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To the Memory of my Professor
Simion Stoilov

A BIFURCATION THEORY INVOLVING A-PROPER MAPPINGS

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It is generally true that the study of nonlinear problems is rather a collection of ad hoc methods depending in a natural manner on the structure of equations and the investigator's ingenuity. However, in the past two decades, many of these procedures have been evolved in unifying theories by introducing appropriate classes of nonlinear mappings. A powerful tool in proving the existence of solutions to operator equations is the topological degree which may be a pivot for other topics in nonlinear analysis, (see e.g. K.Deimling's book [3]). The extensions beyond the Leray-Schauder degree to cover succesively wider classes of operators started with an axiomatic scheme of its basic properties. The fixed point index appears as a natural generalization of the degree for vector fields (perturbations of the identity) and turns out to be useful for applications.

On the other hand, various problems in mathematical physics are modeled by a parameter-dependent operator equation

(1) $$T(\lambda,u) = 0 \qquad (\lambda,u) \in \mathbb{R} \times X,$$

on a Hilbert space X over reals $\mathbb{R}$. Once the existence of solutions is positively solved, the second step in the qualitative study of the last equation will be the behavior and branching of solutions with respect to change of value of parameter λ involved in problem. This is certainly the case in bifurcation theory where values of parameters are sought for which the number of solutions changes.

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1. INTRODUCTION

Suppose now that a branch $(\lambda, u_0(\lambda))$ of solutions of equation (1) is known. Without loss of generality, we may assume that $u_0(\lambda) = 0$ and the equality

$$T(\lambda, 0) = 0 \qquad \forall\ \lambda \in \mathbb{R}$$

holds. Otherwise, we can replace the initial equation by the equivalent one $T(\lambda, u_0(\lambda)+u) = 0$ and later we refer to $u_0(\lambda)$ as a trivial branch of solutions. Thereby, we suppose that equation (1) has the real line $\mathfrak{X} = \{(\lambda,0) \mid \lambda \in \mathbb{R}\}$ as trivial solutions and denote $\mathfrak{B} = \{(\lambda,u) \in \mathbb{R}\times X \mid T(\lambda,u)=0,\ u \neq 0\}$ the set of nontrivial solutions of (1). Thus, $(\mu,0)$ is a **bifurcation point** with respect to the real line of trivial solutions if every neighborhood of $(\mu,0)$ in $\mathbb{R}\times X$ contains nontrivial solutions of (1).

Later on we shall restrict ourselves to the operator $T: \mathbb{R}\times X \mapsto X$ involved by the equation

(2) $$T(\lambda, u) = (I - \lambda L)u - G(\lambda,u) = 0 \qquad (\lambda,u) \in \mathbb{R}\times X,$$

where the principal part L is a bounded linear operator and the nonlinear perturbation G is a higher order map in the sense that $o(\|u\|)$ uniformly in λ as $u \to 0$. Denote by $N(T)$ and $R(T)$ the kernel and range of T, respectively. In that case, the equivalent equation $u = \lambda Lu + G(\lambda,u)$ can be regarded as a perturbation of the linear eigenvalue problem $u = \lambda Lu$, so that our approach is rooted in the spectral theory of linear operators.

Let $L: X \mapsto X$ be a closed, densely defined linear operator with resolvent $\rho(L)$ and spectrum $\sigma(L)$. For any $\lambda \in \sigma(L)$, we define

$$\chi(\lambda) = \dim \bigcup_{j=1}^{\infty} N(I - \lambda L)^j$$

as the algebraic multyplicity of λ. The **essential spectrum** $\sigma_e(L)$ is the set of all points $\lambda \in \sigma(L)$ for which $R(I - \lambda L)$ is not closed or $\chi(\lambda)$ is infinity or λ is a limit point of $\sigma(L)$. The set $\sigma_e(L)$ is close and $r_e(L) = \max\{|\lambda| \mid \lambda \in \sigma_e(L)\}$ the radius of essential spectrum of L. Moreover, we consider

$$\tau(L) = \{ \lambda \in \mathbb{R} \mid \lambda^{-1} \in \sigma(L),\ \ |\lambda| < [r_e(L)]^{-1}\}$$

the set of **characteristic values** of L. For $\lambda \in \tau(L)$ we denote by $\mathfrak{B}(\lambda)$ the component of $\mathfrak{B} \cup \{(\lambda,0)\}$ containing $(\lambda,0)$. By a component we mean a closed connected subset maximal with respect to inclusion. A point $(\mu,0)$ is a bifurcation point for (2) if and only if, given any $\varepsilon > 0$, there exists an element $(\lambda,u) \in \mathfrak{B}$ such that $\| u \|^2 + | \lambda - \mu |^2 < \varepsilon^2$. In that case, $\mathfrak{B}(\mu)$ is called the **bifurcating branch** emanating from $(\mu,0)$.

It is easy to prove (see [1] or [14]) :

PROPOSITION 1. Let $(\mu,0)$ be a bifurcation point of (2) with $|\mu| < [r_e(L)]^{-1}$. Then μ is a characteristic value of L.

However, this necessary condition is not sufficient for bifurcation to occur.

The purpose of the paper is to establish a local and global bifurcation theory for solutions of the equation (2) involving A-proper operator components. These mappings have a prevalent role because of their appropriateness in the study of infinite-dimensional problems via the Galerkin finite-dimensional reduction method. Following R.D.Nussbaum's approach [10], we substitute the solving of the equation (2) by finding the fixed points of the sum $\lambda L + G(\lambda,\bullet)$. First, we construct a fixed point index for A-proper mappings, developing an idea of K. Scholz [13]. This definition using the theory of retracts, we believe to be more intrinsecally linked to our goal than that derived from the generalized topological degree [2] and exploited by F.E.Fitzpatrick and W.V.Petryshyn [7] in nonlinear eigenvalue problems. Our treatment is based on the compactness of bounded branches of solutions to the equation (2), established by J.F.Toland [15]. To prove the existence of bifurcation points and the bifurcation alternative we adapt the techniques given by S.Ackermann [1] and R.D.Nussbaum [10], respectively. Finally, it is worth pointing out that our results refer to operator equations involving a wide class of A-proper mappings which includes compact operators, ball k-set contractions, condensing maps as well as perturbations of strongly monotone operators by ball k-set contractions.

2. FIXED POINT INDEX

In our approach, we rely on the following definition of the fixed point index in finite-dimensional Euclidean spaces. Let U be an open set of $\mathbf{R}^m$, $F:U \mapsto \mathbf{R}^m$ be a continuous function such that $K = fix(F,U) = \{ u \in U \mid F(u)=u \}$ is compact. Denote by $\mathcal{O}_K \in H_m(U, U\backslash K)$ the fundamental class around K, where H_* stands for singular homology with coefficients in $\mathbf{Z}$, the ring of integers. The **fixed point index** of F is

(3) $$i_X(F, U) = (I - F)_* \mathcal{O}_K \in H_m(\mathbf{R}^m, \mathbf{R}^m \backslash 0) \approx \mathbf{Z}.$$

here $(I - F):(U, U\backslash K) \mapsto (\mathbf{R}^m, \mathbf{R}^m \backslash 0)$ is the map $(I - F)(x) = x - F(x)$ (see e.g. [4], p.202). One can see that this definition agrees with the analytical one: if a point $x_0 \in U$ is an isolated fixed point of F and V an open subset of U such that $V \cap K = \{x_0\}$, then the analytical index $i_X(F,x_0) = \deg(I-F,V,x_0)$ of F at x_0 and the fixed point index $i_X(F,V)$ are equal.

On the other hand, a set M of a topological space Y is called a **neighborhood retract** in Y if there is an open set U of Y containing M and a **retraction** of U onto M, i.e. a continuous map $r: U \mapsto M$ such that $r_{|M} = Id_M$, the identity map on M. Thus any continuous mapping $f: M \mapsto X$ can be extended by $f \circ r: U \mapsto X$, where X is another space. Moreover, a metric space X is said to be an **absolute neighborhood retract** (ANR) if, whenever it is homeomorphic to a closed set M of some Banach space Y, then M is a neighborhood retract of Y. Closed convex sets in a Banach space are obviously ANRs.

Given now a topological pair (X,V), $V \subseteq X$, and a continuous map $F: V \mapsto X$, consider its fixed point set $\text{fix}(F, V) = \{ u \in V \mid F(u) = u \}$. By $\bar{V}$ and ∂V we mean the closure and the boundary of V, respectively.

A triple (X, F, U) is **admissible** if X is a metric ANR space, U is a bounded open set of X such that $\text{fix}(F, U)$ is compact. An **admissible homotopy** is a continuous map $H : [0, 1] \times X \mapsto X$ such that the union of the fixed-point sets of $H(t,\bullet)$ for $t \in [0,1]$ is compact. Let $\mathfrak{U}$ denote the class of all admissible triples and $\mathbf{Z}' = \mathbf{Z} \cup \{ \pm\infty \}$. The (possibly multivalued) map

$$i : \mathfrak{U} \mapsto \mathbf{Z}'$$

is called a **fixed point index** if the following axioms are satisfied:

(i) **Normalization:** $i_X(F, U) = 1$ for every constant map F whose image lies in U;

(ii) **Existence**: if $i_X(F, U) \neq \{0\}$ then F has fixed points in U;

(iii) **Additivity**: For every pair of disjoint open subsets U_1 and U_2 of U such that $\text{fix}(F, U) \subset U_1 \bigcup U_2 \subset U$ then

$$i_X(F, U) \subseteq i_X(F, U_1) + i_X(F, U_2) ;$$

with equality holding if either of $i_X(F, U_j)$, $j = 1,2$, is a singleton set.

(iv) **Homotopy** : If H is an admissible homotopy, then

$$i_X(H(t, \bullet), U) = \text{constant} \quad \text{for all } t \in [0, 1].$$

(v) **Commutativity**: Let $F : U \subseteq X \mapsto Y$ and $G : V \subseteq Y \mapsto X$ be continuous maps on then open sets U and V in the ENRs X and Y, respectively. If one of the composites

$$G \circ F : F^{-1}(V) \mapsto X \quad \text{and} \quad F \circ G : G^{-1}(U) \mapsto Y$$

is admissible, then so is the other and, in that case,

$$i_X(G \circ F, F^{-1}(V)) = i_Y(F \circ G, G^{-1}(U)).$$

Now if U is unbounded, we choose a bounded open subset $V \subseteq U$ so that (X,F,V) be admissible and $(U \setminus V) \cap fix(F, U) = \emptyset$ and define $i_X(F,U) = i_X(F, V)$.

Using the Leray - Schauder degree, R. D.Nussbaum [10] extended the fixed point index to locally compact functions. Let X be a metric ANR, U an open subset of X and let $F:U \mapsto X$ be a continuous map such that $K = fix(F,U)$ is compact and there is an open neighborhood V of K for which $F_{|V}$ is compact. According to the Arens - Eells theorem ([5], p. 158), we may assume, in addition, that X is a closed set in a normed linear space. Because X is an ANR, there is an open neighborhood $\mathcal{O}$ of X and a continuous retraction r of $\mathcal{O}$ into X. Then $F \circ r: r^{-1}(U) \mapsto X$ and we define

$$i_X(F, U) = \deg_{LS}(I - F \circ r, r^{-1}(U), U).$$

By means of this, R.D.Nussbaum [8] extended the global bifurcation alternative for equations defined by locally strict-set-contraction perturbations of the identity.

On the other hand, we can prove that the neighborhood retract have an intrinsic character in Euclidean spaces, i.e. if $U_m \subset \mathbf{R}^m$ is a neighborhood retract and $U_n \subset \mathbf{R}^n$ is homeomorphic with U_m, then U_n is also a neighborhood retract, ([5], p.81). So, X is called an **Euclidean neighborhood retract** (ENR), if it is homeomorphic with a neighborhood retract U in $\mathbf{R}^m$. The set U is locally closed ($U = \mathcal{C} \cap \mathcal{O}$, where $\mathcal{C}$ is closed and $\mathcal{O}$ is open) and locally compact (every point of U has a compact neighborhood).

A couple of sequences $\Gamma = (X_n, r_n)$ is called an **approximation scheme** for a separable Hilbert space X with norm $\| \cdot \|$ provided there exists an increasing sequence of ENR topological subspaces (X_n) of X and (r_n) the corresponding retracts $r_n : X_n \mapsto X_n$ such that $\cup X_n$ is dense in X and $r_n v \longrightarrow v$ for each $v \in X$ as $n \to \infty$. For $F : X \mapsto X$, define $F_n : X_n \mapsto X_n$ by $F_n = r_n \circ F_{|X_n}$.

A map $F : X \mapsto X$ is said to be **A- proper** with respect to the scheme Γ if each $F_n : X_n \mapsto X_n$ is continuous and whenever $x_n \in X_n$ is a bounded sequence such that, for a subsequence (x_j), with $F_j (x_j) - x_j \mapsto 0$ in X, there exists a further subsequence (x_m) converging to some $x \in X$ and $F(x) = x$.

This definition actually corresponds to the usual A-properness of $I-F$. For other details of the above topics we refer to the G.Eisenack and C.Fenske book [4].

We consider now a bounded open set U in an ENR Hilbert space X, $F : U \mapsto X$ an A-proper map such that (X, F, U) is an admissible triple. Set $U_n = U \cap X_n$ and it is easy to check (X_n, F_n, U_n) is also admissible for n large enough. An index for triples (X_n, F_n, U_n) is defined by (3). Then, we define $i_X(F, U)$ to equal the set of limit points

$$\{ i_{X_n}(F_n, U_n) \mid (X_n, F_n, U_n) \text{ is admissible } \}.$$

It follows that $i_X(F,U)$ is a non-empty set in $\mathbf{Z}'$ and the above axioms are easily verfied.

In the sequel, on the Cartesian product $E = \mathbb{R} \times X$ we take the norm

$$\|(r,v)\| = \left(|r|^2 + \|v\|^2 \right)^{\frac{1}{2}}.$$

Let $f: \mathfrak{I} \mapsto \mathbb{R}$ be continuous, where $\mathfrak{I}$ is an interval of $\mathbb{R}$ and $A : U \mapsto X$ a continuous map such that (X, A, U) is an admissible triple. In this case, the product map $f \times A : \mathfrak{I} \times U \mapsto \mathbb{R} \times \mathbf{X}$, given by $(f \times A)(r,v) = (f(r,v), A(r, v))$, is also admissible and

(4) $$i_E(f \times A, \mathfrak{I} \times X) = \deg(f, \mathfrak{I}) \times i_X(A, U),$$

where deg denotes Brouwer's degree.

Moreover, we replace axiom (iv) by the following **generalized homotopy invariance**, which is easy to establish :

Let Λ be a set in $[0,1] \times X$ and $\Lambda_t = \{ x \in X \mid (t,x) \in \Lambda \}$ its t-fiber. A homotopy $H: [0,1] \times X \mapsto X$ is **admissible** provided $H(\bullet,v) : [0,1] \mapsto X$ is continuous, uniformly on bounded closed sets of X, and $H(t,\bullet) : X \mapsto X$ is A-proper for each $t \in [0,1]$ such that the union of fixed point sets of $H(t,\bullet)$ over all $t \in [0,1]$ is compact. Then $i_X(H(t,\cdot), \Lambda_t)$ is indepedent of $t \in [0,1]$.

By a fixed point in the case of a homotopy $H:[0,1]\times\mathbf{X} \mapsto E$ we understand a fixed point for its projection onto $\mathbf{X}$ and its index is given by formula (4).

Finally, we recall two useful results in the computation of the fixed point index for linear operators [15]:

LEMMA 1. Let U be a neighborhood of the origin in X and let $L: U \mapsto X$ be a bounded linear A-proper operator with $|\lambda| r_e(L) < 1$ but $\lambda \notin \tau(L)$. Then $i_X(\lambda L, U) = (-1)^\beta$, where β is equal to the sum $\Sigma_j \chi(\lambda_j)$ of the algebraic multiplicities of all of eigenvalues $\lambda_j > 1$ of λL. If there are no such eigenvalues, then let $\beta = 0$.

LEMMA 2. Let $T(\lambda,\bullet):U \mapsto X$ be an A-proper map with $T(\lambda,v) = v$ whose Frechet derivative at v is of the form λL, where L fulfills the hypotheses of the previous lemma. Then v is an isolated fixed point of $T(\lambda,\bullet)$ and

$$i_X(T(\lambda,\bullet),v) = i_X(T'(\lambda,v),0).$$

3. BRANCHING OF SOLUTIONS.

We turn now to the bifurcation of solutions of equation (2) on an open interval J of the real line containing $\left(-[r_e(L)]^{-1}, [r_e(L)]^{-1}\right)$. In the framework of A-proper mappings, we make the following assumptions on the components of T:

(I) $L: X \mapsto X$ is a bounded linear operator;

(II) $G: J \times X \mapsto X$ is contionuous with $G(\lambda,0) = 0$ for all $\lambda \in J$ and

(IIa) $\dfrac{\| G(\lambda, v) \|}{\| v \|} \to 0$, uniformly for λ in bounded intervals and

(IIb) $G(\bullet, v): J \mapsto X$ is continuous for u in bounded sets of X;

(III) Both λL and $\lambda L + G(\lambda, \bullet)$ are A-proper mappings for all $\lambda \in J$.

For any $\lambda \in J$ we define henceforth the map $F_\lambda: X \mapsto X$ where

$$F_\lambda(u) = F(\lambda,u) = \lambda Lu + G(\lambda,u),$$

and $F(t,\lambda,u) = \lambda Lu + tG(\lambda,u)$ for $t \in [0,1]$. As in [10], we replace $\mathfrak{L} \cup \mathfrak{B}$ by

$$S = \{ J \times \{0\}\} \cup \{ (\lambda,u) \in J \times X \mid u \neq 0, \ F(\lambda,u) = u \}.$$

Given $\mu \in \tau(L)$, let S_μ be the component of S containing $(\mu,0)$.

It is not difficult to establish

PROPOSITION 2. ([15], Th.2.3) If assumptions (I) - (III) hold, then any closed bounded subset of $S \cap (J \times X)$ is compact.

This proposition allows us to handle the fixed point index defined above. It follows that S_μ is compact whenever it is bounded and denote

$$\Lambda_\mu = \{ \lambda \in \tau(L) \mid (\lambda,0) \in S_\mu \}.$$

Since $\tau(L)$ has no accumulation points, S_μ meets Λ_μ at the finitely many points $(\lambda_j,0)$, $j=1,\dots,m$, including $(\mu,0)$. After a suitable renumbering, we may assume that $\lambda_j < \lambda_{j+1}$, $j = 1,\dots,m-1$.

Further, $B_r(v)$ denotes the ball of radius r about v while $\|A-B\|$ is the distance between A and B in X or E.

The crucial tool is given by

PROPOSITION 3. Suppose the hypotheses (I) - (III) are fulfilled. If S_μ is bounded, then there exists a bounded open set $\mathcal{O} \subset J \times X$ such that

(j) $S_\mu \subset \mathcal{O}$;

(jj) $\partial\mathcal{O} \cap S = \emptyset$;

(jjj) $\mathcal{O}$ contains no points $(\lambda,0)$ with $\lambda \in \tau(L)$ other than S_μ ;

(jv) Set $d = \min_j \| \{\lambda_j\} - \{\tau(L)\setminus\{\lambda_j\}\} \|$. Then λ_j is the single characteristic number of L in $B_d(\lambda_j)$;

(v) For every $\varepsilon \in (0,d]$ there is a $\omega(\varepsilon) > 0$ such that whenever $(\lambda,u) \in \mathcal{O}$ is a solution of $F(t,\lambda,u) = u$ with $\|u\| < \omega(\varepsilon)$, $\lambda \notin B_\varepsilon(\lambda_j)$ and $t \in [0,1]$ it follows that $u = 0$.

The proof is similar to the case of compact operators, ([8], pp. 78-79).

For the sake of simplicity, set $J(\lambda) = (\lambda-d,\lambda+d)$ and get $J(\lambda) \cap \tau(L) = \{\lambda\}$ for $\lambda\in\Lambda_\mu$. By using properties (iii) -(iv), we establish at once:

LEMMA 3. ([10], p. 98) To any $\lambda\in J(\mu)$, $\lambda\neq\mu$, there corresponds a $\rho(\lambda)>0$ such that $F(\lambda,u) \neq u$ for $\|u\|\leq\rho(\lambda)$. Moreover, $i_X(F_{\mu_1},B_{\rho(\mu_1)})$ is constant for $\mu_1 <\mu$, $\mu_1\in J(\mu)$, and $i_X(F_{\mu_2},B_{\rho(\mu_2)})$ is constant for $\mu_2>\mu$, $\mu_2\in J(\mu)$.

Consequently, we may define the **index jump** of F at μ by

$$\Delta_F(\mu) = i_X(F_{\mu_2},B_{\rho(\mu_2)}) - i_X(F_{\mu_1},B_{\rho(\mu_1)}) ,$$

which is independent of the particular choice of $\mu_i\in J(\mu)$ with $\mu_1 < \mu < \mu_2$. Moreover, $\rho(\mu_i)$ can be replaced by any number $r>0$ such that $F(\lambda_i,u) \neq u$ for $\|u\|\leq r$.

In particular, for any linear operator L satisfying assumptions (I) and (III), we have

$$i_X((\mu+r)L,B_r(0)) = (-1)^{\chi(\mu)}\, i_X((\mu+r)L,B_r(0)),$$

where μ is a characteristic value of L of algebraic multiplicity $\chi(\mu)$. In that case,

$$\Delta_L(\mu) = \begin{cases} 0 & \text{when } \chi(\mu) \text{ is even,} \\ \pm 2 & \text{when } \chi(\mu) \text{ is odd.} \end{cases}$$

The proof proceeds as in [17], p. 619.

We now proceed to state and prove our main results:

LOCAL BIFURCATION THEOREM. If (H_1) - (H_3) are satisfied and μ is a characteristic numer of L of odd multiplicity, then $(\mu,0)$ is a bifurcation point of the equation

$$F(\lambda,u) = \lambda L + G(\lambda,u) = u \qquad (\lambda,u)\in J\times \mathbf{X}. \tag{5}$$

Proof. By the last conclusion in Proposition 2, there exists $\omega=\omega(\varepsilon)>0$ for all $\varepsilon\in(0,d]$ with property that $\|u\|\leq\omega(\varepsilon)$ and $F(t,\mu\pm\varepsilon,u) = u$ imply $u = 0$.

Now, on the open ball $M_\varepsilon(\mu)=\{(\lambda,u)\in J\times X \mid (\lambda-\mu)^2+\|u\|^2<\varepsilon^2+\omega^2\}$, we define a homotopy between λL and $F(\lambda,\bullet)$ by

$$H(t,\lambda,u) = \Big(t(\|u\|^2-\omega^2) - (1-t)((\lambda-\mu)^2-\varepsilon^2),\ F(t,\lambda,u)\Big),$$

Proposition 2 guarantees that $H:[0,1]\times M_\varepsilon(\mu) \mapsto X$ is an admissible homotopy. Indeed, as $u\in\partial M_\varepsilon$ means $(\lambda-\mu)^2+\|u\|^2=\varepsilon^2+\omega^2$ or $\|u\|^2-\omega^2= \varepsilon^2-(\lambda-\mu)^2$, the assumption $H(t,\lambda,u) = u$ on ∂M_ε would imply that

$t(\|u\|^2+\omega^2)+(1-t)(\omega^2-(\lambda-\mu)^2) = \|u\|^2-\omega^2=\varepsilon^2-(\lambda-\mu)^2) = 0$ and $F(t,\lambda,u) = u$,

thus $\lambda=\mu\pm\varepsilon$, $\|u\|=\omega>0$ while $F(t,\mu\pm\varepsilon,u) = u$ entails $u = 0$, a contradiction.

Therefore, $H(0,\lambda,u) = (\varepsilon^2-(\lambda-\mu)^2,\lambda Lu) = (0,u)$ has exactly two fixed points $P_\pm= (\mu\pm\varepsilon,0)$ on $M_\varepsilon(\mu)$ and we have

$$i_E(H(0,\bullet,\bullet),M_\varepsilon) = i_E(H(0,\bullet,\bullet),P_+) + i_E(H(0,\bullet,\bullet),P_-).$$

We consider the F-derivative

$$H'(0,P_\pm,0)(\alpha,v) = (\mp 2\alpha v,(\mu\pm\varepsilon)Lv)$$

and, by using Lemma 2 , we get

$$i_E(H(0,\bullet,\bullet),P_\pm) = i_E(H'(0,P_\pm),(0,0)).$$

Define $g_\pm(v) = \mp 2\varepsilon v$, apply formula (4) and obtain

$$i_E(H'(0,P_\pm),(0,0)) = \deg(g_\pm,0)\times i_E((\mu\pm\varepsilon)L,0) = \mp i_E((\mu\pm\varepsilon)L,0).$$

These relations yield

$$i_E(H(0,\bullet,\bullet),M_\varepsilon(\mu)) = i_X((\mu-\varepsilon)L,0) - i_X((\mu+\varepsilon)L,0).$$

Because $\chi(\mu)$ is odd, we note that $i_E(H(0,\bullet,\bullet),M_\varepsilon(\mu)) = -\Delta_L(\mu) = \mp 2$.

On the other hand, the homotopy invariance implies that

$$i_E(H(0,\bullet,\bullet), M_\varepsilon(\mu)) = i_E(H(1,\bullet,\bullet),M_\varepsilon(\mu)).$$

Thus $H(1,\lambda,u) = (\|u\|^2- \omega^2, F(\lambda,u))$ has at least a fixed point in $M_\varepsilon(\mu)$ for all $\varepsilon\in(0,d]$, i. e. $F(\lambda, u) =u$ with $|\lambda-\mu|<\varepsilon$ and $\|u\| = \omega(\varepsilon)>0$, hence $(\mu,0)$ is a bifurcation point of equation (5). ■

Degree theoretic arguments provide not only existence criteria for bifurcation points but also information regarding the behavior of solution branches.

GLOBAL BIFURCATION THEOREM. Assume that hypotheses (I) - (III) hold. Then either S_μ is unbounded or S_μ is compact, Λ_μ is a finite set and

(5) $$\sum_{\lambda\in\Lambda_\mu} \Delta_F(\lambda) = 0 .$$

Furthermore, if S_μ is compact and $\Delta_F(\mu) \neq 0$, then S_μ contains a point $(0,\lambda)$ with $\mu \neq \lambda \in \Lambda_\mu$.

Proof. Assuming that S_μ is compact, it suffices to prove (5). Let $\mathcal{O} \cap J = (a,b)$ where a,b are finite real numbers and $\mathcal{O}_t = \{u\in X \mid (t,v)\in\mathcal{O}\}$.

First, we select positive numbers δ and ρ, with $\delta < d$, such that

$$[\lambda-\delta,\lambda+\delta\} \times B_\rho(0) \subset \mathcal{O} \quad \text{for all} \quad \lambda\in \Lambda_\mu,$$

and remark that $F(t,u) \neq u$ for $u\in\partial\mathcal{O}$ implies $u = 0$ and $|t - \lambda|<\delta$ for some $\lambda\in\Lambda_\mu$. With $\lambda\in\Lambda_\mu$ and $r\in[0,\rho]$, we associate the open set

$$U = \{ (t,v)\in[\lambda-\delta,\lambda+\delta] \times X \mid (t,v)\in\mathcal{O} \text{ or } \|v\|<r \}.$$

The construction insures that $F(t,u) \neq u$ on ∂U and we take $U_t=\{v\in X|(t,v)\in U\}$. By applying the generalized homotopy property, we find

$$i_X(F_{\lambda+\delta},U_{\lambda+\delta}) = i_X(F_{\lambda-\delta}, U_{\lambda-\delta}),$$

while $U \subset \mathcal{O} \cup B_r(0)$ and the additivity properties give

(6) $$i_X(F_t,U_t) = i_X(F_t,\mathcal{O}_t) + i_X(F_t,B_r(0)) \quad \text{with} \quad t = \lambda \pm \delta.$$

The last two equations yield

$$\Delta_F(\lambda) = i_X(F_{\lambda-\delta},\mathcal{O}_{\lambda-\delta}) - i_X(F_{\lambda+\delta},\mathcal{O}_{\lambda+\delta}).$$

On the other hand, let λ and λ' be two consecutive elements of Λ_μ. As $(\lambda,\lambda') \cap \Lambda_\mu = \emptyset$ and $F(t,u) \neq u$ for $(t,u) \in\partial\mathcal{O}$ and $t\in[\lambda+\delta,\lambda'-\delta]$, the homotopy property implies

(7) $$i_X(F_{\lambda'-\delta}, \mathcal{O}_{\lambda'-\delta}) = i_X(F_{\lambda+\delta},\mathcal{O}_{\lambda+\delta}).$$

Apply repeatedly (6) and (7) and obtain for $1 \leq j < k \leq m$ that

$$\sum_{i=j}^{k} \Delta_F(\lambda_i) = i_X(F_{\lambda_j-\delta}, \mathcal{O}_{\lambda_j-\delta}) - i_X(F_{\lambda_k+\delta},\mathcal{O}_{\lambda_k+\delta}).$$

Choose $j=1$ and $k = m$ and deduce

$$\sum_{\lambda\in\Lambda_\mu} \Delta_F(\lambda) = i_X(F_{\lambda_1-\delta},\mathcal{O}_{\lambda_1-\delta}) - i_X(F_{\lambda_m+\delta},\mathcal{O}_{\lambda_m+\delta}).$$

Finally, the generalized homotopy invariance implies that $i_X(F_t,\mathcal{O}_t)$ is constant when $t\leq \lambda_1-\delta$ and when $t \geq\lambda_m+\delta$. As $\mathcal{O}_t = \emptyset$ for $t\notin[a,b]$, we infer that

$$i_X(F_t,\mathcal{O}_t) = 0 \quad \text{for} \quad t = \lambda_1-\delta \quad \text{or} \quad t = \lambda_m+\delta,$$

and (5) follows. ☐

In conclusion, we mention some other results related to our subject. The first investigations concerning bifurcation theory for A-proper mappings were carried out by W.V.Petryshyn [11] and J.F.Toland [15], a decade ago, independently. They used the generalized topological degree and outlined some implementations to semilinear elliptic eigenvalue problems. A suggestive example is when $G(\lambda,\bullet) = R(\lambda,\bullet) + S(\lambda,\bullet)$. To obtain a global alternative, we may assume for instance that λL is a ball k-set contraction, $R(\lambda,\bullet)$ is compact and $S(\lambda,\bullet)$ is strongly monotone, i.e. $(S(\lambda,u) - S(\lambda,v)) \geq c\|u-v\|^2$ with $k-c < 1$. In particular, the bifurcation results are extended to the operator equation

$$Lu - \lambda Ku + G(\lambda,u) = 0$$

where L is an A-proper linear map and K is a linear compact operator. Appropriate characteristic values for the couple (F,K) are defined.

On the other hand, J.R.L.Webb and S.C.Welsh [16] gave a global alternative for solutions of nonlinear equations in the form

$$u - T(\lambda)u + G(\lambda,u) = 0$$

under the assumptions (II)-(III). Here $T(\lambda)$ is an odd degree polynomial in λ with operator coefficients. Their approach is based on the multiplication property of the degree mapping on a direct sum of spaces.

Due to the importance of bifurcation theory in the interpretation of physical phenomena, a great number of sophisticated pure mathematical techniques have been developed, some of them belonging to the algebraic topology or group representation theory. However, to proliferate the bifurcation principles, simpler approaches based only on elementary point set topology and basic results of operator theory has been recently revealed. By exploiting the concept of essentiality, the structure of global solution branches to multiparameter operator equations are obtained in [8]. These results apply to 0-maps and in particular to any class of operators which have reasonable degree theory. P.Milojevic [9] introduced a corresponding approximation-essential mapping and established new fixed point and surjectivity results.

R E F E R E N C E S

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