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PERTURBATION OF COMPACT SEMIGROUPS GENERATORS

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0. Introduction

Compact semigroups generators A are important in the study of existence and asymptotic behaviour of solutions of abstract differential equations (inclusions) $x' \in Ax$. We refer to the monographs [7-9] for details. In this paper we are concerned with some perturbations B of A having the "compactness preserving property". More precise, we look for several conditions which guarantee that the compactness of the semigroup S_A generated by A via the exponential formula of Crandall-Liggett, implies the compactness of S_{A+B}. In particular, we extend a result of Pazy [8].

1. Statement of main results

Throughout this paper X is a general (real) Bannach space of norm $\|\cdot\|$. Recall some other notations:

$$\langle y, x\rangle_i = \lim_{h\uparrow 0} \frac{\|x+hy\|^2 - \|x\|^2}{2h}; \quad \langle y, x\rangle_- = \lim_{h\uparrow 0} \frac{\|x+hy\| - \|x\|}{h} \tag{1.1}$$

$$< y, x >_s = \lim_{h\downarrow 0} \frac{\|x+hy\|^2 - \|x\|^2}{2h}; \quad < y, x >_+ = \lim_{h\downarrow 0} \frac{\|x+hy\| - \|x\|}{h}$$

for $x, y \in X$. Clearly

$$\langle y, x\rangle_i = \|x\|\|\langle y, x\rangle_- \leq \|x\|\langle y, x\rangle_+ \leq \|x\|\|y\|, x, y \in X \tag{1.2}$$

A nonlinear (possible multivalued) operator $A: D(A) \subset X \to 2^X$ is said to be w-dissipative (w - a real number) if

$$\langle y_1 - y_2, \ x_1 - x_2 \rangle_i \ \leq \ w\|x_1 - x_2\|^2, \tag{1.3}$$

for all $x_j \in D(A)$ and $y_j \in Ax_j$, $j = 1, 2$, "0-dissipativity" is simply called "dissipativity". The operator A is said to be m-dissipative, if it is dissipative and $R(I - A) = X$. Roughly speaking (Crandall-Liggett's theorem asserts that if A is m-dissipative, then the family $S = \{S(t), S(t) : \overline{D(A)} \to \overline{D(A)}\}$ defined by the exponential formula

$$\lim_{n \to \infty} (I - \frac{t}{n} A)^{-n} x \equiv S(t)x, \ x \in \overline{D(A)} \tag{1.4}$$

is a nonexpansive semigroup. In order to avoid confusions, denote S by S_A and call it "the semigroup generated by A via the exponential formula".

The semigroup S_A is said to be compact if for every $t > 0$, $S_A(t)$: $\overline{D(A)} \to \overline{D(A)}$ is compact (i.e. maps bounded subsets of $\overline{D(A)}$ into precompact subsets of X). We are now in a position to state the main result of this paper.

Theorem 1.1. Let $A : D(A) \subset X \to 2^X$ be w-dissipative satisfying the range condition $R(I - A) = X$. Suppose that S_A is compact. If $B : D(B) \subset X$ (with $D(B) \supseteq \overline{D(A)}$) is continuous and

1) $A + B$ is $\tilde{w}$-dissipative (for some $\tilde{w} \in R$)

2) B maps bounded subsets of $\overline{D(A)}$ into bounded subsets,

then S_{A+B} is also compact.

Clearly, if $D(B) = X$ and B is a linear bounded operator (from X into itself) of norm $\|B\|$, then $A + B$ is $\tilde{w}$-dissipative, with $\tilde{w} = w + \|B\|$. Indeed

$$\langle Ax - Ay + Bx - By, \ x - y \rangle_i \ \leq \langle Ax - Ay, \ x - y \rangle_i \ + \|Bx - By\|\|x - y\|$$

$$\leq w\|x - y\|^2 + \|B\|\|x - y\|^2, \ \forall x, y \in D(A)$$

Therefore, Theoreom 1.1 implies

Corollary 1.1. Let A be as in Theorem 1.1 (i.e. $A - wI$ is m-dissipative). If S_A is compact and B is a linear bounded operator from X into itself, then S_{A+B} is also compact.

Remark 1.1 If in addition to the hypothesis that $A - wI$ is m-dissipative we assume that A is linear, then Corollary 1.1 is just a result of Pazy [8, p.79, Proposition 1.4]. Note also that: $A - wI$ m-dissipative, $B : \overline{D(A)} \to X$ continuous and $A + B$ $\tilde{w}$- dissipative implies that $A + B - \tilde{w}I$ is m-dissipative, i.e. $A + B$ generates (via the exponential formula) a nonlinear semigroup S_{A+B} (see Kaboyaski [4] or the book [7] for details).

Remark 1.2 Condition 2, in Theorem 1.1 is not necessary. This fact follows from

Theorem 1.2 Let H be a real Hilbert space and let $f, g : H \to]-\infty, +\infty]$ be two lower semi-continuous convex functions such that $0 \in Int(D(f) - D(g))$. Set $A = -\partial f$ and $B = -\partial g$. If S_A is compact and g is positive-valued, then S_{A+B} is also compact.

Recall that $D(f) = \{x \in H, f(x) < +\infty\}$ i.e. $D(f)$ is "the effective domain"of f and $\partial f(x) = \{x^* \in H, \ f(y) - f(x) \geq \langle x^*, \ y - x\rangle, \ \forall y \in H\}$ is the subdifferential of f at x.

2. Proof of the results

We first recall that if f and g are continuous (or only in $L_1([0,T], X)$) from an interval $[0,T]$, $T > 0$ into X and if u and v are the integral solutions of the problems

$$u'(t) \in Au(t) + f(t), \ u(t_o) = u_o \in \overline{D(A)}, \ 0 \leq t_o \leq t \leq T \tag{2.1}$$

$$v'(t) \in Av(t) + g(t), \ v(t_o) = v_o \in \overline{D(A)}, \ 0 \leq t_o \leq t \leq T$$

(with A-dissipative) then

$$\|u(t) - v(t)\| \leq \|u(s) - v(s)\| + \int_s^t \|f(x) - g(x)\| d\tau \tag{2.2}$$

for all $0 \leq t_o \leq s \leq t \leq T$ (see e.g. [7], Ch.2, (4.28))

Proof of Theorem 1.1 Denote by $S = S_{A+B}$ the semigroup generated by $A + B$ via the exponential formula. For simplicity of writing choose $w = \tilde{w} = 0$ and set

$$u(t) = S_{A+B}(t)x \equiv S(t - t_o)S(t_o)x, \quad v(t) = S_A(t - t_o)y, \quad 0 \le t_o \le t \tag{2.3}$$

for $x, y \in \overline{D(A)}$. Then u and v are the integral solutions of the problems

$$u'(t) \in Au(t) + Bu(t); \; u(t_o) = u_o = S(t_o)x; \; 0 \le t_o \le t$$

$$v'(t) \in Av(t); \; v(t_o) = y, \; 0 \le t_o \le t$$

In view of (2.2) it follows

$$\|S_{A+B}(t)x - S_A(t - t_o)y\| \le \|S(t_o)x - y\| + \int_{t_o}^{t} \|Bu(s)\|ds \tag{2.4}$$

for all $0 \le t_o \le t \le T$ and $x, y \in \overline{D(A)}$. Fix $t > 0$. We are now prepared to prove that $S(t)$ is compact. Therefore, let M be a bounded subset of $\overline{D(A)}$. We will prove that $S_{A+B}(t)M$ is precompact. First, we note that the function $(s, x) \to S_{A+B}(s)x$ is bounded on bounded sets so there exists a constant $C = C(t, M) > 0$ such that

$$\|u(s)\| = \|S_{A+B}(s)x\| \le C, \; \forall x \in M, \; s \in [0, t]$$

which implies the boundedness of $Bu(s) \equiv BS_{A+B}(s)x$ on $[0, t] \times M$. Say $\|Bu(s)\| \le \tilde{C}$, $\forall s \in [0, t]$, $x \in M$. Now take an arbitrary $\varepsilon \in]0, t[$. With $t_o = t - \varepsilon$ and $y \in u(t_o) = S_{A+B}(t - \varepsilon)x$, $x \in M$, (2.4) yields

$$\|S_{A+B}(t)x - S_A(\varepsilon)u(t_o)\| \le \int_{t-\varepsilon}^{t} \|Bu(s)\|ds \le \varepsilon\tilde{C} \tag{2.5}$$

for all $x \in M$. Since $S_{A+B}(t - \varepsilon)M$ is a bounded subset of $\overline{D(A)}$ and $S_A(\varepsilon)$ is compact, it follows that $S_A(\varepsilon)S_{A+B}(t - \varepsilon)M = K_\varepsilon$ is precompact. This implies (on the basis of (2.5)) the precompactness of $S_{A+B}(t)M$, q.e.d.

Proof of Theorem 1.2 According to a result of Brezis [2], the compactness of S_A with $A = -\partial f$ is equivalent to the precompactness of the level sets

$$C_r^f = \{x \in H, \quad \|x\| \le r, \quad f(x) \le r\}, \quad r > 0$$

Note that in our case here we have $\partial f + \partial g = \partial(f + g)$ (see [7], Ch.2, Th. 8.10' due to Attouch).

Moreover, we have

$$C_r^{f+g} = \{x \in H, \ \|x\| \leq r, \ f(x) + g(x) \leq r\} \subset$$

$$\{x \in H, \ \|x\| \leq r, \ f(x) \leq r\} \equiv C_r^f$$

so C_r^{f+g} is also precompact, that is the semigroup generated by $\partial(f + g) = A + B$ is compact, q.e.d.

Remark2.1 According to Theorems 1.1, 1.2, and Remark 1.2, it is interesting to give an example of a convex function $g : H \to R_+$ of class C^1, such that the Frechet derivative $x \to \dot{g}(x)$ is unbounded on some bounded subsets of H. See also Appendix in [7].

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