LIBERTAS MATHEMATICA , VOL. VII (1987)

INTEGRATION OF POISSON STABLE FUNCTIONS OF TWO VARIABLES*

To the memory of S. Stoilow

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Abstract. In this paper we consider Poisson stable functions from $\mathbb{R}^2$ into a Banach space. If $u = u(x,y)$ is a function from $\mathbb{R}^2$ into a Banach space which has relatively compact range, and if the partial (strong) derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ are Poisson stable functions with a common returning sequence, then u is also a Poisson stable function.

Introduction. A classical theorem of Bochner (see for instance the monographs [2] and [5] establishes almost-periodicity of functions from R into a Banach space which have relatively compact range and almost-periodic strong derivative. The same kind of result holds true also in the more general situation of almost-automorphic functions (see [5]). On the other hand, the classical theorem of Bohr-Bohl (almost-periodicity of scalar-valued functions which are bounded and have almost-periodic derivative, see again [2], [5]) has been extended to functions of two variables in [1] and quite recently, when the range of the functions is a Banach space, in the paper [4].

The concept of functions which are stable in the sense of Poisson can be found in the recent monograph [3, p. 80] and an analogous to Bochner's theorem is given as Proposition 1, p. 81 in [3].

(*) We acknowledge support of the Natural Sciences and Engineering Research Council of Canada.

In this short paper we shall prove a similar result for Poisson stable functions in two variables, in the case where the range is a Banach space.

1. Extending in a natural way the definition in [3] we consider strongly continuous functions $f = f(x,y)$ from $\mathbb{R}^2$ into the Banach space E and say that a sequence $(x_m, y_m) \subset \mathbb{R}^2$ such that $x_m^2 + y_m^2 \to +\infty$ is f-returning if the relation

$$\lim_{m\to\infty} f(x+x_m,\ y+y_m) = f(x,y) \tag{1.1}$$

holds true, strongly in E and uniformly on compact subsets of $\mathbb{R}^2$.

We give now the following

Definition. A continuous function $f = f(x,y)$, from $\mathbb{R}^2$ into the Banach space E is said to be stable in the sense of Poisson if it has at least one returning sequence.

Consider now strongly continuous functions $u = u(x,y)$, from $\mathbb{R}^2$ into E, which have partial (strong) derivatives, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ defined in the usual way. Assume also that the functions $f = f(x,y)$ and $g = g(x,y)$ from $\mathbb{R}^2$ into E are Poisson stable functions having at least one common returning sequence. Then the following is valid:

Theorem. If $u = u(x,y)$, $\mathbb{R}^2 \to E$ has relatively compact range and if $\frac{\partial u}{\partial x} = f$, $\frac{\partial u}{\partial y} = g$, then u is also Poisson stable.

Proof. We begin with a few preliminary remarks

From the equality $\frac{\partial u}{\partial x} = f$ we deduce: $\frac{\partial}{\partial x}\left(u(x,y) - \int_0^x f(s,y)ds\right) = \theta$, hence $u(x,y) - \int_0^x f(s,y)ds = u(0,y)$. Next, from the equality $\frac{\partial u}{\partial y} = g(x,y)$ we find that $\frac{\partial}{\partial y}\left(u(x,y) - \int_0^y g(x,s)ds\right) = \theta$, hence $u(x,y) - \int_0^y g(x,s)ds = u(x,0)$; accordingly we get $u(0,y) = u(0,0) + \int_0^y g(0,s)ds$ and finally the representation formula

$$(1.2)\qquad u(x,y) = u(0,0) + \int_0^x f(s,y)ds + \int_0^y g(0,s)ds, \qquad \forall (x,y) \in \mathbb{R}^2.$$

Note now that for any $a,b \in R$, the translated function $u_{a,b}(x,y) = u(x+a, y+b)$ verifies the system of partial differential equations

$$\frac{\partial}{\partial x} u_{a,b}(x,y) = f_{a,b}(x,y) \quad \text{and} \quad \frac{\partial}{\partial y} u_{a,b}(x,y) = g_{a,b}(x,y)$$

where $f_{a,b}$ and $g_{a,b}$ are translated functions of f and g respectively. Consequently, from the above remarks we obtain the representation

$$(1.3)\qquad u(x+a, y+b) = u(a,b) + \int_0^x f(s+a, y+b)ds + \int_0^y g(a, s+b)ds,$$

for all $(x,y) \in \mathbb{R}^2$ and $(a,b) \in \mathbb{R}^2$.

Let us take now a common returning sequence for the functions f and g, say (x_m,y_m); next, we write the representation formula following from (1.3):

$$(1.4)\quad u(x+x_m, y+y_m) = \int_0^x f(s+x_m, y+y_m)ds + \int_0^y g(x_m, s+y_m)ds + u(x_m,y_m).$$

From the relative compactness of the range of u in the space E we see that it is possible to extract a subsequence of (x_m,y_m), which we denote by (x_{m_k},y_{m_k}), with the property that

$$(1.5)\qquad \lim_{k\to\infty} u(x_{m_k},y_{m_k}) = \tilde{u} \in E.$$

It follows readily that the formula

$$(1.6)\quad \lim_{k\to\infty} u(x+x_{m_k},y+y_{m_k}) = \int_0^x f(s,y)ds + \int_0^y g(0,s)ds + \tilde{u} = u(x,y) - u(0,0) + \tilde{u}$$

is satisfied (<u>the limit being uniform on compact subsets of $\mathbb{R}^2$</u>, as easily seen): $\int_0^x [f(s+x_{m_k},y+y_{m_k}) - f(s,y)]ds \to \theta$ uniformly on compacts in $\mathbb{R}^2$ for (if $|x|^2 + |y|^2 \le A^2$, then $\|f(s+x_{m_k},y+y_{m_k}) - f(s,y)\| < \frac{\varepsilon}{A}$ for $k > \bar{k}_1(\varepsilon)$ and accordingly $\|\int_0^x [f(s+x_{m_k},y+y_{m_k}) - f(s,y)]ds\| \le \frac{\varepsilon}{A}|x| \le \varepsilon$ for $k > \bar{k}_1(\varepsilon)$; also we see that

$\|\int_0^y [g(x_{m_k}, s+y_{m_k}) - g(0,s)]ds\| \le |y| \cdot \frac{\varepsilon}{A} \le \varepsilon$, due to the fact that

$\|g(x_{m_k}, s+y_{m_k}) - g(0,s)\| < \varepsilon/A$ for $k > \bar{k}_2(\varepsilon)$, and $|s| < |y| < A$).

Let us denote by $v = v(x,y)$ the function given as $\lim_{k\to\infty} u(x+x_{m_k}, y+y_{m_k})$. We obtain, using (1.2) the relation

$$v(x,y) = u(x,y) - u(0,0) + \tilde{u} = u(x,y) + C,$$

where C is some element in E.

It follows that: $v(x+x_{m_k}, y+y_{m_k}) = u(x+x_{m_k}, y+y_{m_k}) + C$ and we deduce

$$\lim_{k\to\infty} v(x+x_{m_k}, y+y_{m_k}) = v(x,y) + C = u(x,y) + 2C.$$

If we denote by T the generalized translation operator given by

$$T\,u(x,y) = \lim_{k\to\infty} u(x+x_{m_k}, y+y_{m_k})$$

we get that $T\,u$ belongs to the closure of the range of u in E.

Note also the relations:

$Tu = v = u + C$; $T(Tu) = T\,v = T\,u + C = u + 2C$; $\dots\ T^n u = u + nC$.

Furthermore the obvious estimates hold:

$$\sup_{\mathbb{R}^2} \|T\,u\| \le \sup_{\mathbb{R}^2} \|u\|; \quad \dots \sup_{\mathbb{R}^2}\|T^n u\| \le \sup_{\mathbb{R}^2} \|u\|, \qquad n = 1,2,\dots$$

Hence, we obtain, for any $(x,y) \in \mathbb{R}^2$ and any natural number n, the estimate

$$\|u(x,y) + nC\| \le \sup_{\mathbb{R}^2}\|u(x,y)\|.$$

On the other hand, if C is a non-zero element of E we get also the lower bound

$$\sup_{\mathbb{R}^2} \|u(x,y)\| \geq \|u(x,y) + nC\| \geq n\|C\| - \|u(x,y)\| \geq n\|C\| - \sup_{\mathbb{R}^2} \|u(x,y)\|.$$

It follows that, $\forall n = 1,2,\ldots,$ the inequality $n\|C\| \leq 2 \sup_{\mathbb{R}^2} \|u(x,y)\|$ must be verified, which is impossible.

Consequently, the element C must be zero and $T u = u$, which means that the sequence (x_{m_k}, y_{m_k}) is a returning sequence for u.

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