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ON THE EXISTENCE OF A WEAK SOLUTION TO AN EQUATION OF VOLTERRA-SKOROHOD TYPE

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We denote by $[0,T]$ a bounded interval of the real line, by R the real line and let $E_o = R - \{0\}$. We denote by $(\Omega, F, (F_t)_{t\in T}, P)$ a fixed stochastic basis, and by $B(R)$, $B(E_o)$, $B(T)$ the class of Borel sets, respectively from R, E and T.

We suppose the following elements given:

- A $(F_t)_{t\in T}$ real Wiener process $w(t)$, $_{t\in T}$, with independent increments with respect to F_t.
- A real $(F_t)_{t\in T}$ adapted process $\zeta(t)$ with independent increments with respect to $(F_t)_{t\in T}$, homogeneous, with $\zeta(0) = 0$, with the characteristic function,

$$E[\exp iz\,\zeta_t] = \exp\{t \int_R [izu - iI_{|u|\leq 1}\, zu]\frac{du}{u^2}\} \tag{1}$$

where $I_A = \begin{cases} 1 & x\varepsilon A \\ 0 & x\varepsilon A \end{cases}$. The process $\zeta(t)$ has a cadlag version [11].

- The random measure $\{p(\omega,dt,du)\ \omega \in \Omega\}$ is the measure of "saltus" of the process $\zeta(t)$, that is for all $\omega \in \Omega$

$$p(\omega,dt,du) = \sum_{s>0} I_{\Delta\zeta_s(\omega)\neq 0}\ \delta(s,\Delta\zeta_s(\omega))(dt,du) \tag{2}$$

where $\delta(s,u)$ indicates the Dirac measure concentrated in (s,u),

$$\Delta\zeta_s(\omega) = \zeta_s(\omega) - \zeta_{s\_}(\omega).$$

It is proved in Skorohod [9] and Jacod [6] that p results in a F_t random Poisson stationary measure whose Lévy measure is defined on E_o by $\alpha(du) = \frac{1}{u^2}\,du$.

Let λ be the product measure of the Lebesque measure on T and α on E_o, then the measure q is defined by

$$q(\omega,ds,du) = p(\omega,ds,du) - \lambda(ds,du)$$

It is well known that for a large class of functions, the integrals with respect to the measures p and q may be considered [9].

We consider the following stochastic equation.

$$x(t) = f(t) + \int_0^t k(t,s,x(s))ds + \int_0^t g(t,s,x(s))dw(s) + \int_0^t \int_{|u|\leq 1} h(t,s,x(s),u)q(ds,du)$$

$$+ \int_0^t \int_{|u|>1} h(t,s,x(s),u)p(ds,du) \qquad (3)$$

where $f(t)$ is a real, F_t adapted, cadlag process, $k(t,s,x)$, $g(t,s,x)$ are real functions defined on $\Delta \times R$, $\Delta = \{(s,t) \cdot 0 \leq s < t \leq T\}$. The function $h(t,s,x,u)$ is defined on $\Delta \times R \times E_o$ with values in R.

We are interested in proving the existence of a weak solution to the equation (3), when the coefficients satisfy conditions of Carathéodory type, that is the functions are continuous in x and measurable with respect to (t,x), (t,s,u) respectively.

This paper is based on Zonzotto's work [13], which is dealing with the same problem for differential stochastic equations.

Remark. The same problem may be studied in R^m. We are working with real valued functions for simplifying the exposure.

Definition 1. A system $(\Omega,F,P,F_t,x(t), w(t), q(dt,du), p(dt,du))_{t\geq 0}$ is said to be a weak solution (without explosion) of the equation (3), if

i) $w(t))_{t\geq 0}$ is a F_t Wiener process on [0,T]

ii) p(dt,du), q(dt,du) are random Poisson measures with the associated Lévy measure $\alpha(du) = \frac{du}{u^2}$

iii) x(t) is a F_t cadlag process which verifies the equation (3) w.p.1 in all its points of continuity.

Theorem If:

a) f(t) is a real, stochastically continuous F_t-adapted cadlag process

b) k(t,s,x), g(t,s,x), are real functions defined on ΔxR, measurable with respect to (t,s) and continuous with respect to x.

c) the function h(t,s,x,u) if defined on Δ x R x E, continuous with respect to x, measurable with respect to (t,s,u). If u takes values in a compact of R, $h(\cdot,\cdot,\cdot, u)$ is uniformly bounded with respect to (t,s,x). It is supposed

$$\lim_{\substack{x\to x_1\\ s\to s_1\\ t\to t_1}} \int_{|u|\le 1} |h(t,s,x,u) - h(t_1, s_1, x_1, u| \frac{du}{u^2} = 0$$

d) A growth condition takes place, that is, there is a positive constant M, so that

$$T|k(t,s,x)|^2 + |g(t,s,x)|^2 + \max(\frac{h(t,s,x,u)|^2}{u^2}, \frac{|h(t,s,x,u)}{u^2}) \le K(1 + |x|^2),$$

for all $(t,s,x,u) \in \Delta \times R \times E$.

e) The functions g,k, satisfy the condition: there is a L_1, $L_1 > 0$, for which $|k(t_1,s,x) - k(t_2,s,x)| \le L_1 \mid t_1 - t_2|^{1/2}$

$$|g(t_1,s,x) - g(t_2,s,x)| \le L_1 |t_1 - t_2|^{1/2} \text{ for all } t_1, t_2, \in [0,T]$$

f) There is a $L_2 < 0$, $|h(t_1, s,x,u) - h(t_2,s,x,u)| \le L_2 |t_1 - t_2| u^2$, for all $t_1,t_2 \in [0,T)$ Then, the equation (3) has a weak solution.

Proof. The idea of the proof is to construct a sequence of approximate solutions $x_2(t)$ which makes us able to apply Skorohod theorem of representation [12] for the Polish spaces C(0,T],R), (The space of continuous functions endowed with the topology of uniform convergence), and D([0,T],R), the space of cadlag functions endowed with J_1, Skorohod topology [2]. After that, we show that the limit of the sequence $\tilde{x}_n(t)$ on the new probability space $(\tilde{\Omega},\tilde{F},\tilde{P})$ satisfies the equation (3) in all the points of continuity of $\tilde{x}_o(t)$, w.p.1.

We use the Tonnelli's approximation procedure [3]. Let $t_J = \frac{JT}{n}$, $J = \overline{0,n-1}$, a partition of the interval [0,T].

We consider the following sequence of approximate solutions, formally

$$\begin{cases} x_n[t,\omega) = f(t,\omega) \qquad t\in[0, \frac{T}{n}] \\ x_n[(t,\omega) = f(t,\omega) + \int_0^{t-\frac{T}{n}} k(t,s,x_n(s))ds + \int_0^{t-\frac{T}{n}} g(t,s,x_n(s))dw(s) \\ + \int_0^{t-\frac{T}{n}} \int_{|u|\leq 1} h(t,s,x_n(s),u)q(ds,du) + \int_0^{t-\frac{T}{n}} \int_{|u|\geq 1} h[t,s,x_n(s),u)p(ds,du) \\ = f(t,\omega) + {}_n(t) + {}_n(t) + {}_n(t) + {}_n(t) \\ \text{for } t\in(\frac{JT}{n}, \frac{(J+1)}{n}] \;, \; J = \overline{1,n-1} \end{cases} \tag{4}$$

Introducing the function $\psi_N(x) = \begin{cases} 1 & |x|\leq N \\ |x|+1-N & \text{for } |x|\in[N,N+1] \\ 0 & |x|>N+1, \quad N\in R \end{cases}$

We denote by $k_N(t,s,x) = k(t,s,x)\psi_N(x)$; $g_N(t,s,x) = g(t,s,x)\psi_N(x)$;

$h_N(t,s,x,u) = h(t,s,x,u)\psi_N(x)|\psi_N(u)$; $f_N(t) = \begin{cases} f(t) & \text{for } |f(t,\omega)|\leq N \\ 0 & |f(t,\omega)|>N \end{cases}$

We shall use the following "truncated approximations"

$$\xi_n(t) = f_N(t) \qquad t\in[0,\tfrac{T}{n}]$$

$$\xi_n(t) = f_N(t) + \int_0^{t-\frac{T}{n}} k_N[t,s,\xi_n(s))ds + \int_0^{t-\frac{T}{n}} g_N(t,s,\xi_n(s))dw(s)$$

$$
\begin{cases}
+ \int_0^{t-\frac{T}{n}} \int_{|u|\le 1} h_N(t,s,\xi_n(s),u)q(ds,du) \\
+ \int_0^{T-\frac{T}{n}} \int_{|u|>1} h_N(t,s,\xi_n(s),u)p(ds,du) \\
= f_N(t) + A_{n,N}(t) + B_{n,N}(t) + C_{n,N}(t) + D_{n,N}(t)
\end{cases}
$$

$$t \in (\frac{JT}{n}, \frac{(J+1)T}{n}) \qquad (5)$$

Lemma 1 $\xi_n(t)$ makes sense for all $t \in [0,T]$, $\xi_n(t)$ is a random element of the space $D[0,T]$.

Proof. For $t \in [0,\frac{T}{n}]$, $E\xi_n^2(t) \le N$ and moreover $E\xi_n^2(t)$ is a continuous function of t. Using the conditions of the theorem, we get $E\xi_n^2(t)$ is continuous on $[\frac{T}{n}, \frac{2T}{n}]$, which implies $P\{\omega, \int_0^T \xi_n^2(t) < \infty\}$, so all the integrals of the second side of (5) make sense for $t \in [\frac{2T}{n}, \frac{3T}{n}]$. We continue in this way by induction.

Let's prove that $\xi_n(t)$ is a cadlag process, also by induction. Using a result of C. Corduneanu [4], the measurability of $k(t,s,x)$ in (t,s), its continuity in x and the growth condition (d) imply the continuity w.p.1 of $A_{n,N}(t)$. $B_{n,N}$ satisfies the condition of Kolmogorov's theorem [9,pg3), so has continuous trajections w.p.1 for $t \in [\frac{T}{n}, \frac{2T}{n}]$. We continue by induction.

We have to prove that $C_n(t)$ is cadlag on $[\frac{T}{n}, \frac{2T}{n}]$. Let $S \subset T$, a set at most countable, and $a < b$, $a, b \in R$.

Let $S_m = \{s_1, s_2 \ldots, s_m\} \subset S \subset [\frac{T}{n}, \frac{2T}{n}]$, a finite set from $[\frac{T}{n}, \frac{2T}{n}]$. We denote by A (S,a,b) the number of upcrossings of the interval [a,b] by the family $_{n,N}(t)$ relatively to S. Using the fact that $\int_0^{t-\frac{T}{n}} \int_{|u|\le|} h_N(v,s,f_N(s),u)q(ds,du)$ is F_s

martingale for every fixed v, we get $E[\int_{s_{k-1}}^{s_k}\int_{|u|\le 1} h_N(s_k,s,f_N(s),u)q(ds,du)/F_{s_{k-1}}]=0$

The previous relation and the condition (f) implies

$$E[(C_{n,N}(s_k)-C_{n,N}(s_{k-1}))/F_{s_{k-1}}] = \int_0^{s_{k-1}}\int_{|u|\le 1}(h_N(s_k,s,f_N(s),u) - h_N(s_{k-1},s,f_N(s),u)q(ds,du) \tag{6}$$

and finally

$$E|C_{n,N}(s_k)-C_{n,N}(s_{k-1})| \le \sqrt{s_{k-1}}(s_k-s_{k-1}) \tag{7}$$

By a similar reasoning like in [11,pg 34-35, 47] using (6) and (7), we prove that $A\,C(S,a,b) < \infty$ w.p.1.

$AC(S,a,b) < \infty$ implies that $C_{n,N}(t)$ does not have discontinuities of saltus on $s \subset [\frac{T}{n}, \frac{2T}{n}]$ for any s countable, dense in $[\frac{T}{n}, \frac{2T}{n}]$.

The stochastic continuity of $C_{n,N}(t)$ or $[\frac{T}{n}, \frac{2T}{n}]$, results in the existence of a cadlag version of $C_{n,N}(t)$ on $[\frac{T}{n}, \frac{2T}{n}]$. $\mathcal{D}_{n,N}(t)$ may be considered as the sum of two integrals,

$$\mathcal{D}_{n,N}(t) = \int_0^{t-\frac{T}{n}}\int_{|\le u\le N} h_N(t,s,f_N(s,u)q(ds,du) + \int_0^{t-\frac{T}{n}}\int_{|\le u|zN} h_N(t,s,f_N(s,u)\frac{duds}{u^2}$$

Taking into account the proof of a cadlag version of $C_{n,N}(t)$ and the continuity of the second integral of (8), we get the existence of a cadlag version of $\mathcal{D}_{n,N}(t)$ on $[\frac{T}{n}, \frac{2T}{n}]$.

Applying the induction, we obtain finally a cadlag version for the processes $C_{n,N}(t)$, $\mathcal{D}_{n,N}(t)$ or $[\frac{JT}{n}, \frac{(J+1)T}{n}]$ for all $J = \overline{1,n-1}$. Hence, using Jonescu-Tulcea theorem of adhesion [11], we prove the existence of a cadlag version of $\xi_n(t)$.

Lemma 2 If the hypothesis of the theorem are fulfilled, then

(i1) $\lim_{c\to\infty} \overline{\lim_{n}} P(\sup_{0\leq t\leq T} |\mathcal{A}_{n,N}(t)| > c) = 0$

(i2) $\lim_{c\to\infty} \overline{\lim_{n}} P(\sup_{0\leq t\leq T} |\mathcal{B}_{n,N}(t)| > c) = 0$

(i3) $\lim_{c\to\infty} \overline{\lim_{n}} P(\sup_{0\leq t\leq T} |\mathcal{C}_{n,N}(t)| > c) = 0$

(i4) $\lim_{c\to\infty} \overline{\lim_{n}} P(\sup_{0\leq t\leq T} |\mathcal{D}_{n,N}(t)| > C) = 0$

uniformly with respect to n,N.

We consider the process $\overline{\xi}_n(t)$, defined exactly like $\xi_n(t)$, with the only difference that $f_N(t)$ is substituted by $f_{N_o}(t)$. By induction, it is easy to show $E|\overline{\xi}_n(t)|^2 \leq KN_o^2T$, using the Gronwall inequality. Then, taking into account the growth condition, we get

$$E(\sup_{0\leq t\leq T} |\int_0^{t-\frac{T}{n}} k_N(t,s,\overline{\xi}_n(s)ds)| \leq E\ T \sup_{0\leq t\leq T} \int_0^{t-\frac{T}{n}} k_N^2(t,s,\overline{\xi}_n(s)ds \leq ETK^2 \sup_{0\leq t\leq T} \int_0^{t-\frac{T}{n}} \overline{\xi}_n^2(s)ds \leq K^2TN_o^2 \tag{8}$$

Relation 8) implies i_1)

Taking into account the properties of the Itô integral, and the hypothesis e), we get

$$E[|\mathcal{B}_{n,N}(t) - \mathcal{B}_{n,N}(s)|^4] \leq B|t-s|^2 \text{ for } 0\leq s<t\leq T, \tag{9}$$

B depending only on T and N_o.

The continuity of $\mathcal{B}_{n,N}(t)$ with probability 1 and 9) enables us to apply the result of Strook and Varadhan [15, Corollary 2.1.4 p. 49], which leads to

(i_2) uniformly with respect to n and N.

For proving $i_3)$ $i_4)$ we need to show first, that for any simple F_t Markov time $\tau < T$,

$$E|\, C_{n,N}(\tau)| \leq MN_o^2T, \text{ M not depending on N and n} \tag{10}$$

Because τ is a simple F_t Markov time,

$$|C_{n,N}(\tau)| \leq \sum_{s_i} \left| I_{\tau=s_i} \int_0^{s_i-\frac{T}{n}} \int_{|u|\leq 1} h_N(s_i,s,\xi_n(s),u)p(ds,du) \right|$$

$$+ \sum_{s_i} I_{\tau=s_i} \int_0^{s_i-\frac{T}{n}} \int_{|u|\leq 1} |h_N(s_i,s,\xi_n(s),u)|\frac{duds}{U^2} = C^1_{n,N}(\tau) + C^2_{n,N}(\tau)$$

The positivity of the measure p(ds,du) and the assumption (d) leads to

$$|C^1_{n,N}(\tau)| \leq \sum_{s_i} I_{\tau=s_i} \int_0^{s_i-\frac{T}{n}} |h_N(s_i,s,\xi_n(s),u)|p(ds,du) \leq K \sum_{s_i} I_{\tau=s_i}$$

$$\int_0^{s_i-\frac{T}{n}} \int_{|u|\leq 1} u^2\xi_n^2(s)p(ds,du) = K \int_0^{\tau-\frac{T}{n}} \int_{|u|\leq 1} u^2\xi_n^2(s)p(ds,du)$$

Hence, due to the postivity of the measure p, we have

$$E|C^1_{n,N}(\tau)| \leq K\, E \int_0^{\tau-\frac{T}{n}} \int_{|u|\leq 1} u^2\xi_n^2(s)p(ds,du) \leq K\, E \int_0^T \int_{|u|\leq 1} I_{|s|\leq\tau-\frac{T}{n}} u^2\xi_n^2(s)p(ds,du)$$

$$\leq K \int_0^T \int_{|u|\leq 1} E\, \xi_n^2(s)dsdu \leq KN_o^2T$$

Now, it is easy to deduce 10).

The continuity to the right of the process $\xi_{n,N}(t)$ allows (10) be true for any Markov time $\tau\leq T$.

If we denote by τ_c the exit time from $S_c = \{x, x\in R, |x|\leq c\}$, of the process $C_{n,N}(t)$, the continuity to the right implies,

$$\{\omega, \sup_{0\leq t\leq T} |C_{n,N}(t)| > c) \subset \bigcup_{t_r\in Q^{\cdot}} \{|\omega, C_{n,N}(t_r) > c\} \subset \bigcup_{t_r\in Q^{\cdot}} \{\omega, \tau_c \leq t_r\}, Q^{\cdot} \tag{11}$$

representing the rationals from [0,T] [5].

The relation (10) leads to $\lim_{c\to\infty} P(\tau_c < t) = 0$, uniformly with respect to n and N and because $(\tau_c \leq t_r)$ may be considered an increasing sequence with, respect to r, of F measurable sets, we have likewise

$$\lim_{c\to\infty} P(\bigcup_{t_r\in Q^{\cdot}} \{\tau_c \leq t_r\}) = 0, \tag{12}$$

uniformly with respect to n and N. Relations (11) and (12) imply (i_3) holds true.

In a similar way, we prove i_4).

<u>Lemma 3</u> <u>The process $x_n(t)$ has a cadlag version and</u> $\mathcal{A}_n(t)$, $\mathcal{B}_n(t)$, $\mathcal{C}_n(t)$, $\mathcal{D}_n(t)$ <u>satisfy the relations (1), (2), (3), (4)</u>

<u>Proof</u> The previous lemma results in $\lim_{c\to\infty} \overline{\lim_{n-\infty}} \sup_N P\{\sup_{0\leq t\leq T} |\xi_n(t)| > c\} = 0$.
Hence, we can choose on increasing sequence N_K, such that

$$P\{\sup_{0\leq t\leq T} |\xi_n(t)| > N_K\} \leq \frac{1}{K^2} \text{ for all n, and } N > N_K.$$

The Borel-Contelli lemma implies that $\xi_{n,N_K}(t)$ converges w.p.1 to some process $\eta_n(t)$ uniformly with respect to n,N. Then the process $\eta_n(t)$ also, will have no discontinuities of the second kind, w.p.1.

By induction, we show that for $t\varepsilon[\frac{JT}{n}, \frac{(J+1)T}{n}]$, $J = \overline{1,n-1}$, $x_n(t)$ makes sense and $\xi_n(t)$ converges in probability to $x_n(t)$ for $N_o \to \infty$.

So, $x_n(t)$ coincides on every $[\frac{JT}{n}, \frac{(J+1)T}{n}]$ with the process $\eta_n(t)$, and we conclude that $x_n(t)$ coincides with $\eta_n(t)$ [0,T]. The stochastic continuity of $x_n(t)$ results in the existence of a cadlag version for $x_n(t)$.

Because for every $t\varepsilon[\frac{JT}{n}, \frac{(J+1)T}{n}]$,

$$P\{\omega, \sup_{0\leq t\leq \frac{(J+1)T}{n}} | \xi_n(t) - x_n(t)| > 0 \} \leq P\{\omega, \sup_{0<t\leq \frac{JT}{n}} |\xi_n(t)| \geq N\},$$

We have

$$P\{\omega, \sup_{0\leq t\leq \frac{(J+1)T}{n}} | x_n(t) > C\} \leq P\{\omega, \sup_{0<t\leq \frac{(J+1)T}{n}} |\xi_n(t)| > C\}$$

$$+ P\{\omega, \sup_{0\leq t\leq \frac{(J+1)T}{n}} | x_n(t) - \xi_n(t)| > 0\} \leq 2P\{\omega, \sup_{0\leq t\leq \frac{(J+1)T}{n}} |\xi_n(t)| > C\} \quad (13)$$

Relation 13) results in the boundedness in probability, uniform with respect to n of $\sup_{0\leq t\leq \frac{(J+1)T}{n}} |x_n(t)| > C\}$

<u>Lemma 4 For any ε and δ these exists $h > 0$ and n_o, such that for all stopping time of the filtration $(F_t)_{t\varepsilon T}$ with values in $[0,T]$ we have</u>

$$\sup_{n\geq n0} \sup_{Q\varepsilon[0,h]} P\{|C_n(\tau + Q \wedge T - C_n(\tau)|>\delta\} \leq \varepsilon \quad (14)$$

$$\sup_{n\geq no} \sup_{Q\varepsilon[0,h]} P\{|C_n(\tau + Q \wedge T - C_n(\tau)|>\delta\} \leq \varepsilon \quad (15)$$

It's enough to prove

$$EI_{t+Q<T}|C^1_{n,N}((\tau+Q)\wedge T - C^1_{n,N}(\tau)| \leq Q \quad (16)$$

for a simple Markov time τ.

For τ, a simple stopping time, we get

$$EI_{\tau+Q\leq T} |C_{n,N}(\tau+Q) - C_{n,N}(\tau)| \leq \sum_{s_i} EI_{s_i+Q\leq T} I_{\tau=s_i} \int_0^{s_i} \int_{|u|\leq|} |h_{N,n}(s_i +Q,s,\xi_n(s),u)$$

$$- h_{N,n}(s_i,s,\xi_n(s),u)|p(ds,du) + \sum_{s_i} EI_{s_i+Q\leq T}I_{\tau=s_i} \int_{s_i}^{s_i+Q} |h_{N,n}(s_i+Q,s,\xi_n(s),u)|p(ds,du)$$

$$\leq LQE \sum_{s_i} I_{\tau=s_i} I_{s_i+Q\leq T} \int_0^{s_i} u^2 p(ds,du) + KE \sum_{s_i} I_{\tau=s} I_{s_i+Q\leq T} \int_{s_i}^{s_i+h} u^2\xi_n^2(s)p(ds,du)$$

$$= LQE \sum I_{\tau+Q\leq T} \int_0^{\tau} u^2 p(ds,du) + KE\, I_{\tau+Q\leq T} \int_{\tau}^{\tau+Q} u^2\xi_n^2(s)p(ds,du) \leq (L+KN_o^2)Q$$

The continuity to the right of the processes $C^1_{n,N}(t)$, $C^2_{n,N}(t)$, $\mathcal{D}_{n,N}(t)$ and the convergence of $C_{n,N}(t)$, $\mathcal{D}_{n,N}(t)$ to $C_n(t)$, $\mathcal{D}_n(t)$, imply the result.

Proof of the theorem

A_n, B_n obviously are random elements of C([0,T],R). The construction of $A_{n,N}$, $B_{n,N}$ and the relation 13), result in the fulfilling of hypothesis of the Kolmogorov's tightness criterion [2,6], by A_n, B_n .

So, if we denote by μA_n, μB_n the laws of A_n, respectively B_n on C([0,T],R), Kolmogorov's criterion implies the relative compactness of $\mu_{\ n}$, $\mu_{\ n}$ in the weak topology of probability measures on C([0,T],R).

The relation (3) and Lemma 4 show that $C_n(t)$ satisfies the condition of Aldous' theorem [13,18], hence the law of C_n, $\mu_{\ n}$, on D([0,T],R) is relatively compact in the weak topology of probability measures on D(0,T],R).

Applying the Skorohod's theorem of representation [3,p6; 13], it follows that the process (f, A_n, B_n, C_n, $\mathcal{D}_n$, w, ζ), which has values in the Polish space $D([0,T] \times R) \times C([0,T] \times R)^2 \times D([0,T]R)^2 \times C([0,T],R) \times D([0,T],R)$, possesses a Skorohod version, ($\tilde{f}_n$, $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$, $\tilde{\mathcal{D}}_n$, $\tilde{w}_n$, $\tilde{\zeta}$) and the last converges w.p.1 to a limit ($\tilde{f}$, $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{\mathcal{D}}$, $\tilde{w}$, $\tilde{\zeta}$).

Because the uniform and Skorohod topologies coincide, when relotivizized to C[1,p.150], likewise we can consider A_n on B_n random elements of D([0,T],R).

We define the mapping Λ: $f(\omega) \times A_n(\omega) \times B_n(\omega) \times C_n(\omega) \times \mathcal{D}_n(\omega) \to D$, by

$$\Lambda(\omega) = f(\omega) + A_n(\omega) + B_n(\omega) + C_n(\omega) + \mathcal{D}_n(\omega)$$

where by G(ω), usually we denoted the trajectory of the process G corresponding to ω.

Using Bafico's lemma [1], we conclude that $\tilde{x}_n$ has the same probability low as x_n, hence $\tilde{x}_n$ is a random element of D([0,T],R) and moreover

$$\tilde{x}_n(t,\tilde{\omega}) = \tilde{f}_n(t,\tilde{\omega}) + \tilde{A}_n(t,\tilde{\omega}) + \tilde{B}_n(t,\tilde{\omega}) + \tilde{C}_n(t,\omega) + \tilde{D}_n(t,\omega), \text{ w.p.1}$$

for all $t\in[0,T]$.

Also, because the Skorohod version $(\tilde{f}_n, \tilde{A}_n, \tilde{B}_n, \tilde{C}_n, \tilde{D}_n)$ has a limit w.p.1 in D, $\tilde{x}_n$ also will converge to process $\tilde{x}_o$ belonging to D, w.p.1.

Taking into account the properties of the natural projection $\pi_t : D \to R$, defined as usual $\pi_t(G) = G(t)$, we get that the converges of $\tilde{x}_n$ to $\tilde{x}_o$ in D, implies $\lim_{n\to\infty} \tilde{x}_n(t,\omega) = \tilde{x}_o(t,\omega)$, w.p.1 for all t, points of continuity of x_o [2,13].

The coincidence of probability laws of w, $\tilde{w}_n$ in C([0,t],R) implies that $\tilde{w}_n(t)$ is a Wiener process in the new space $(\tilde{\Omega}, \tilde{F}, \tilde{P}))$, $\tilde{\zeta}_n(t)$ is a process with independent increments having the characteristic function (1). $\tilde{q}_n$, $\tilde{p}_n$ has the usual meaning, referring to the process $\tilde{\zeta}_n(t)$.

Following the same arguments as in [19,p. 89], we define $\tilde{F}^n_t$ the completion of $\sigma(\tilde{x}_n(s), \tilde{w}_n(s), \tilde{\zeta}_n(s), \tilde{f}(s), s\leq t)$. Obviously, from the construction of $\tilde{F}^n_t$ $\tilde{x}_n(t)$ is $\tilde{F}^n_n$ adapted and being cadlag, $\tilde{x}_n(t)$ is progressively measurable with respect to $\tilde{F}^n_t$.

We need to prove that in $(\tilde{\Omega}, \tilde{F}, \tilde{P})$

$$J_1)\ \tilde{A}_n(t) = \int_0^{t-\frac{T}{n}} k(t,s,\tilde{x}_n(s))ds$$

$$J_2)\ \tilde{B}_n(t) = \int_0^{t-\frac{T}{n}} g(t,s,\tilde{x}_n(s)d\tilde{w}_n(s)$$

$$J_3)\ \tilde{C}_n(t) = \int_0^{t-\frac{T}{n}} \int_{|u|\leq 1} h(t,s,\tilde{x}_n(s)u)\tilde{q}_n(ds,du)$$

$$J_4)\ \tilde{D}_n(t) = \int_0^{t-\frac{T}{n}} \int_{|u|\geq 1} h(t,s,\tilde{x}_n(s)u)\tilde{p}_n(ds,du)$$

For proving (J_1) we use the same argument given in [13].

The conditions (b)(e),(d) of the theorem enable us to find a sequence of step functions $g_r(t,s,\tilde{x}_n(s))$, such that

$$\lim_{r\to\infty} \int_0^T |g(t,s,\tilde{x}_n(s)) - g_r(t,s,\tilde{x}_n(s))|^2\, ds = 0, \text{ uniformly with respect to t and}$$

n, w.p.1 ([12],[17,[8]). (16)

Using a similar proof like [15], [1], we get (J_2), (J_3), (J_4). We now have to prove that $\tilde{x}_o(t)$ verifies the equation (3) in $(\tilde{\Omega}, \tilde{F}, \tilde{p})$, w.p.1, for all t, points of continuity of $\tilde{x}_o(t)$.

We have to prove that $\tilde{A}_n(t)$ converges in probability to $\int_0^t k(t,s,\tilde{x}_o(s)ds$ $\tilde{B}_n(t)$ converges in probability to $\int_0^t g(t,s,\tilde{x}_o(s))d\tilde{w}_o(s)$, $\tilde{C}_n(t)$ converges in probability to $\int_0^t \int_{|u|\leq 1} h(t,s,\tilde{x}_o(s),u)\tilde{q}_o(ds,du)$, and $\tilde{D}_n(t)$ to $\int_0^t \int_{|u|\geq 1} h(t,s,x_o(s),u)$ $\tilde{p}_o(ds,du)$.

We prove that $\tilde{C}_n(t)$ converges in probability to $\int_0 \int_{|u|\leq 1} h(t,s,\tilde{x}_o(s),u)$ $\tilde{q}_o(ds,du)$. We know that for any $\bar{\varepsilon} > 0$, there is $N > 0$, such that

$\tilde{P}\{\omega, \sup_{0\leq t\leq T} |\tilde{f}_n(t) - \tilde{f}_{n,N}(t)|\} < \varepsilon$, so which implies, taking into account that $\tilde{x}_n(t)$ and $x_n(t)$ have the same finite-dimensional distributions,

$$\lim_{n\to\infty} \tilde{P}\{\omega, \sup_{0\leq y\leq T} |\tilde{x}_n(t) - I_{|\tilde{x}_n|\leq N} \tilde{x}_n(t)| > 0\} < \varepsilon \quad (17)$$

Taking into account the hypothesis (c) we have for h a relation similar to (16), that is, for all (t,s,x,u) in D x R X E, there is a continuous function $h_{N,\rho}$, for which

$$\lim_{\rho\to\infty} \int_0^T \int_{|u|\leq 1} |h_N(t,s,\tilde{x}_o(s),u) - h_{N,\rho}(t,s,x_o(s),u)|^2 \frac{duds}{u^2} = 0 \quad \text{w.p.1} \qquad (18)$$

Then

$$Y = |\int_0^{t-\frac{T}{n}} \int_{|u|\leq 1} h(t,s,\tilde{x}_n(s),u)\tilde{q}_n(ds,du) - \int_0^t \int_{|u|\leq 1} h(t,s,\tilde{x}_o(s),u)\tilde{q}_o(ds,du)|$$

$$\leq |\int_0^{t-\frac{T}{n}} \int_{|u|\leq 1} h(t,s,\tilde{x}_n(s),u)\tilde{q}_n(ds,du) - \int_0^{t-\frac{T}{n}} \int_{|u|\leq 1} h_N(t,s,\tilde{x}_n(s),u)\tilde{q}_n(ds,du)|$$

$$+|\int_{t-\frac{T}{n}}^t \int_{|u|\leq 1} h_N(t,s,\tilde{x}_n(s),u)\tilde{q}_n(ds,du)| + |\int_0^t \int_{0\leq|u|\leq\varepsilon} h_N(t,s,\tilde{x}_n(s),u)\tilde{q}_n(ds,du)|$$

$$+|\int_0^t \int_{\varepsilon\leq|u|\leq 1} (h_N(t,s,\tilde{x}_n(s),u) - h_N(t,s,\tilde{x}_o(s),u))\tilde{q}_n(ds,du)|$$

$$+|\int_0^t \int_{\varepsilon\leq|u|\leq 1} (h_N(t,s,\tilde{x}_o(s),u) - h_{N,\rho}(t,s,\tilde{x}_o(s),u))\tilde{q}_n(ds,du)|$$

$$+|\int_0^t \int_{\varepsilon\leq|u|\leq 1} h_{N,\rho}(t,s,\tilde{x}_o(s),u)\tilde{q}_n(ds,du) - \int_0^t \int_{\varepsilon\leq|u|\leq 1} h_{N,\rho}(t,s,\tilde{x}_o(s),u)\tilde{q}_o(ds,du)|$$

$$+|\int_0^t \int_{\varepsilon\leq|u|\leq 1} (h_N(t,s,\tilde{x}_o(s),u) - h_{N,\rho}(t,s,\tilde{x}_o(s),u))\tilde{q}_o(ds,du)|$$

$$+|\int_0^t \int_{0\leq|u|\leq\varepsilon} h_N(t,s,\tilde{x}_o(s),u)\tilde{q}_o(ds,du)| + \Big|\int_0^t \int_{|u|\leq 1} h_N(t,s,x_o(s),u)$$

$$- h(t,s,\tilde{x}_o(s),u))\tilde{q}_o(ds,du)\Big|$$

$$= Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6 + Y_7 + Y_8 + Y_9 \tag{19}$$

$$E(I_2)^2 \leq \int_0^{t-\frac{T}{n}} \int_{|u\leq 1} E\, h_N^2(t,s,\tilde{x}_n(s),u)\, \frac{duds}{u^2} \leq N^2 \frac{T}{n}\ ,\ \text{so} \lim_{n\to\infty} E(I_2)^2 = 0 \tag{20}$$

$$\lim_{n\to\infty} E(I_4^2) \leq \lim_{n\to\infty} \int_0^t \int_{\varepsilon\leq|u|\leq 1} \frac{E(h_N(t,s,\tilde{x}_n(s),u) - h_N(t,s,\tilde{x}_o(s),u))^2}{u^2}\, duds = 0$$

because h_N is measurable in (t,s,u), continuous in x and bounded, and $\tilde{x}_n(s)$ converges to $\tilde{x}_o(s)$ w.p.1, almost everywhere in s (in all s, points of continuity for $\tilde{x}_o(s)$).

The continuity of $h_{N,\rho}(t,s,x,u)$, the fact that $\tilde{x}(s)$ is cadlag imply, applying Skorohod's lemma [9,p. 20], that

$$I_6 = \int_0^{t-\frac{T}{n}} \int_{\varepsilon\leq u\leq 1} h_{N,\rho}(t,s,\tilde{x}_o(s,u)\tilde{q}_n(ds,du) - \int_0^t \int_{\varepsilon\leq|u|\leq 1} h_{N,\rho}(t,s,x_o(s),u)\tilde{q}_o(ds,du)$$

converges in probability to 0, when $n\to\infty$.

Because $\tilde{x}_n(t)$ converges to $\tilde{x}_o(t)$ in the J_1 Skorohod topology [[2],(17)] using (13) we get

$$P\{\omega, \sup_{0\leq t\leq 1} \left| \tilde{x}_o(t) - I_{|\tilde{x}_o|\leq N} \cdot \tilde{x}_o(t)\right| > 0\} < \bar{\varepsilon} \tag{22}$$

The properties of the stochastic integral with respect to $\tilde{q}$ implies

$$P(I_1 > 0) \leq \overline{\lim}\, P\{\omega, \sup_{0\leq t\leq 1} \left| \tilde{x}_n(t) - I_{|\tilde{x}_n|\leq N}\, \tilde{x}_n(t)\right| > 0\},\ \text{and}$$

$$P(I_9 > 0) \leq P\{\omega, \sup_{0\leq t\leq 1} | \tilde{x}_o(t) - I_{|\tilde{x}_o|\leq N}\, \tilde{x}_o(t)| > 0\}$$

We have $\lim_{\rho\to\infty} E(I_5)^2 = 0$, and $\lim_{\rho\to\infty} E(I_7)^2 = 0$, for all n, N, $\bar{\varepsilon}$. We have also, that

$$\lim_{\varepsilon-0} E(I_3^2) \leq \lim_{\varepsilon\to 0} \int_0^t \int_{0\leq|u|\leq\varepsilon} \frac{E(h_N^2(t,s,\tilde{x}_n(s),u)duds}{u^2} = 0, \text{ and also } \lim_{\varepsilon\to 0} E(I_8)^2 = 0.$$

Taking into account all these consideration, taking in (19) the limit for $n\to\infty$ in probability, after that the limit for $\rho\to\infty$, the limit for $\varepsilon\to 0$, and finally the limit when $N\to\infty$, we get that $\int_0^{t-\frac{T}{n}} h(t,s,\tilde{x}_n(s),u)\tilde{q}_n(ds,du)$ converges in probability to $\int_0^t h(t,s,\tilde{x}_o(s),u)\tilde{q}_o(ds,du)$, for all t, points of continuity of the process $\tilde{x}_o(t)$.

As $\tilde{F}_t$ we consider the completion of $\sigma(\tilde{x}_o(t), \tilde{w}_o(t), \tilde{f}_o(t)\ \tilde{\zeta}_o(t))$.

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Bibliography

[1] R. Bafico, Una estensione del teorema di esistneza de Skorohod; Boll. Un. Mat. Ital, 16.B (1979) 134-153.

[2] P. Billingsley, Convergence of Probability Measures, (1968), Wiley Series in Probability and Mathematical Statistics.

[3] C. Corduneanu, "An existence theorem for functional equations of Volterra type," Libertas Mathematica, Vol. 6, 1986.

[4] C. Corduneanu, "Principles of Differential and Integral equations, Chelsea Publ. Co., New York, 1977.

[5] T. Morozan, Stabilitatea Sistemelor Cu Parametri Aleatori, Bucuresti, 1969, Edit R.S.R.

[6] J. Jacod, "Calcul stochastique et problemes de Martingales," (1979) Springer-Verlag.

[7] Métivier I. Y. Pellaumail, Stochastic integration," (1980) Academic Press.

[8] Cora Sadosky, "Interpolation of operators and singular integrals," 1979, Pure and Applied Mathematics

[9] A. V. Skorokhod, "Studies in the theory of random processes," (1965), Adiwes International Series in Mathematics.

[10] C. Tudor, "On Volterra Stochastic equations," Boll. U.M.I.(6) 5-A, 335-344.

[11] C. Tudor, "Probabilitati si procese stocastice," (2 volumes 1979) Edit Acad R.S.R.

[12] K. Yosida, "Functional Analysis," Springer-Verlag (1978).

[13] A. Zanzotto, "Soluzioni deboli per equazioni stocastiche del tipo di Skorokhod," Boll U.M.I (5) (1985).

[14] A. Zanzotto, "Compacité en loi des solutions d'une famille d' équations stochastiques," C. R. Acad Sci, Paris a 291 (1980) 52-60.

[15] D. W. Stroock-S.R. Varadhan, "Multidimensional diffusion processes," Springer-Verlag, New York, 1979.

[16] II Gihman, A. V. Skorohod, "Stochastic differential equations, equations," Springer-Verlag, New York, 1972.

[17] A. Friedman, "Stochastic differential equations and applications," Vol. 1,2, 1975. Academic Press.

[18] A. Aldous, "Stopping times and tightness," Ann Probab. 6, 2 (1978) (335,340).

[19] N. V. Krylov, "Controlled Diffusion Processes," Applications of Mathematics," 1980, Springer-Verlag, New York, Inc.