

AN AVERAGE PER CAPITA FORMULA FOR THE SHAPLEY VALUE

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A new formula for the Shapley value is given which does not require the storage of the $2^n - 1$ values of the characteristic function in the computer, and avoids the search in the memory for such data.

1. In computing the Shapley value of a cooperative TU game, the $2^n - 1$ values of the characteristic function should be introduced in the computing tool anyway. Therefore, if these data should be kept and retrieved, the computation may become impossible for large n . We give a formula for the Shapley value which is allowing the computation of the value by manipulating two $n+1$ -dimensional vectors; each value of the characteristic function is given as an $n+1$ -dimensional vector which is immediately used and deleted. In this way, the difficulty of computing the Shapley value reduces to the correct typing of data. We give a simple combinatorial proof in the next section, then a more elegant one using the Hart/Mas-Colell potentials and an example. The list of references contains papers devoted to the applications ([1] and [5]) and papers in which the interpretations of the Shapley value as a sound fair allocation are emphasized ([4] and [7]); the computational aspect is discussed mainly in two papers ([2] and [6]).

2. The Shapley value introduced in [8], for a cooperative TU game with the set of players N and the characteristic function v , is given by the well-known formula

$$(1) \quad \Phi_i(v) = \sum_{S/i \in S} \frac{(s-1)!(n-s)!}{n!} [v(S) - v(S - \{i\})], \quad \forall i \in N,$$

where $s = |S|$. Some notations are needed: for $k = 1, \dots, n-1$ denote

$$(2) \quad v_k = \binom{n}{k}^{-1} \cdot \sum_{S/|S|=k} v(S), \quad v_k^i = \binom{n-1}{k}^{-1} \cdot \sum_{S/i \notin S, |S|=k} v(S), \quad \forall i \in N.$$

In words, v_k is the average worth of coalitions of size k and v_k^i is the average worth of coalitions of size k which do not contain the player i ; it is natural to denote $v_n = v(N)$ and convenient to take $v_n^i = 0$.

Proposition 1: The Shapley value of a cooperative TU game v with the set of players N is

$$(3) \quad \Phi_i(v) = \sum_{k=1}^{k=n} k^{-1} (v_k - v_k^i), \quad \forall i \in N.$$

Proof: We rearrange the terms in (1); therefore, we consider those terms in (1) corresponding to coalitions T of size k , where k is fixed $k \in N - \{n\}$. Any coalition T containing i will give a term $v(T)$ with the same coefficient $\frac{(k-1)!(n-k)!}{n!}$ equal to $k^{-1} \cdot \binom{n}{k}^{-1}$, so that the part of the sum corresponding to such coalitions of size k is $k^{-1} \cdot \binom{n}{k}^{-1} \cdot \sum_{T/i \in T, |T|=k} v(T)$. The terms collected in this sum are provided by the first numbers in the brackets. Any coalition T which does not contain i will give a term $v(T)$ with the same coefficient $-\frac{k!(n-k-1)!}{n!}$, because this term corresponds to $S = T \cup \{i\}$ which has the size $k+1$. Obviously, these coefficients equal to $-(n-k)^{-1} \cdot \binom{n}{k}^{-1}$, so that the part of the sum corresponding to such coalitions of size k is $-(n-k)^{-1} \cdot \binom{n}{k}^{-1} \cdot \sum_{T/i \notin T, |T|=k} v(T)$. The terms collected in this sum are provided by the second numbers in the brackets. So, all terms in (1) for coalitions of size k give the sum

$$(4) \quad k^{-1} \cdot \binom{n}{k}^{-1} \cdot \sum_{T/i \in T, |T|=k} v(T) - (n-k)^{-1} \cdot \binom{n}{k}^{-1} \cdot \sum_{T/i \notin T, |T|=k} v(T).$$

Now, the sum of worth of coalitions of size k which contain i equals the difference between the sum of worth of all coalitions of size k and the sum of worth of coalitions which do not contain i . If this change is made in (4), we get

$$(5) \quad k^{-1} \cdot \binom{n}{k}^{-1} \cdot \sum_{T/|T|=k} v(T) - nk^{-1}(n-k)^{-1} \cdot \binom{n}{k}^{-1} \cdot \sum_{T/i \notin T, |T|=k} v(T).$$

Now, the coefficient of the second sum equals $-k^{-1} \cdot \binom{n-1}{k}^{-1}$ and we got from (5) exactly $k^{-1}(v_k - v_k^i)$. Obviously, for $k=n$ we have in (1) the unique term $n^{-1}v(N)$, which can be written as $n^{-1}(v_n - v_n^i)$, hence we proved (3).

Note that by the same kind of reasoning (3) can be obtained from the Kleinberg/Weiss formula given in [4] or from a Rothblum formula given in [7]. A more elegant proof can be done by using the Hart/Mas-Colell potentials introduced in [3].

3. The potential, as introduced in [3], is a function P defined on the space of cooperative TU games with the set of players N , which associates a real number $P(S, v)$ with each subgame of a game v defined by a coalition $S \subseteq N$. It is shown in [3], that for a game v the potential function can be computed by the recursion formula

$$(6) \quad P(S, v) = (|S|)^{-1} [v(S) + \sum_{i \in S} P(S - \{i\}, v)], \quad \forall S \subseteq N,$$

where by definition $P(\emptyset, v) = 0$. It is also shown that the Shapley value $\Phi(v)$ is given in terms of potentials by

$$(7) \quad \Phi_i(v) = P(N) - P(N - \{i\}, v), \quad \forall i \in N.$$

However, a game usually being given by the function v , the computation of the Shapley value by formula (7) is as difficult as the computation by formula (1). The potential function will allow us to give another proof of (3); some notations are needed: for $k=1, \dots, n-1$ denote

$$(8) \quad p_k = \binom{n}{k}^{-1} \cdot \sum_{S/|S|=k} P(S, v), \quad p_k^i = \binom{n-1}{k}^{-1} \cdot \sum_{S/i \notin S, |S|=k} P(S, v), \quad \forall i \in N.$$

In words, p_k is the average potential associated with coalitions of size k and p_k^i is the average potential associated with coalitions of size k which do not contain the player i ; it is natural to denote $p_n = P(N, v)$.

Proposition 2: The potential averages and the worth averages are related by formula

$$(9) \quad p_k = k^{-1} \cdot v_k + p_{k-1}, \quad k = 2, \dots, n.$$

Proof: Sum up the recursion formulas (6) for all coalitions of size k , $2 \leq k \leq n$, to get

$$(10) \quad \sum_{S/|S|=k} P(S, v) = k^{-1} \cdot \sum_{S/|S|=k} v(S) + k^{-1} \cdot \sum_{S/|S|=k} \left[\sum_{j \in S} P(S - \{j\}, v) \right].$$

Note that a coalition $S - \{j\}$, which has the size $k-1$ if $|S| = k$, is a subset of $n-k+1$ coalitions of size k , those obtained by adding to $S - \{j\}$ each of the remaining players; therefore, the double sum equals $(n-k+1) \cdot \sum_{S/|S|=k-1} P(S, v)$. By dividing to $\binom{n}{k}$ and using the equality

$$(n-k+1)/k \cdot \binom{n}{k} = \binom{n}{k-1}^{-1}, \text{ we get (9).}$$

Proposition 3: For any $i \in N$, the averages of potentials p_k^i and the averages of worth v_k^i are related by formula

$$(11) \quad p_k^i = k^{-1} \cdot v_k^i + p_{k-1}^i, \quad k = 2, \dots, n-1.$$

Proof: Sum up the recursion formulas (6) for all coalitions of size k , $2 \leq k \leq n-1$, which do not contain i , to get

$$(12) \quad \sum_{\substack{S/i \notin S \\ |S|=k}} P(S, v) = k^{-1} \cdot \sum_{\substack{S/i \notin S \\ |S|=k}} v(S) + k^{-1} \cdot \sum_{\substack{S/i \notin S \\ |S|=k}} \left[\sum_{j \in S} P(S - \{j\}, v) \right].$$

Note that a coalition $S - \{j\}$ which does not contain i and has the size $k-1$ if

$|S| = k$, is a subset of $n - k$ coalitions of size k which do not contain i , those obtained by adding to $S - \{j\}$ each of the remaining players different of i ; therefore, the double sum equals $(n - k) \cdot \sum_{S/i \notin S, |S| = k-1} P(S, v)$. By dividing to $\binom{n-1}{k}$ and using the equality $(n - k)/k \binom{n-1}{k} = \binom{n-1}{k-1}^{-1}$, we get (11).

Proof of Proposition 1: For any $i \in N$, sum up all equalities (9) given by Proposition 2 for $k = 2, \dots, n-1$, and subtract the sum of all equalities (11) given by Proposition 3 for $k = 2, \dots, n-1$; after simplification, we have

$$(13) \quad p_{n-1} - p_{n-1}^i = \sum_{k=2}^{k=n-1} k^{-1}(v_k - v_k^i) + p_1 - p_1^i, \quad \forall i \in N.$$

The last two terms can be included in the sum to make it from $k = 1$ to $k = n-1$, because $P(\{i\}, v) = v(\{i\})$, $\forall i \in N$. By using the Hart/Mas-Colell result (7), then the recursion formula (6) for $S = N$ and formula (13), we get

$$(14) \quad \begin{aligned} \Phi_i(v) &= P(N, v) - P(N - \{i\}, v) = n^{-1} \cdot v(N) + n^{-1} \cdot \sum_{S/|S|=n-1} P(S, v) + \\ &- P(N - \{i\}, v) = n^{-1}(v_n - v_n^i) + p_{n-1} - p_{n-1}^i = \sum_{k=1}^{k=n} k^{-1}(v_k - v_k^i), \quad \forall i \in N, \end{aligned}$$

hence (3) is proved.

4. Discussion.

a) The algorithm. To compute the Shapley value Φ for a game v with the set of players N by using formula (3), notice first that each term of the sum is independent of the others, so that the computation can be done in parallel. If a sequential approach is used, then the computation will be done in n cycles of steps, each cycle k computing the term $k^{-1}(v_k - v_k^i)$ for all $i \in N$. To outline the operations of a step h in cycle k denote

$$(15) \quad w_k = \sum_{S/|S|=k} v(S), \quad w_k^i = \sum_{S/i \notin S, |S|=k} v(S), \quad a_k = k \binom{n}{k}, \quad b_k = k \binom{n-1}{k},$$

so that our formula (3) for the Shapley value becomes

$$(16) \quad \Phi_i(v) = \sum_{k=1}^{k=n} \left(\frac{w_k}{a_k} - \frac{w_k^i}{b_k} \right), \quad \forall i \in N.$$

Note that $a_1 = n$, $b_k = [1 - (k/n)]a_k$, and $a_{k+1} = [(n/k) - 1]a_k$, for $k = 1, \dots, n-1$, a fact which helps in updating a_k and b_k ; so, the numbers w_k and w_k^i , $\forall i \in N$, are the ones to be computed from the data.

Before starting the cycle k we have the current n -vector Φ^k , which contains the partial sums of $k-1$ terms of the sum (16) and the number a_k ; obviously, $\Phi^1 = 0$ and $a_1 = n$. Before starting the step h of cycle k we have the current $n+1$ -vector $(w_k; w_k^1, \dots, w_k^n)$ which contains the partial sums of $h-1$ terms of the sums (15); this is a zero vector for $h=1$. The step h will consider a new coalition S with $|S| = k$ and $v(S) \neq 0$, if any; the following operations will be needed:

(A) Computing the sums: Choose a coalition S with $|S| = k$ and $v(S) \neq 0$; if none can be found, go to (B), otherwise an $n+1$ -vector $(v(S); s_1, \dots, s_n)$, where (s_1, \dots, s_n) is the characteristic vector of S , will be typed. The computer will do: $w_k := w_k + v(S)$ and for each $i = 1, \dots, n$, if $s_i = 0$, then $w_k^i := w_k^i + v(S)$. Return to (A) for the next step.

(B) Computing one term: Compute $w_k := w_k / a_k$, $b_k := [1 - (k/n)]a_k$, and for each $i = 1, \dots, n$, find $w_k^i := w_k^i / b_k$, $w_k^i := w_k - w_k^i$, go to (C).

(C) Updating: For each $i = 1, \dots, n$, compute $\Phi_i^k := \Phi_i^k + w_k^i$, and check whether $k < n$. If yes, find $a_k := [(n/k) - 1]b_k$, make $k := k + 1$, $w_k := 0$, $w_k^i := 0$, $\forall i \in N$, and go to the next cycle; if no, stop, Φ is the Shapley value.

b) Example: Consider the 3-person game $v(1) = 100$, $v(2) = 200$, $v(3) = 300$, $v(12) = 400$, $v(13) = 500$, $v(23) = 600$, and $v(123) = 900$. The data are given in Table 1:

h	1	2	3	1	2	3	1
k	1	1	1	2	2	2	3
$v(S)$	100	200	300	400	500	600	900
s_1	1	0	0	1	1	0	1
s_2	0	1	0	1	0	1	1
s_3	0	0	1	0	1	1	1

Table 1: On each column the step number, the cycle number and the vector of data in that step are given.

The computation of the Shapley value is shown in Table 2:

h	1	2	3	$a_1=3, b_1=2$			1	2	3	$a_2=6, b_2=2$			1	$a_3=3, b_3=arb.$		
k	1	1	1				2	2	2				3			
op.	A	A	A	B	B	C	A	A	A	B	B	C	A	B	B	C
w_k	100	300	600	200	---	---	400	900	1500	250	---	---	900	300	---	---
w_k^1	0	200	700	250	-50	-50	0	0	600	300	-50	-100	0	0	300	200
w_k^2	100	100	400	200	0	0	0	500	500	250	0	0	0	0	300	300
w_k^3	100	300	300	150	50	50	400	400	400	200	50	100	0	0	300	400

Table 2: On each column the step number, the cycle number, the operation of that step and the current vector $(w_k; w_k^1, \dots, w_k^n)$ or $(-; w_k - w_k^1, \dots, w_k - w_k^n)$, or $(-; \Phi_1^k, \dots, \Phi_n^k)$ are given.

The Shapley value of the above game is $\Phi_1 = 200$, $\Phi_2 = 300$, and $\Phi_3 = 400$.

c) Other Algorithms: As [6] is presenting an algorithm which requires the presence of all the values of the characteristic function in the computer, the difficulties arising in the use of the algorithm are the same as those occurring in applying formula (1). The method proposed in [2] is based on a new formula for the Shapley value which is allowing, like in our method based on formula (3), the use of each value of the characteristic function only once and independently of the other values. In our notations, the Gambarelli/Zambruno formula is:

$$(17) \quad \Phi_i(v) = v(\{i\}) + \sum_{k=2}^{k=n} k^{-1} \cdot \binom{n}{k}^{-1} \cdot \sum_{S/|S|=k} \left[v(S) - \sum_{j \in S} v(\{j\}) \right] +$$

$$- \sum_{k=2}^{k=n-1} k^{-1} \cdot \binom{n-1}{k}^{-1} \cdot \sum_{S/i \notin S, |S|=k} \left[v(S) - \sum_{j \in S} v(\{j\}) \right], \quad \forall i \in N.$$

Notice that this formula is approaching in a different way the worth of the singletons, namely from each of the $2^n - n - 1$ numbers $v(S)$ with $|S| > 1$ we should subtract $|S|$ numbers and this again makes the formula inefficient for large n . Therefore, formula (17) is usable only for zero-normalized games when (17) reduces to formula (3).

References

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