w.r.t. $(x,\mathcal{S})$. A bargaining proposal with no negative excess coalition is a trivial bargaining proposal.

As in [2] we suppose: (A) there is no trivial bargaining proposal w.r.t.$(x,\mathcal{S})$.

Definition 1.2: If $\mathcal{T}$ is a bargaining proposal w.r.t. $(x,\mathcal{S})$, then any $y \in F_{\mathcal{T}}$ such that $y_h \geq x_h$, $\forall h \in P$, and $y_h > x_h$ for some $h \in P_i$, $\forall$ $i=\overline{1,\pi}$, will be called a bargaining distribution of the gain provided by $\mathcal{T}$. The set of bargaining distributions for $\mathcal{T}$ will be denoted by $D(\mathcal{T})$.

Let $\mathcal{T}$ be a bargaining proposal w.r.t. $(x,\mathcal{S})$. Consider any other bargaining proposal $\mathcal{T} = (P_1^*,\dots,P_{\pi^*}^*;\dots)$ w.r.t. $(x,\mathcal{S})$. We suppose: (B) $P^*\cap P \neq \emptyset$, where P^* is the union of all $P_{i^*}^*$, $i^*=\overline{1,\pi^*}$. We can suppose without loss of generality that there is an integer r, $1\leq r\leq \pi^*$, such that

(1.1) $P_{i^*}^*\cap P \neq \emptyset$, $i^*=\overline{1,r}$; $P_{i^*}^*\cap P = \emptyset$, $i^*=\overline{r+1,\pi^*}$, if $r<\pi^*$.

We suppose: (B^*) at least one of $P_1^*,\dots,P_r^*$ is different of all $P_1,\dots,P_\pi$.

Consider $y\in D(\mathcal{T})$. The double excess of any $S\in P(I)$ w.r.t. $(x,\mathcal{S})$ and $(y,\mathcal{T})$ is defined by

(1.2) $e(x,S;y,\mathcal{T}) = e(x,S) - [\,y(S\cap P) - x(S\cap P)\,]$,

which is the amount available to S after paying the players of $S\cap P$ as in y and the players of $S - P$ as in x.

Definition 1.3: Any bargaining proposal $\mathcal{T}^*$ subject to (B) and (B^*) that satisfies

(1.3) $e(x,P_{i^*}^*;y,\mathcal{T}) \geq 0$, $i^*=\overline{1,r}$,

for all $y\in D(\mathcal{T})$, will be called a bargaining counter proposal w.r.t. $(x,\mathcal{S})$ and $\mathcal{T}$.

Definition 1.4: If $\mathcal{T}^*$ is a bargaining proposal subject to (B) and (B^*), then any $z\in E_{\mathcal{T}^*}$ such that

(1.4) $z_h \geq y_h$, $\forall\, h\in P^*\cap P$; $z_h \geq x_h$, $\forall\, h\in P - P^*$ for $y\in D(\mathcal{T})$

will be called a bargaining counter distribution w.r.t. y.

These two concepts are related as follows:

Theorem 1.5, ([2],Th.2.4): Consider a pair $(\mathcal{T},\mathcal{T}^*)$ of bargaining proposals subject to (B) and (B^*). Then, $\mathcal{T}^*$ is a bargaining counter proposal w.r.t. $(x,\mathcal{S})$ and $\mathcal{T}$, if and only if for every $y\in D(\mathcal{T})$ there exists a bargaining counter distribution $z\in E_{\mathcal{T}^*}$ w.r.t. y.

A combinatorial characterization of the bargaining counter proposals has also been proved:

Theorem 1.6, ([2],Th.2.7): Consider a pair $(\mathcal{T},\mathcal{T}^*)$ of bargaining proposals subject to (B) and (B^*), where r is defined by (1.1). Then $\mathcal{T}^*$ is a bargaining counter proposal w.r.t. $(x,\mathcal{S})$ and $\mathcal{T}$, if and only if

$$\text{(C)} \qquad e(x,P_{i^*}^*) \geq \sum_{i \in I(P_{i^*}^*)} e(x,P_i), \quad i^*=\overline{1,r},$$

where $I(P_{i^*}^*) = \{ i \mid P_{i^*}^* \cap P_i \neq \emptyset, \ i=\overline{1,\pi} \}$, $i^*=\overline{1,r}$.

Definition 1.7: The multicoalitional bargaining set M_o is the set of all stable $x \in F$. Some $x \in F$ is stable, if either $x \in C(G)$, or for any bargaining proposal $\mathcal{T}$ w.r.t. $(x,\mathcal{S})$, there exists a bargaining counter proposal $\mathcal{T}^*$ w.r.t. $(x,\mathcal{S})$ and $\mathcal{T}$.

From this definition and Theorem 1.6 follows a combinatorial characterization of the payoff vectors belonging to $M_o - C(G)$:

Theorem 1.8: Some $x \in F_{\mathcal{S}}$, $x \notin C(G)$, belongs to M_o, if and only if for each bargaining proposal $\mathcal{T}$ w.r.t. $(x,\mathcal{S})$, there exists another bargaining proposal $\mathcal{T}^*$ w.r.t. $(x,\mathcal{S})$, such that the pair $(\mathcal{T},\mathcal{T}^*)$ satisfies (B), (B^*), (C).

Definition 1.9: Any bargaining proposal of the form $\mathcal{T} = (P, I-P)$ will be called an onecoalitional bargaining proposal. Any bargaining counter proposal of the form $\mathcal{T}^* = (P^*, I-P^*)$ will be called an onecoalitional bargaining counter proposal. $\mathcal{T}^*$ may counter a multicoalitional bargaining proposal.

Definition 1.1o: The onecoalitional bargaining set M is the set of all c-stable $x \in F$. Some $x \in F_{\mathcal{S}}$ is c-stable, if either $x \in C(G)$, or for any onecoalitional bargaining proposal $\mathcal{T}$ w.r.t. $(x,\mathcal{S})$, there exists an onecoalitional bargaining counter proposal $\mathcal{T}^*$ w.r.t. $(x,\mathcal{S})$ and $\mathcal{T}$.

Obviously, the above definitions show that $C(G) \subseteq M_o \subseteq M$. It has been shown in [2] that Theorem 1.8 still holds if we confine ourselves to onecoalitional bargaining counter proposals. Then the natural question is whether or not $M_o = M$. Such an equality would show that we can confine ourselves to onecoalitional bargaining proposals and counter proposals. Moreover, it has been proved in [3] that:

<u>Theorem</u> 1.11, ([3],Th.2.4): If G is convex, then $M_o = M$.

The main result of the present paper is that there exists non convex games such that $C(G) \subset M_o \subset M$. This fact is motivating the study of the multicoalitional bargaining proposals and counter proposals.

2. <u>A family of non convex games with $C(G) \subset M_o \subset M$</u> .

Consider the family of 6 person games G defined by:

(a) $v(\emptyset) = o$;

(b) $v(12)$, $v(34)$, $v(13)$, $v(24)$ any numbers subject to

$$\lambda = 1/2 \left[v(12) + v(34) - v(13) - v(24) \right] > o \; ; \tag{2.1}$$

(c) $v(5)$, $v(6)$, $v(23)$ any numbers;

(d) $v(123)$ and $v(234)$ given in terms of the above numbers by:

$$\begin{aligned} v(123) &= 1/2 \left[2v(13) + v(23) + v(24) - v(34) \right] + 2\lambda , \\ v(234) &= 1/2 \left[2v(24) + v(13) + v(23) - v(12) \right] + 2\lambda ; \end{aligned} \tag{2.2}$$

(e) $v(1)$, $v(2)$, $v(3)$, $v(4)$, $v(14)$, $v(124)$, $v(134)$ and $v(S)$ with $|S| > 3$, any numbers small enough to satisfy some conditions imposed below.

Consider the coalition structure $\mathcal{S} = \{ 13, 24, 5, 6 \}$ and the set of admissible payoff vectors for $\mathcal{S}$:

$$F_{\mathcal{S}} = \{ x \mid x \in R^6 , x_1 + x_3 = v(13) , x_2 + x_4 = v(24) , x_5 = v(5) , x_6 = v(6) \} . \tag{2.3}$$

Consider the admissible payoff $\tilde{x} \in F_{\mathcal{S}}$ given by:

$$\begin{aligned} \tilde{x}_1 &= 1/2 \left[v(12) + v(13) - v(23) \right] , & \tilde{x}_2 &= 1/2 \left[v(23) + v(24) - v(34) \right] , \\ \tilde{x}_3 &= 1/2 \left[v(13) + v(23) - v(12) \right] , & \tilde{x}_4 &= 1/2 \left[v(24) + v(34) - v(23) \right] , \\ \tilde{x}_5 &= v(5) , & \tilde{x}_6 &= v(6) . \end{aligned} \tag{2.4}$$

The values of the excesses are

$$e(\tilde{x},13) = e(\tilde{x},24) = e(\tilde{x},5) = e(\tilde{x},6) = o , \tag{2.5}$$

because $\tilde{x} \in F_{\mathcal{S}}$, then

(2.6) $e(\tilde{x},12) = e(\tilde{x},34) = e(\tilde{x},23) = \lambda > o$,

follow from (b), then

(2.7) $e(\tilde{x},123) = e(\tilde{x},234) = 2\lambda > o$,

follow from (d); the excesses $e(\tilde{x},1)$, $e(\tilde{x},2)$, $e(\tilde{x},3)$, $e(\tilde{x},4)$, $e(\tilde{x},14)$, $e(\tilde{x}, 124)$, $e(\tilde{x},134)$ and $e(\tilde{x},S)$ with $|S| > 3$, will be negative, if the values of v mentioned in (e) are small enough. In fact, the negativity of these excesses and the conditions (2.1) , (2.2) define the family of games G .

From the identity

$$e(x,S\cap T) + e(x,S\cup T) - e(x,S) - e(x,T) = \tag{2.8}$$
$$= v(S\cap T) + v(S\cup T) - v(S) - v(T) \ , \quad \forall\ x \in R^n \ , \forall\ S,T \in P(I) \ ,$$

and the inequality

(2.9) $e(\tilde{x},123) + e(\tilde{x},2) - e(\tilde{x},12) - e(\tilde{x},23) = e(\tilde{x},2) < o$,

that follows from (2.6) and (2.7) we get

(2.1o) $v(S\cap T) + v(S\cup T) - v(S) - v(T) < o$ for $S=12$, $T=23$,

hence any game of the family is not convex.

As there are positive excesses, we get $\tilde{x} \notin C(G)$.

There are 6 possible bargaining proposals, namely:

5 onecoalitional bargaining proposals: $\mathcal{T}_1 = \{12 , 3456\}$, $\mathcal{T}_2 = \{34 , 1256\}$, $\mathcal{T}_3 = \{23 , 1456\}$, $\mathcal{T}_4 = \{123 , 456\}$, $\mathcal{T}_5 = \{234 , 156\}$;

1 multicoalitional bargaining proposal: $\mathcal{T}_o = \{12 , 34 , 56\}$.

By using Theorem 1.6 we can show that all these bargaining proposals can be blocked as follows: $\mathcal{T}_1$ by $\mathcal{T}_5$; $\mathcal{T}_2$ by $\mathcal{T}_4$; $\mathcal{T}_3$ by $\mathcal{T}_1$ or $\mathcal{T}_2$; $\mathcal{T}_4$ by $\mathcal{T}_5$; $\mathcal{T}_5$ by $\mathcal{T}_4$; $\mathcal{T}_o$ by either $\mathcal{T}_4$ or $\mathcal{T}_5$. Therefore, we get $\tilde{x} \in M_o$. As $\tilde{x} \notin C(G)$, we have shown that for our class of 6 person games the strict inclusion $C(G) \subset M_o$ holds. We have proved the following:

<u>Theorem</u> 2.1: There exist non convex games such that $C(G) \subset M_o$.

Consider the same class of games, the same coalition structure $\mathcal{S}$, but another admissible payoff $x^* \in F_{\mathcal{S}}$:

(2.11) $x_1^* = \tilde{x}_1 - a$, $x_2^* = \tilde{x}_2 + a$, $x_3^* = \tilde{x}_3 + a$, $x_4^* = \tilde{x}_4 - a$, $x_5^* = \tilde{x}_5$, $x_6^* = \tilde{x}_6$, $o < a < \lambda/2$.

The excesses are:

(2.12) $e(x^*,13) = e(x^*,24) = e(x^*,5) = e(x^*,6) = o$,

because $x^* \in F_{\mathcal{S}}$, then

(2.13) $e(x^*,12) = e(x^*,34) = \lambda$, $e(x^*,23) = \lambda - 2a$,

follow from (2.6) and (2.11) , then

(2.14) $e(x^*,123) = e(x^*,234) = 2\lambda - a$,

follow from (2.7) and (2.11). We may choose a subclass of our class of games such that all the other excesses are negative, let us call it G^*.

As there are positive excesses, we have $x^* \notin C(G)$.

There are again 6 possible bargaining proposals, the same as above. The same Theorem 1.6 with the new values of the excesses, shows that all 5 onecoalitional bargaining proposals are blocked as mentioned above, hence $x^* \in M$.

However, now $\mathcal{T}_o$ can not be blocked by any other bargaining proposal:

$e(x^*,23) = \lambda - 2a < e(x^*,12) + e(x^*,34) = 2\lambda$ hence $\mathcal{T}_3$ does not block;

$e(x^*,123) = 2\lambda - a < e(x^*,12) + e(x^*,34) = 2\lambda$ hence $\mathcal{T}_4$ does not block;

$e(x^*,234) = 2\lambda - a < e(x^*,12) + e(x^*,34) = 2\lambda$ hence $\mathcal{T}_5$ does not block.

Further, $\mathcal{T}_1$ and $\mathcal{T}_2$ can not be taken into consideration, because both do not satisfy (B^*). Hence, $x^* \notin M_o$. As $x^* \in M$, we have shown that for our class of games G^* the strict inclusion $M_o \subset M$ holds. We have proved:

<u>Theorem</u> 2.2: There exist non convex games such that the strict inclusion $M_o \subset M$ holds.

In fact, the class G^* is included in G , therefore we have obtained:

<u>Theorem</u> 2.3: There exist non convex games such that the strict inclusion $C(G) \subset M_o \subset M$ holds.

The last result motivates the study of the multicoalitional bargaining proposals and counter proposals done in [2] and [3]. On the other hand, the same result shows that the modified bargaining set defined in [2] has a real meaning, i.e. at least for some games M_o is not empty. Obviously, the existence problem for any game is an open question.

R E F E R E N C E S

[1] Aumann,R.J., and M. Maschler, "The bargaining set for cooperative games", in Advances in Game Theory V , (M.Drescher, L.S.Shapley, A.W.Tucker, eds.), Annals of Math. Study, No.52, Princeton, N.J.,1964, 443-476.

[2] Dragan,I. "A combinatorial approach to the theory of the bargaining sets", Libertas Math.,5,(1985), 133-15o.

[3] Dragan,I. "The bargaining set M_o for convex games", Libertas Math.,6, (1986), 175-181.

[4] Maschler,M. "The inequalities that determine the bargaining set $M_1^{(i)}$", Israel Journal of Math.,4,(1966), 127-134.

[5] Maschler,M., Peleg,B., and Shapley,L.S., "The kernel and the bargaining set for convex games", Int. Journal of Game Theory,1,(1972), 73-97.