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WELL-POSED CAUCHY PROBLEM AND RELATED SEMIGROUPS OF OPERATORS FOR THE EQUATION $Bu'(t) = Au(t)$, $t \geq 0$ IN BANACH SPACES

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INTRODUCTION. In this paper we present an extension of some statements in [1], [2] concerning the Cauchy problem for differential equations in abstract (here Banach) spaces, in strong form; the main novelty is the passage from the equation $u'(t) = Au(t)$ with a (single) unbounded operator A to an equation involving two operators: $Bu'(t) = Au(t)$, where B can also be unbounded and non-invertible operator. Under assumptions of well-posedness of the Cauchy problem on the half-line $[0, \infty)$ it is shown how to associate the semigroup transporting the initial data into the solution at time t; next we prove several properties, mainly relating the operators B, A and the Laplace transformation of the semigroup. The final result appears to be formula (3.5) which, when B = Identity becomes a very classical formula relating the Laplace transform of the semigroup with the resolvent $(\lambda I-A)^{-1}$

1. Consider a Banach space X and two linear operators A and B, with domains D(A) and D(B) and range in X. (the domains are linear subspaces of X) .

Definition 1.1 *A solution* $u(\cdot)$ *of the equation*, $Bu'(t) = Au(t)$, $t \geq 0$ *is a function* $u(\cdot)$, $[0,\infty] \to D(A) \cap D(B)$, *such that* $Bu(\cdot) \in C([0, \infty); X)$, $u'(t)$ *exists* $\forall t \geq 0$ and $u'(t) \in D(B)$ $\forall t \geq 0$, $Bu'(t) = Au(t)$ *holds*, $\forall t \geq 0$.

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Remarks: From existence of u' it follows that $u(\cdot) \in C([0, \infty); X)$; the case $B = I$ (Identity operator) appears in [2] - p.6.

Defintion 1.2 *The Cauchy problem for* $Bu' = Au$ *on* $[0, \infty)$ *consists in: finding a solution* $u(\cdot)$ *on* $[0, \infty)$ *such that* $u(0) = u_0$ *- a given element in* $D(A) \cap D(B)$.

Definition 1.3 *The above Cauchy problem is well posed if,* $\forall\ u_0 \in D(A) \cap D(B)$ *there is one and only one solution and if, given any sequence* $(u_{0,n})_1^\infty \subset D(A) \cap D(B)$ *such that* $u_{0,n} \to \theta$ *it follows that* $u_n(t) \to \theta\ \forall t > 0$, *where* $u_n(\cdot)$ *is the solution on* $[0, \infty)$ such that $u_n(0) = u_{0,n}$.

Remark: From the existence of a solution $\forall\ u_0 \in D(A) \cap D(B)$ it follows that the relation

$$\{Ax,\ x \in D(A) \cap D(B)\} \subset \{By,\ y \in D(B)\} \tag{1.1}$$

holds.

In fact, let $z = Ax$, for some $x \in D(A) \cap D(B)$. Then, take a solution $u(\cdot)$ such that $u(0) = x$. Thus $Bu'(0) = Au(0) = Ax = z$, $z \in R(B)$.

Consider now a well-posed Cauchy problem and associate to it a family of mappings $U(t)$, depending on the non-negative parameter t, acting from $D(A) \cap D(B)$ into itself, defined by the relation

$$U(t)u_0 = \{u(t), \text{ the solution on } [0, \infty) \text{ such that } u(0) = u_0\} \tag{1.2}$$

We see that $U(0) = \text{Identity}$

We have then the following.

Proposition 1.1 *The mapping* $U(t)$ *is linear,* $D(A) \cap D(B) \to D(A) \cap D(B)$, $\forall t \geq 0$ *and if* $(x_n)_1^\infty \subset D(A) \cap D(B)$ and $x_n \to x_0 \in D(A) \cap D(B)$, *then* $U(t)x_n \to U(t)\, x_0$, $\forall t \geq 0$.

This proposition contains Prop. 2.1 and 2.2 in [2] - p.7 (just take B=I) and is proved in a similar way.

We also have

Proposition 1.2 *The semi-group property:*

$$U(t_1+t_2)x = U(t_1)U(t_2)x,\ x \in D(A) \cap D(B),\ t_1, t_2 \in [0, \infty)$$

holds true.

This contains Prop. 2.3, p. 8 in [2] and is proved in a similar way.

Next we state

Proposition 1.3 *Let us assume that* $D(A) \cap D(B)$ *is dense in* X. *Then there exists one and only one mapping* $V(t)$, $\mathbb{R}^+ \to \mathcal{L}(X)$, *such that* $V(t)x = U(t)x$ $\forall x \in D(A) \cap D(B)$ *and also such that* $V(t+s) = V(t)V(s)\ \forall s, t \in \mathbb{R}^+$, $V(0) = I$.

The result contains Prop. 2.4, p.8 in [2] and is proved in the same way.

Now we first see that the function $V(t)x = U(t)x$ is continuous on $[0, \infty)$, $\forall x \in D(A) \cap D(B)$. Also, following the "Main Lemma" p.9 in [2], we see that an estimate $\|V(t)\|_{\mathcal{L}(X)} \le M_\delta$, $0 < \delta \le t \le \frac{1}{\delta}$ holds, $\forall \delta > 0$ and consequently the function $V(t)x$ is continuous on $]0, \infty[$, $\forall x \in X$.

We also have

Proposition 1.4 *Assume that the Cauchy problem is well-posed* (Definitions 1.1, 1.2, 1.3) *and that* $D(A) \cap D(B)$ *is dense in* X. *Then the derivative* $u'(t)$ *is continuous* $\forall t \in\ ]0, \infty[$.

Proof. We have in fact the relations

$$u'(t) = \frac{d}{dt} U(t)\, u(0) = \lim_{h \downarrow 0} U(t) \frac{u(h) - u(0)}{h} = \lim_{h \downarrow 0} V(t) \frac{u(h) - u(0)}{h} = V(t)\, u'_+(0)$$

Another result, related to Remark 1, p.13 in [2] is stated as

Proposition 1.5. *Assume conditions in Prop. 1.4 and also:* V(t) *maps* D(B) *into* D(B) *and* BV(t)x = V(t)Bx holds, $\forall x \in D(B)$. *Then the commutativity relation*

$$V(t)Ax = AV(t)x = AU(t)x, \quad \forall x \in D(A) \cap D(B) \tag{1.3}$$

holds true .

Proof: Take $x \in D(A) \cap D(B)$; then U(t) x is the solution u(t) to the equation $Bu'(t) = Au(t),\ t \geq 0,\ u(0) = x$. Therefore one gets

$$AU(t)x = B\frac{d}{dt}U(t)x = B \lim_{h \downarrow 0} U(t)\frac{u(h) - u(0)}{h} = B \lim_{h \downarrow 0} V(t)\frac{u(h) - u(0)}{h}$$

$$= B\, V(t)\, u'(0) = V(t)\, Bu'(0) = V(t)\, Au(0) = V(t)\, Ax$$

Corollary *Under assumptions of Prop. 1.5, it follows that* $Bu' \in C\,(]0, \infty[, X)$.

(in fact, $Bu'(t) = Au(t) = AU(t)u(0) = V(t)\, Au(0)$).

Remark Both Prop. 1.4 and Corollary to Prop. 1.5 contain Th. 4.1 in [2] - p.12

Proposition 1.6 *Assume: The Cauchy problem is well-posed and* $D(A) \cap D(B)$ *is dense is* X. *Then, there exists* $\omega \geq -\infty$, $\omega < +\infty$, *such that*

$$\lim_{t \to \infty} \frac{1}{t} \ln \|V(t)\|_{\mathcal{L}(X)} = \omega \tag{1.4}$$

The proof is the same as in [1] - p. 28-29. The number ω is the type of the C-P and of the related semigroup V(t). The following then holds.

Proposition 1.7 *Assume that the* C-P *is well-posed and* $D(A) \cap D(B)$ *is dense in* X. *Then,* if $\lambda \in \mathbb{C}$, $\operatorname{Re} \lambda > \omega$, *the operator* $\lambda B - A$ *is invertible.*

Proof. Take $z \in D(A) \cap D(B)$ such that $\lambda Bz - Az = \theta$. Next, consider the function $u(t) = \exp(\lambda t)z$, $[0, \infty) \to X$. We see that: $u(t) \in D(A) \cap D(B)\ \forall t \geq 0$;

$Bu(t) = e^{\lambda t} Bz \in C([0,\infty),X)$; $u'(t) = \lambda e^{\lambda t} z \in D(B)$ $\forall t \geq 0$; $Bu'(t) = \lambda e^{\lambda t} Az = Au(t)$, $t \geq 0$; $u(0) = z$. Therefore: $u(t) = U(t)z = V(t)z = e^{\lambda t} z$ and $\|e^{\lambda t} z\| = e^{t \Re e \lambda} \|z\| \leq \|V(t)\| \, \|z\|$. If $z \neq \theta$ it follows that: $\Re e\, \lambda \leq \frac{1}{t} \ln \|V(t)\|$, $t > 0$, and as $t \to \infty$, we get $\Re e\, \lambda \leq \omega$. Therefore z must be θ.

2. In this chapter we investigate the Laplace transformation of solutions:

$$\hat{u}(\lambda) = \int_0^\infty e^{-\lambda t} u(t)dt, \text{ for complex } \lambda.$$

We start with the following

Proposition 2.1 *Assume that: the* C-P *is well-posed* on $[0,\infty)$ *and the intersection* $D(A) \cap D(B)$ *is dense in* X. *Then the Laplace transforms:*

$$\int_0^\infty e^{-\lambda t} u(t)\, dt \text{ and } \lim_{\epsilon \downarrow 0} \lim_{N \to \infty} \int_\varepsilon^N e^{-\lambda t} u'(t)dt = \int_{0+}^{+\infty} e^{-\lambda t} u'(t)dt$$

where $u(\cdot)$ *is a solution to the* C-P *both exist for* $\Re e\, \lambda > \omega$ *and the relation*

$$\int_{0+}^{+\infty} e^{-\lambda t} u'(t)dt = \lambda\, \hat{u}(\lambda) - u(0), \; \Re e\, \lambda > \omega \qquad (2.1)$$

holds true.

Proof. The solution $u(\cdot)$ can be written as $u(t) = V(t)\, u(0)$, $t \geq 0$
It is obvious that $\forall \omega_1 \geq \omega$, $\exists R_1$, such that $\|V(t)\| \leq e^{t\omega_1}$, for $t \geq R_1$
Therefore $\|u(t)\| \leq e^{t\omega_1} \|u(0)\|$, $t \geq R_1$ and this shows the absolute convergence of the integral

$$\hat{u}(\lambda) = \int_0^\infty e^{-\lambda t} u(t)dt, \text{ for } \Re e\, \lambda > \omega. \qquad (2.2)$$

We know (Prop. 1.4) that $u'(\cdot) \in C(]0, \infty[; X)$. By partial integration we get

$$\int_\varepsilon^N e^{-\lambda t} u'(t)dt = e^{-\lambda N} u(N) - e^{-\lambda \varepsilon} u(\varepsilon) + \lambda \int_\varepsilon^N e^{-\lambda t} u(t)dt \qquad (2.3)$$

As $\mathbb{R}e\,\lambda > \omega$, we obtain

$$\|e^{-\lambda N}u(N)\| \leq e^{-(\mathbb{R}e\lambda)N}e^{N\omega_1}\|u(0)\| \leq e^{N(\omega_1-\omega)}\|u(0)\| \to 0 \text{ as } N \to \infty$$

Thus

$$\lim_{N\to\infty}\int_\varepsilon^N e^{-\lambda t}u'(t)dt \text{ exists and } = -e^{-\lambda\varepsilon}\,u(\varepsilon) + \lambda\int_\varepsilon^\infty e^{-\lambda t}u(t)dt$$

Using also continuity of $u(\cdot)$ on $[0,\infty)$, we obtain

$$\lim_{\epsilon\downarrow 0}\lim_{N\to\infty}\int_\varepsilon^N e^{-\lambda t}u'(t)dt = -u(0) + \lambda\int_0^\infty e^{-\lambda t}\,u(t)dt \tag{2.4}$$

Note also that, in view of the representation: $u'(t) = V(t)\,u'_+(0)$ (used in the proof of Proposition 1.4) we get that

$$\int_\varepsilon^\infty e^{-\lambda t}u'(t)dt \text{ is absolutely convergent for } \mathbb{R}e\,\lambda > \omega.$$

Therefore, the above formula (2.4) may also be written as

$$\lim_{\epsilon\downarrow 0}\int_\varepsilon^\infty e^{-\lambda t}u'(t)dt = \int_{0+}^\infty e^{-\lambda t}u'(t)dt = -u(0) + \lambda\hat{u}(\lambda) \tag{2.5}$$

Our next statement is expressed as

Proposition 2.2 *Let us assume: the* C-P *is well-posed on* $[0,\infty)$; *the intersection* $D(A)\cap D(B)$ *is dense in* X; B *is a closed operator which commutes with* $V(t)$: $V(t)Bx = BV(t)x\ \forall x \in D(B)$. *Then the following formula holds*

$$\int_{0+}^\infty e^{-\lambda t}Au(t)dt \textit{ exists and } = \lambda B\hat{u}(\lambda) - Bu(0),\ \mathbb{R}e\,\lambda > \omega \tag{2.6}$$

Proof

We start with the equality $Bu'(t) = Au(t)$ on $[0, \infty)$; we know that Bu' and Au are continuous on $]0, \infty[$, and also that u' is continuous on $]0, \infty[$; Using closedness of B we find ($\varepsilon>0$, $N>0$).

$$\int_\varepsilon^N e^{-\lambda t}u'(t)dt \in D(B)$$

and

$$B\int_\varepsilon^N e^{-\lambda t}u'(t)dt = \int_\varepsilon^N e^{-\lambda t}Bu'(t)dt = \int_\varepsilon^N e^{-\lambda t}Au(t)dt.$$

Hence, using also (2.3) above, we obtain:

$$\int_\varepsilon^N e^{-\lambda t}Au(t)dt = B[e^{-\lambda N}u(N) - e^{-\lambda\varepsilon}u(\varepsilon) + \lambda\int_\varepsilon^N e^{-\lambda t}u(t)dt] \tag{2.7}$$

Note also, because u and Bu are continuous, that $\int_\varepsilon^N e^{-\lambda t}u(t)dt \in D(B)$ and

$B\int_\varepsilon^N e^{-\lambda t}u(t)dt = \int_\varepsilon^N e^{-\lambda t}Bu(t)dt$. Therefore, from (2.7) we derive

$$\int_\varepsilon^N e^{-\lambda t}Au(t)dt = e^{-\lambda N}Bu(N) - e^{-\lambda\varepsilon}Bu(\varepsilon) + \lambda\int_\varepsilon^N e^{-\lambda t}Bu(t)dt \tag{2.8}$$

Use now relation $V(t)Bx = BV(t)x$, $\forall x \in D(B)$ and also $u(t) = V(t)u(0)$; it follows that $Bu(t) = V(t)\,Bu(0)$, hence $\|Bu(t)\| \leq e^{t\omega_1}\|(Bu)(0)\|$ if $t \geq R_1$ where

$\omega_1 > \omega$. It follows then that $e^{-\lambda N}Bu(N) \to \theta$, $N \to \infty$ and $\int_\varepsilon^\infty e^{-\lambda t}Bu(t)dt$ is absolutely convergent, if $\mathbb{R}e\,\lambda > \omega$.

Also $Bu'(t) = BV(t)u'(0) = V(t)Bu'(0)$, $Au(t) = V(t)Bu'(0)$ has a similar exponential estimate for large t, and $\int_\varepsilon^\infty e^{-\lambda t}Au(t)dt$ also converges absolutely for $\mathbb{R}e\,\lambda > \omega$.

Hence the relation

$$\int_{\varepsilon}^{\infty} e^{-\lambda t} Au(t)dt = - e^{-\lambda\varepsilon} Bu(\varepsilon) + \lambda \int_{\varepsilon}^{\infty} e^{-\lambda t} Bu(t)\, dt,\ \mathbb{R}e\, \lambda > \omega \tag{2.9}$$

holds true.

Remember (Defin. 1.1) that $Bu \in C([0,\infty);X)$; we obtain ,

$$\int_{0+}^{\infty} e^{-\lambda t} Au(t)dt \text{ exists and } = -Bu(0) + \lambda \int_{0}^{\infty} e^{-\lambda t} Bu(t)dt,\ \mathbb{R}e\, \lambda > \omega \tag{2.10}$$

Now again, from existence of both $\hat{u}(\lambda)$ and $(Bu)^\wedge(\lambda)$ for $\mathbb{R}e\, \lambda > \omega$, we derive that $\hat{u}(\lambda) \in D(B)$ and $B\hat{u}(\lambda) = (Bu)^\wedge(\lambda)$. Finally (2.10) becomes

$$\int_{0+}^{\infty} e^{-\lambda t} Au(t)dt = -Bu(0) + \lambda B\hat{u}(\lambda),\ \mathbb{R}e\, \lambda > \omega.$$

Next, in the special case when operator A is closable (admits a closed linear extension) the above formula (2.6) changes slightly as indicated in next

Proposition 2.3. *Assume conditions in* Prop. 2.2. *and also*: A *is closable. Then* $\hat{u}(\lambda) \in D(\overline{A})$. ($\overline{A}$ being the minimal closed linear extension of A.)

and

the relation

$$(\lambda B-\overline{A})\, \hat{u}(\lambda) = Bu(0),\ \mathbb{R}e\, \lambda > \omega \tag{2.11}$$

holds true

Proof

We have $A \subset \overline{A}$, $e^{-\lambda t} u(t) \in D(\overline{A})$, $\int_{0}^{\infty} e^{-\lambda t} u(t)dt$ exists (for $\mathbb{R}e\, \lambda > \omega$) and also

$\int_{0+}^{\infty} e^{-\lambda t} \overline{A}u(t)dt$ exists (as seen in previous Proposition), Consequently, $\hat{u}(\lambda) \in$

$D(\overline{A})$ and $\overline{A}\,\hat{u}(\lambda) = \int_{0+}^{\infty} e^{-\lambda t}Au(t)dt = \lambda\, B\hat{u}(\lambda) - Bu(0)$ holds, $\mathbb{R}e\,\lambda > \omega$. But this is (2.11).

Corollary 1. *If A is closed,* $\hat{u}(\lambda) \in D(\overline{A})$ *and*

$$(\lambda B\text{-}A)\,\hat{u}(\lambda) = Bu(0) \text{ holds, } \forall\lambda, \mathbb{R}e\,\lambda > \omega \tag{2.12}$$

Corollary 2. *The formula*

$$\int_{0}^{\infty} e^{-\lambda t}\,U(t)x\,dt = (\lambda B\text{-}A)^{-1}\,Bx, \forall x \in D(A) \cap D(B) \tag{2.13}$$

holds true if $\mathbb{R}e\,\lambda > \omega$, A *is closed and the assumptions of Prop. 2.2 are satisfed.*

Proof of Corollary 2. First note that in (2.12) u(0) can be any element $x \in D(A) \cap D(B)$ (this because the C-P. is well-posed). Next, the same formula (2.12) indicates that $Bu(0) \in \mathcal{R}\,(\lambda B\text{-}A)$; therefore, the set $\{Bx, x \in D(A) \cap D(B)\}$ is contained in the range of the operator λB - A; we also know (prop. 1.7) that for $\mathbb{R}e\,\lambda > \omega$, the inverse $(\lambda B\text{-}A)^{-1}$ exists; hence $\{Bx, x \in D(A) \cap D(B)\} \subset D\,((\lambda B\text{-}A)^{-1})$.

From (2.12) we get readily: $\hat{u}(\lambda) = (\lambda B\text{-}A)^{-1}\,Bu(0)$, that is, using also:

$u(t) = U(t)u(0)$, $\int_{0}^{\infty} e^{-\lambda t}U(t)x\,dt = (\lambda B\text{-}A)^{-1}\,Bx.$, $\forall x \in D(A) \cap D(B)$, $\mathbb{R}e\,\lambda > \omega$

which is (2.13).

Remark: If B = Identity, (2.13) becomes (1.8) in [1] - p.30.

3. In this section we elaborate about formula (2.13) under some more assumptions on our operators A and B. We first have.

Lemma 3.1 *Assume: conditions in* Prop. 2.2, *A is closed, and* $\exists\, \lambda_0 \in \mathbb{C}$ *such that* $(\lambda_0 B\text{-}A)^{-1} \in \mathcal{L}(X)$. *Then the range* $\mathcal{R}(\lambda B\text{-}A) = X$, $\forall \lambda$ *such that* $\mathbb{R}e\,\lambda > \omega$.

Proof

Note that, by definition, the domain of the operator $\lambda_0 B\text{-}A$ is $D(A) \cap D(B)$ therefore $(\lambda_0 B\text{-}A)^{-1} x \in D(A) \cap D(B)$ $\forall x \in X$. As seen in "Proof of Corollary 2" the elements $B(\lambda_0 B\text{-}A)^{-1}y$ belong to $\mathcal{R}(\lambda B\text{-}A)$, for $\mathbb{R}e\,\lambda > \omega$, $\forall y \in X$. Let us write now following representation formula, for an arbitrary $x \in X$: $x = (\lambda_0 B\text{-}A)(\lambda_0 B\text{-}A)^{-1}x = (\lambda_0\text{-}\lambda)B\,(\lambda_0 B\text{-}A)^{-1}x + (\lambda B\text{-}A)[(\lambda_0 B\text{-}A)^{-1}x]$. The first term right-hand side is $B(\lambda_0 B\text{-}A)^{-1}\,(\lambda_0\text{-}\lambda)x$, hence $\in \mathcal{R}(\lambda B\text{-}A)$, $\mathbb{R}e\,\lambda > \omega$. The second term right is also (obviously) in $\mathcal{R}(\lambda B\text{-}A)$. Thus, $x \in \mathcal{R}(\lambda B\text{-}A)$ and $\mathcal{R}(\lambda B\text{-}A) = X$, for $\mathbb{R}e\,\lambda > \omega$.

Corollary 3.1 *If* λB - A *is closed for* $\mathbb{R}e\,\lambda > \omega$, $(\lambda B\text{-}A)^{-1} \in \mathcal{L}(X)$.

In fact, $(\lambda B\text{-}A)^{-1}$ exists, is everywhere defined and is closed, hence the result.

We need also a preliminary result of a general nature:

Lemma: *Let* $\lambda, \mu \in \mathbb{C}$, *such that* $(\lambda B\text{-}A)^{-1} \in \mathcal{L}(X)$, $(\mu B\text{-}A)^{-1} \in \mathcal{L}(X)$ (mapping X into $D(A) \cap D(B)$).

Then the equality:

$$(\lambda B\text{-}A)^{-1} - (\mu B\text{-}A)^{-1} = (\mu-\lambda)\,(\lambda B\text{-}A)^{-1}\,B(\mu B\text{-}A)^{-1} \quad \textit{holds true.} \tag{3.1}$$

Proof:

We can write $(\lambda B\text{-}A)^{-1} = (\lambda B\text{-}A)^{-1}\,(\mu B\text{-}A)\,(\mu B\text{-}A)^{-1} = (\lambda B\text{-}A)^{-1}\{(\mu\text{-}\lambda)B + (\lambda B\text{-}A)\}(\mu B\text{-}A)^{-1}\} = (\mu-\lambda)\,(\lambda B\text{-}A)^{-1}\,B(\mu B\text{-}A)^{-1} + (\mu B\text{-}A)^{-1}$.

Remark: If we replace λ with μ, we get

$$(\mu B-A)^{-1} - (\lambda B-A)^{-1} = (\lambda-\mu)(\mu B-A)^{-1}B(\lambda B-A)^{-1}$$

or

$$(\lambda B-A)^{-1} - (\mu B-A)^{-1} = (\mu-\lambda)(\mu B-A)^{-1}B(\lambda B-A)^{-1}$$

It follows that $(\lambda B-A)^{-1} B(\mu B-A)^{-1} = (\mu B-A)^{-1} B(\lambda B-A)^{-1}$ which could be called a B - commutativity.

We now prove the following

Proposition 3.1. *Assume conditions in Lemma 3.1, and also that* $\lambda B - A$ *is closed for* $\mathbb{R}e\,\lambda > \omega$. *Then the formula*

$$(\lambda B-A)^{-1} y = (\lambda_0 B-A)^{-1} y + (\lambda-\lambda_0)\int_0^\infty e^{-\lambda t} V(t)(\lambda_0 B-A)^{-1} y\,dt,$$
$$\forall y \in X,\ \mathbb{R}e\,\lambda > \omega \qquad (3.2)$$

is valid

Proof:

First apply above Lemma, in λ_0 and λ with $\mathbb{R}e\lambda > \omega$. We get accordingly, $\forall y \in X$, the relation

$$(\lambda B-A)^{-1}y = (\lambda_0 B-A)^{-1}y + (\lambda-\lambda_0)(\lambda B-A)^{-1}B(\lambda_0 B-A)^{-1}y. \qquad (3.3)$$

Next, we shall use formula (2.13), taking $x = (\lambda_0 B-A)^{-1}y$. It follows that

$$(\lambda B-A)^{-1}y = (\lambda_0 B-A)^{-1}y + (\lambda-\lambda_0)\int_0^\infty e^{-\lambda t}U(t)(\lambda_0 B-A)^{-1}y\,dt$$
$$= (\lambda_0 B-A)^{-1}y + (\lambda-\lambda_0)\int_0^\infty e^{-\lambda t}V(t)(\lambda_0 B-A)^{-1}y\,dt,\ \mathbb{R}e\,\lambda > \omega.$$

Corollary 3.1 *Under previous assumptions we have also the equality*

$$(\lambda B-A)^{-1}y = (\lambda_0 B-A)^{-1}y + (\lambda-\lambda_0)\int_0^\infty e^{-\lambda t}(\lambda_0 B-A)^{-1} V(t)y\,dt,$$
$$\mathbb{R}e\,\lambda > \omega,\ y \in X \qquad (3.4)$$

In order to prove this Corollary, it suffices to establish commutativity between $V(t)$ and $(\lambda_0 B-A)^{-1}$; we have in fact

$$V(t)(\lambda_0 B-A)x = (\lambda_0 B-A)V(t)x, \ \forall x \in D(A) \cap D(B) \qquad \text{(sec Prop. 1.5)}$$

Take here $x = (\lambda_0 B-A)^{-1}y$, $y \in X$. We obtain

$$V(t)y = (\lambda_0 B-A)\, V(t)(\lambda_0 B-A)^{-1}y, \ \forall y \in X$$

and then

$$(\lambda_0 B-A)^{-1}\, V(t)\, y = V(t)(\lambda_0 B-A)^{-1}y, \ \forall y \in X.$$

Our final result is given as

Proposition 3.2 *Assume conditions in* Prop. 3.1, *and take* $y \in X$ *such that* $V(t)y$ *is Bochner integrable on* $[0, T]$. *Then we have the equality*

$$B(\lambda B-A)^{-1}y = \int_0^\infty e^{-\lambda t}V(t)y\,dt, \ \mathbb{R}e\, \lambda > \omega \tag{3.5}$$

Remark For $B = I$ this is the classical formula expressing the resolvent $(\lambda-A)^{-1}$ as Laplace transform of the associated semigroup (see [1] - formula 1.10).

Proof. For the special choice of y we have that $\int_0^\infty e^{-\lambda t}V(t)y\, dt$, exists (absolutely) when $\mathbb{R}e\, \lambda > \omega$. Then (3.4) becomes

$$(\lambda B-A)^{-1}\, y = (\lambda_0 B-A)^{-1}y + (\lambda-\lambda_0)(\lambda_0 B-A)^{-1}\int_0^\infty e^{-\lambda t}V(t)y\, dt \tag{3.6}$$

Use also (3.3) and get

$$(\lambda B-A)^{-1}y = (\lambda_0 B-A)^{-1}y + (\lambda-\lambda_0)(\lambda_0 B-A)^{-1}B(\lambda B-A)^{-1}y \tag{3.7}$$

Taking difference between (3.6) and (3.7) we derive

$$\theta = (\lambda-\lambda_0)(\lambda_0 B-A)^{-1}\Big[B(\lambda B-A)^{-1}y - \int_0^\infty e^{-\lambda t}V(t)y\, dt\Big], \ \mathbb{R}e\, \lambda > \omega$$

Apply $(\lambda_0 B-A)$ in both sides (for $\lambda \neq \lambda_0$) and obtain

$$\theta = (\lambda-\lambda_0)[B(\lambda B-A)^{-1}y - \int_0^\infty e^{-\lambda t}V(t)y\,dt] \tag{3.8}$$

which obviously implies (3.5).

References

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