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SINGULAR BOUNDARY VALUE PROBLEMS ON THE SEMI-INFINITE INTERVAL

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Abstract

This paper examines the existence of solutions to the nonlinear boundary value problem $\frac{1}{p}(py')' = \frac{f(t,y)}{\psi}$, $0 < t < \infty$, $y(0) = a$, $y(t)$ bounded on $[0, \infty)$. In addition we deduce possible values for $\lim_{t\to\infty} y(t)$ if such a limit exists.

1 INTRODUCTION

This paper establishes the existence of a bounded solution to singular and nonsingular second order differential equations on the semi-infinite interval of the form

$$\frac{1}{p(t)}(p(t)y'(t))' = \frac{f(t,y(t))}{\psi(t)},\ 0 < t < \infty$$

with the solution satisfying a specified boundary condition at $t = 0$ and at infinity. Here the function $f : [0, \infty) \times \Re \to \Re$, $p : [0, \infty) \to [0, \infty)$ and $\psi : [0, \infty) \to [0, \infty)$ are continuous. The theory on boundary value problems on the semi-infinite interval to date is not very well developed. Most of the results concern the nonsingular second order problem ($p \equiv 1 \equiv \psi$) ; see [1,2,3,6] for discussion and references.

Our discussion of existence will be in two parts. Section 2 will specifically discuss the singular nonlinear boundary value problem

$$\begin{cases} \frac{1}{p}(py')' = \frac{f(t,y)}{\psi},\ 0 < t < \infty, \\ y(0) = a,\ y(t) \text{ bounded on } [0, \infty). \end{cases} \tag{1.1}$$

By placing a monotonicity condition on f and integrability conditions on p and ψ we will show that (1.1) has a bounded solution $y \in C[0, \infty) \cap C^2(0, \infty)$. We remark here as well that the linear problem

$$\begin{cases} y'' = y - M,\ 0 < t < \infty \\ y(0) = a \end{cases}$$

with M and a constants, has exactly one bounded solution $y(t) = (a - M)e^{-t} + M$. For this solution we have $\lim_{t\to\infty} y(t) = M$. In section 3 we will use results of section 2 to deduce possible values for $\lim_{t\to\infty} y(t)$ if such a limit exists. The results reported in this section were motivated by the nonlinear singular Thomas-Fermi equation [5],

$$y'' = t^{\frac{-1}{2}} y^{\frac{3}{2}},\ 0 < t < \infty$$

subject to the boundary condition corresponding to the isolated neutral atom

$$y(0) = 1,\ \lim_{t\to\infty} y(t) = 0.$$

Finally in this introduction for notational purposes let $BC^2[0, \infty)$ denote the space of bounded, continuous functions u on $[0, \infty)$ with u'' continuous on $(0, \infty)$.

2 GLOBAL SOLVABILITY

This section examines the semi-infinite interval boundary value problem

$$\begin{cases} \frac{1}{p}(py')' = \frac{f(t,y)}{w}, \ 0 < t < \infty, \\ y(0) = a, \ y(t) \text{ bounded on } [0,\infty) \end{cases} \tag{2.1}$$

with $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$, $p : [0,\infty) \to [0,\infty)$ and $w : [0,\infty) \to [0,\infty)$ continuous. By a solution to (2.1) we mean a function $y \in BC^2[0,\infty)$ with y satisfying the differential equation on $(0,\infty)$ together with $y(0) = a$. To show that (2.1) has a solution we begin by examining the family of two point boundary value problems

$$\begin{cases} \frac{1}{p}(py')' = \frac{f(t,y)}{w}, \ 0 < t < n, \\ y(0) = a, \ y(n) = 0 \end{cases} \tag{2.2^n}$$

for each $n \in N^+ = \{1, 2, \ldots\}$. We remark here that by a solution to (2.2^n), n fixed, we mean a function $y \in C[0,n] \cap C^2(0,n]$ which satisfies the differential equation on $(0,n]$ and the above stated boundary conditions.

The following existence principle is needed; see [7] for a proof.

Theorem 2.1 *Fix $n \in N^+$ and suppose $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous. Assume*

$$\begin{cases} p \in C[0,\infty) \cap C^1(0,\infty), \ w \in C[0,\infty) \text{ together with} \\ p(t) > 0 \text{ and } w(t) > 0 \text{ on } (0,\infty) \end{cases} \tag{2.3}$$

and

$$\int_0^b \frac{ds}{p(s)} < \infty \text{ and } \int_0^b \frac{1}{p(u)} \int_u^b \frac{p(s)}{w(s)} dsdu < \infty \text{ for any } b > 0. \tag{2.4}$$

Now suppose there is a constant J, independent of λ, such that $\sup_{[0,n]} |y(t)| \le J$ for each solution y to

$$\begin{cases} \frac{1}{p}(py')' = \lambda\frac{f(t,y)}{w}, \ 0 < t < n, \\ y(0) = a, \ y(n) = 0 \end{cases} \tag{2.5$^n_\lambda$}$$

for each $\lambda \in (0,1)$. Then (2.2^n) has at least one solution in $C[0,n] \cap C^2(0,n]$.

This theorem immediately produces an existence result for singular boundary value problems on the finite interval.

Theorem 2.2 *Suppose $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous with conditions (2.3) and (2.4) being satisfied. In addition*

$$\begin{cases} \text{suppose there is a constant } M \ge 0 \text{ such that } yf(t,y) > 0 \\ \text{for } |y| > M \text{ and for all } t \in [0,\infty). \end{cases} \tag{2.6}$$

Then (2.2^n), for each n, is at least one solution $y_n \in C[0,n] \cap C^2(0,n]$ with $|y_n(t)| \le \max\{M, |a|\} \equiv M_0$ for $t \in [0,n]$.

If in addition

$$p \text{ is nondecreasing on } (0,\infty) \text{ and } \frac{p^2}{w} \text{ is nonincreasing on } (0,\infty) \tag{2.7}$$

$$\int_0^b \frac{p(s)}{\psi(s)}\,ds < \infty \text{ for any } b > 0 \tag{2.8}$$

$$\begin{cases} \text{for any } n \in N^+, \text{ there exists a constant } Q \\ \text{independent of } n \text{ with } \frac{p(n)}{n} \le Q \end{cases} \tag{2.9}$$

and

$$\text{for } 0 \le t < \infty \text{ and } u \text{ in a bounded interval } f(t,u) \text{ is bounded.} \tag{2.10}$$

Then there exist constants M_1, M_2 and M_3 independent of n with

$$|y_n'(t)| \le \frac{M_1}{p(t)} + \frac{M_2}{(\psi(t))^{\frac{1}{2}}} + \frac{M_3}{p(t)}\int_0^t \frac{p(s)}{\psi(s)}\,ds + |a| \tag{2.11}$$

for $t \in (0,n)$.

PROOF: Fix $n \in N^+$. To show that (2.2^n) has at least one solution we will apply Theorem 2.1. Let y be a solution to (2.5^n_λ). We claim $|y(t)| \le M_0$ for $t \in [0,n]$. If $|y|$ achieves a maximum at 0 or n then the claim is trivially true. Suppose now that $|y|$ achieves a positive maximum at $t_0 \in (0,n)$ with $|y(t_0)| > M$. Then $y'(t_0) = 0$ and $y(t_0)y''(t_0) \le 0$. However the differential equation yields

$$p(t_0)y''(t_0) + p'(t_0)y'(t_0) = \frac{\lambda p(t_0) f(t_0, y(t_0))}{\psi(t_0)}$$

and this together with (2.6) yields $p(t_0)y(t_0)y''(t_0) > 0$, a contradiction. Thus $|y(t_0)| \le M_0$ and our claim is established. Existence of a solution y_n to (2.2^n) follows immediately from Theorem 2.1.

Also since $y_n(0) = a$ and $y_n(n) = 0$ there exists $z \in (0,n)$ with $y_n'(z) = \frac{-a}{n}$. Now for each $t \in (0,n)$ with $|y_n'(t)| > \frac{|a|}{n}$ there exists $(\mu,\nu) \subset (0,n)$ with $t \in (\mu,\nu)$ and y_n' has a fixed sign on (μ,ν) with $|y_n'(\mu)|$ and/or $|y_n'(\nu)|$ equal to $\frac{|a|}{n}$. There are four cases to consider

CASE(a). Suppose $y_n' \ge 0$ on (μ,ν) and $y_n'(\nu) = \frac{a}{n}$.

Then with $K = \sup |f(t,u)|$, where the supremum is taken over $[0,\infty) \times [-M_0, M_0]$, we have $-(py_n')' \le \frac{pK}{\psi}$ since $|(py_n')'| = |\frac{pf(t,y_n)}{\psi}|$. Multiplying by py_n' yields $-py_n'(py_n')' \le \frac{p^2K(y_n')}{\psi}$ so integrate from t to ν to obtain

$$(p(t)y_n'(t))^2 \le \frac{p^2(\nu)a^2}{n^2} + \frac{4p^2(t)KM_0}{\psi(t)} \le \frac{p^2(n)a^2}{n^2} + \frac{4p^2(t)KM_0}{\psi(t)} \le Q^2a^2 + \frac{4p^2(t)KM_0}{\psi(t)}$$

using assumptions (2.7) and (2.9). Consequently

$$|y_n'(t)| \le \frac{Q|a|}{p(t)} + 2\left(\frac{KM_0}{\psi(t)}\right)^{\frac{1}{2}}. \tag{2.12}$$

CASE(b). Suppose $y_n' \ge 0$ on (μ,ν) and $y_n'(\mu) = \frac{a}{n}$.

Now $(py_n')' \le \frac{Kp}{\psi}$ so integration from μ to t yields

$$p(t)y_n'(t) \le \frac{p(n)a}{n} + \frac{K}{p(t)}\int_\mu^t \frac{p(s)}{\psi(s)}\,ds \le Q|a| + \frac{K}{p(t)}\int_0^t \frac{p(s)}{\psi(s)}\,ds.$$

Consequently

$$|y_n'(t)| \le \frac{Q|a|}{p(t)} + \frac{K}{\psi(t)}\int_0^t \frac{p(s)}{\psi(s)}\,ds. \tag{2.13}$$

CASE(c). Suppose $y_n' \leq 0$ on (μ, ν) and $y_n'(\nu) = \frac{-a}{n}$.

Then $(py_n')' \leq \frac{pK}{\psi}$ so multiplying by $-py_n'$ and integrating from t to ν yields (2.12).

CASE(d). Suppose $y_n' \leq 0$ on (μ, ν) and $y_n'(\mu) = \frac{-a}{n}$.

Then $(py_n')' \leq \frac{pK}{\psi}$ so integrating from μ to t yields (2.13).

Consequently (2.12) and (2.13) will yield (2.11).□

The above results together with the *Arzela-Ascoli* theorem will now imply the solvability of the boundary value problem (2.1).

Theorem 2.3 *Let $f : [0, \infty) \times \Re \to \Re$ be continuous with conditions (2.3), (2.4), (2.6), (2.7), (2.8), (2.9) and (2.10) satisfied. In addition assume*

$$\int_0^b \frac{ds}{(\psi(s))^{\frac{1}{2}}} < \infty \text{ for any } b > 0. \tag{2.14}$$

Then (2.1) *has at least one solution $y \in BC^2[0, \infty)$ with $|y(t)| \leq \max\{M, |a|\} \equiv M_0$ for $t \in [0, \infty)$.*

PROOF: By Theorem 2.2 there exists a solution $y_n \in C[0, n] \cap C^2(0, n]$ to (2.2^n) with $|y_n(t)| \leq M_0$ for $t \in [0, n]$. In addition there exist constants M_1, M_2 and M_3 independent of n with

$$|y_n'(t)| \leq \frac{M_1}{p(t)} + \frac{M_2}{(\psi(t))^{\frac{1}{2}}} + \frac{M_3}{p(t)} \int_0^t \frac{p(s)}{\psi(s)} ds + |a|$$

for $t \in (0, n)$. Consequently for $t, s \in [0, n]$ we have

$$|y_n(t) - y_n(s)| \leq M_1 \left| \int_s^t \frac{dz}{p(z)} \right| + M_2 \left| \int_s^t \frac{dz}{(\psi(z))^{\frac{1}{2}}} \right| + M_3 \left| \int_s^t \frac{1}{p(z)} \int_0^z \frac{p(u)}{\psi(u)} du\, dz \right| + |a||t - s|.$$

Now define functions u_n on $[0, \infty)$ by $u_n(x) = y_n(x),\ x \in [0, n]$ and $u_n(x) = 0,\ x > n$. Then u_n is continuous on $[0, \infty)$. In addition $|u_n(x)| \leq M_0,\ x \in [0, \infty)$ and for $t, s \in [0, \infty)$ we have

$$|u_n(t) - u_n(s)| \leq M_1 \left| \int_s^t \frac{dz}{p(z)} \right| + M_2 \left| \int_s^t \frac{dz}{(\psi(z))^{\frac{1}{2}}} \right| + M_3 \left| \int_s^t \frac{1}{p(z)} \int_0^z \frac{p(u)}{\psi(u)} du\, dz \right| + |a||t - s|.$$

Let $S = \{u_n\}_1^\infty$. By the *Arzela-Ascoli* theorem there is a subsequence N_1 of N^+ and a continuous function z_1 on $[0, 1]$ such that $u_n(x) \to z_1(x)$ uniformly on $[0, 1]$ as $n \to \infty$ through N_1. Applying the *Arzela-Ascoli* theorem again guarantees the existence of a subsequence N_2 of N_1 and a continuous function z_2 on $[0, 2]$ such that $u_n(x) \to z_2(x)$ uniformly on $[0, 2]$ as $n \to \infty$ through N_2. Note $z_2 = z_1$ on $[0, 1]$ since $N_2 \subset N_1$. Proceed inductively to obtain for $k = 1, 2, \ldots$ a subsequence $N_k \subset N^+$ with $N_k \subset N_{k-1}$ and a continuous function z_k on $[0, k]$ such that $u_n(x) \to z_k(x)$ uniformly on $[0, k]$ as $n \to \infty$ through N_k. Note since $N_k \subset N_{k-1}$, $z_k = z_{k-1}$ on $[0, k-1]$.

Now we define a function y as follows. Fix $x \in [0, \infty)$ and let $k \in N^+$ with $x \leq k$. Define $y(x) = z_k(x)$. Then y is well defined with $y \in C[0, \infty)$. Now fix x and choose and fix $k \geq x$, $k \in N^+$. Then

$$\begin{aligned} u_n(x) &= a + \left\{ \int_0^k \frac{1}{p(v)} \int_v^k \frac{p(s) f(s, u_n(s))}{\psi(s)} ds\, dv - a \right\} \left(\int_0^k \frac{ds}{p(s)} \right)^{-1} \int_0^x \frac{ds}{p(s)} \\ &- \int_0^x \frac{1}{p(v)} \int_v^k \frac{p(s) f(s, u_n(s))}{\psi(s)} ds\, dv \end{aligned}$$

for $n \in N_k$. Since u_n converges uniformly to z_k on $[0,k]$, let $n \to \infty$ through N_k to obtain

$$\begin{aligned} z_k(x) &= a + \{\int_0^k \frac{1}{p(v)} \int_v^k \frac{p(s)f(s,z_k(s))}{\psi(s)}\, ds\, dv - a\}(\int_0^k \frac{ds}{p(s)})^{-1} \int_0^x \frac{ds}{p(s)} \\ &- \int_0^x \frac{1}{p(v)} \int_v^k \frac{p(s)f(s,z_k(s))}{\psi(s)}\, ds\, dv. \end{aligned}$$

Thus

$$\begin{aligned} y(x) &= a + \{\int_0^k \frac{1}{p(v)} \int_v^k \frac{p(s)f(s,y(s))}{\psi(s)}\, ds\, dv - a\}(\int_0^k \frac{ds}{p(s)})^{-1} \int_0^x \frac{ds}{p(s)} \\ &- \int_0^x \frac{1}{p(v)} \int_v^k \frac{p(s)f(s,y(s))}{\psi(s)}\, ds\, dv \end{aligned}$$

and so $(p(x)y'(x))' = \frac{p(x)f(x,y(x))}{\psi(x)}$. Consequently $y \in C^2(0,\infty)$ with $\frac{1}{p}(py')' = \frac{f(t,y)}{\psi}$, $0 < t < \infty$ and in addition $y(0) = a$ with $|y(t)| \le M_0$, $0 \le t < \infty$.□

In many situations, for example the Thomas-Fermi and generalised Emden-Fowler equations, we can improve the results of Theorem 2.3 by replacing (2.6) with a more restrictive condition.

Theorem 2.4 *Suppose $f : [0,\infty) \times \Re \to \Re$ is continuous with conditions (2.3) and (2.4) being satisfied. In addition*

$$\textit{suppose } yf(t,y) > 0 \textit{ for all } t \in [0,\infty) \textit{ and } |y| > 0. \tag{2.15}$$

Then (2.2^n), with $a > 0$, has at least one solution $y_n \in C[0,n] \cap C^2(0,n]$ with $0 \le y_n(t) \le a$ for $t \in [0,n]$.

Moreover if conditions (2.7), (2.9) and (2.10) hold, then there exist constants M_1 and M_2 independent of n with for $t \in (0,n)$,

$$-y_n'(t) = |y_n'(t)| \le \frac{M_1}{p(t)} + \frac{M_2}{(\psi(t))^{\frac{1}{2}}} + a.$$

PROOF: Now (2.15) implies (2.6) is satisfied with $M = 0$. Consequently (2.2^n) with $a > 0$, has at least one solution $y_n \in C[0,n] \cap C^2(0,n]$ with $|y_n(t)| \le a$ for $t \in [0,n]$. In fact $y_n \ge 0$ on $[0,n]$, for if not then y_n would assume a negative minimum on $(0,n)$, say at z. Then $y_n'(z) = 0$ and $y_n''(z) \ge 0$. However the differential equation yields

$$p(z)y_n''(z)y_n(z) = [p(z)y_n''(z) + p'(z)y_n'(z)]y_n(z) = \frac{y_n(z)p(z)f(z,y_n(z))}{\psi(z)} > 0,$$

a contradiction. Consequently $y_n \ge 0$ on $[0,n]$.

Also we claim $p(t)y_n'(t) \le 0$ for $t \in (0,n)$. To see this notice (2.15) and the differential equation together with the fact that $y_n \ge 0$ on $[0,n]$ implies $(p(t)y_n'(t))' \ge 0$ on $(0,n)$ i.e $p(t)y_n'(t)$ is nondecreasing on $(0,n)$. Now if there exists a $t_0 \in (0,n)$ with $p(t_0)y_n'(t_0) > 0$ then $p(t)y_n'(t) > 0$ for $t \ge t_0$ which contradicts the fact that $y_n(n) = 0$. Consequently $p(t)y_n'(t) \le 0$ for $t \in (0,n)$. Now since $y_n(0) = a$ and $y_n(n) = 0$ there exists a $z \in (0,n)$ with $y_n'(z) = \frac{-a}{n}$. Since $(py_n')' \ge 0$ we have $p(t)y_n'(t) \ge p(z)y_n'(z)$ for $t \ge z$ and so $-y_n'(t) \le \frac{p(z)a}{p(t)n} \le a$, using assumption (2.7). Thus

$$0 \le -y_n'(t) \le a, \; t \in [z,n]. \tag{2.16}$$

Now for each point $t \in (0,z)$ with $-y_n'(t) > \frac{a}{n}$ there exists $(\mu,\nu) \subset (0,z)$ with $t \in (\mu,\nu)$ and $y_n' \le 0$ on (μ,ν), $y_n'(\nu) = \frac{-a}{n}$. Let $K = \sup|f(t,u)|$ where the supremum is computed over $[0,\infty) \times [0,a]$. Then $(py_n')' \le \frac{pK}{w}$ and so $-py_n'(py_n')' \le \frac{p^2K(-y_n')}{w}$. Integration from t to ν now yields

$$(p(t)y_n'(t))^2 \le \frac{p^2(\nu)a^2}{n^2} + \frac{4p^2(t)Ka}{w(t)}$$

and so

$$-y_n'(t) \le \frac{Qa}{p(t)} + 2\Big(\frac{Ka}{w(t)}\Big)^{\frac{1}{2}},\ t \in (0,z]. \tag{2.17}$$

Now (2.16) and (2.17) yields the required result. □

Similarly we have the following.

Theorem 2.5 *Suppose $f : [0,\infty) \times \Re \to \Re$ is continuous with conditions (2.3), (2.4) and (2.15) satisfied. Then (2.2^n), with $a < 0$, has at least one solution $y_n \in C[0,n] \cap C^2(0,n]$ with $0 \le -y_n(t) \le -a$ for $t \in [0,n]$.*

Moreover if conditions (2.7), (2.9) and (2.10) hold, then there exist constants M_1 and M_2 independent of n with for $t \in (0,n)$,

$$y_n'(t) = |y_n'(t)| \le \frac{M_1}{p(t)} - \frac{M_2}{(w(t))^{\frac{1}{2}}} - a.$$

The above two theorems now give a new existence theorem for problems on the semi-infinite interval.

Theorem 2.6 *Let $f : [0,\infty) \times \Re \to \Re$ be continuous with conditions (2.3), (2.4), (2.7), (2.9), (2.10), (2.14) and (2.15) satisfied.*
(i). If $a > 0$ then (2.1) has at least one solution $y \in BC^2[0,\infty)$ with $0 \le y(t) \le a$ for $t \in [0,\infty)$.
(ii). If $a < 0$ then (2.1) has at least one solution $y \in BC^2[0,\infty)$ with $0 \le -y(t) \le -a$ for $t \in [0,\infty)$.

PROOF: Essentially the same argument as in Theorem 2.3 yields the result. □

Remark. If $a = 0$ and (2.15) holds then $y \equiv 0$ satisfies (2.1). This follows immediately since assumption (2.15) implies $f(t,0) = 0$.

Examples.
(i). Consider the generalised Emden-Fowler equation

$$\begin{cases} y'' + \frac{j}{t}y' = ct^r y^q,\ 0 < t < \infty, \\ y(0) = a > 0,\ y(t) \text{ bounded on } [0,\infty) \end{cases} \tag{2.18}$$

with $0 \le j < 1$, $c > 0$, $q > 1$, $r > -2$ and $2j + r \le 0$. Then (2.18) has a solution $y \in BC^2[0,\infty)$ with $y \ge 0$ on $[0,\infty)$.

To see this look at

$$\begin{cases} \frac{1}{t^j}(t^j y')' = ct^r y|y|^{q-1},\ 0 < t < \infty, \\ y(0) = a > 0,\ y(t) \text{ bounded on } [0,\infty). \end{cases} \tag{2.19}$$

Let $p(t) = t^j$, $f(t,y) = cy|y|^{q-1}$ and $w(t) = t^{-r}$. Then clearly (2.3), (2.4), (2.7) since $2j+r \le 0$, (2.9) with $Q \equiv 1$, (2.10), (2.14) and (2.15) are satisfied. Thus (2.19) has a solution $y \in BC^2[0,\infty)$ with $y \ge 0$ on $[0,\infty)$ by Theorem 2.6. Consequently y is a solution to (2.18).
(ii). Consider the boundary value problem

$$\begin{cases} \frac{1}{t^j}(t^j y')' = t^r f(t,y), \ 0 < t < \infty, \\ y(0) = a, \ y(t) \text{ bounded on } [0,\infty) \end{cases} \tag{2.20}$$

with $f(t,y) = y-1$ or $y+1$ or y^3+1, $0 \le j < 1$, $r > -2$, $j+r > -1$ and $2j+r \le 0$.

Taking $p(t) = t^j$, $w(t) = t^{-r}$ we see that (2.3), (2.4), (2.6) with $M = 1$, (2.7), (2.8) since $j+r > -1$, (2.9), (2.10) and (2.14) are satisfied. Consequently (2.20) has a solution $y \in BC^2[0,\infty)$ by Theorem 2.3.

3 SEMI-INFINITE PROBLEM

The results of section 2 are now used to establish existence results for the semi infinite problem

$$\begin{cases} \frac{1}{p}(py')' = \frac{f(t,y)}{w}, \ 0 < t < \infty, \\ y(0) = a, \ \lim_{t\to\infty} y(t) \text{ exists.} \end{cases} \tag{3.1}$$

Theorem 3.1 *Let $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be continuous with conditions (2.3), (2.4), (2.7), (2.9), (2.10), (2.14) and (2.15) satisfied. In addition assume*

$$\begin{cases} \lim_{x\to\infty}\left[A\int_b^x \frac{1}{p(u)}\int_b^u \frac{p(s)}{w(s)}ds\,du - B\int_b^x \frac{ds}{p(s)}\right] = +\infty \text{ for} \\ \text{any constants } A > 0,\ B \ge 0 \text{ and } b > 0 \end{cases} \tag{3.2}$$

and

$$\begin{cases} \text{for all } u \ge \beta > 0 \text{ with } u \le a \text{ (as defined in (3.4)) and } x \ge c > 0 \text{ there exists} \\ \text{a constant } K > 0 \text{ (which may depend on } \beta) \text{ such that } f(t,u) \ge K. \end{cases} \tag{3.3}$$

Then the nonlinear boundary value problem

$$\begin{cases} \frac{1}{p}(py')' = \frac{f(t,y)}{w}, \ 0 < t < \infty, \\ y(0) = a > 0, \ \lim_{t\to\infty} y(t) = 0 \end{cases} \tag{3.4}$$

has at least one solution $y \in BC^2[0,\infty)$ with $0 \le y(t) \le a$ for $t \in [0,\infty)$.

PROOF: Theorem 2.6 implies there exists $y \in BC^2[0,\infty)$ which satisfies $\frac{1}{p}(py')' = \frac{f(t,y)}{w}$, $0 < t < \infty$ with $y(0) = a$. In addition $0 \le y(t) \le a$ on $[0,\infty)$. We now claim that $\lim_{t\to\infty} y(t)$ exists. To see this notice (2.15) together with the differential equation and $y \ge 0$ on $[0,\infty)$ implies $(p(t)y'(t))' \ge 0$ for $t > 0$ i.e. py' is nondecreasing on $(0,\infty)$. There are two cases to consider.

Case(a): Suppose $y'(t) \ne 0$ for $t \in (0,\infty)$.

Then y is monotonic and this together with $0 \le y(t) \le a$ on $[0,\infty)$ implies $\lim_{t\to\infty} y(t)$ exists.

Case(b): Suppose $y'(z) = 0$ for some $z \in (0, \infty)$.

Then since py' is nondecreasing on $(0, \infty)$ we have $p(t)y'(t) \geq p(z)y'(z) = 0$ for $t \geq z$ i.e $y'(t) \geq 0$ for $t \geq z$. Consequently y is monotonic for $t \geq z$ and once again $\lim_{t\to\infty} y(t)$ exists.

Thus $\lim_{t\to\infty} y(t) = \alpha$ with $\alpha \in [0, a]$. It remains to show that $\alpha = 0$. To do this we assume that $0 < \alpha \leq a$.

Remark. Notice assumption (2.15) implies $f(t, 0) = 0$ and $\lim_{t\to\infty} f(t, q) \neq 0$ for $q \in (0, a]$ by assumption (3.3).

Now since $\lim_{t\to\infty} y(t) = \alpha > 0$ there exists a $c > 0$ with $y(x) \geq \frac{\alpha}{2}$ for $x \geq c$. Then assumption (3.3) guarantees the existence of a constant $K > 0$ with $f(x, y(x)) \geq K$ for $x \geq c$. Consequently for $x \geq c$, $(py')' \geq \frac{pK}{w}$ and integration from c to x yields

$$y'(x) \geq \frac{p(c)y'(c)}{p(x)} - \frac{K}{p(x)} \int_c^x \frac{p(s)}{w(s)} ds.$$

Another integration from c to x gives

$$y(x) \geq y(c) - p(c)y'(c) \int_c^x \frac{ds}{p(s)} + K \int_c^x \frac{1}{p(u)} \int_c^u \frac{p(s)}{w(s)} ds\, du. \tag{3.5}$$

Assumption (3.2) together with (3.5) implies $y(x)$ is unbounded on $[0, \infty)$, a contradiction. Thus $\lim_{t\to\infty} y(t) = 0$.□

Similarly we have the following.

Theorem 3.2 *Let $f : [0, \infty) \times \Re \to \Re$ be continuous with conditions (2.3), (2.4), (2.7), (2.9), (2.10), (2.14), (2.15) and (3.2) satisfied. In addition assume*

$$\left\{ \begin{array}{l} \text{for all } u \leq \beta < 0 \text{ with } u \geq a \text{ (as defined in (3.7)) and } x \geq c > 0 \text{ there exists} \\ \text{a constant } K > 0 \text{ (which may depend on } \beta) \text{ such that } f(t, u) \leq -K. \end{array} \right. \tag{3.6}$$

Then the nonlinear boundary value problem

$$\left\{ \begin{array}{l} \frac{1}{p}(py')' = \frac{f(t,y)}{w},\ 0 < t < \infty, \\ y(0) = a < 0,\ \lim_{t\to\infty} y(t) = 0 \end{array} \right. \tag{3.7}$$

has at least one solution $y \in BC^2[0, \infty)$ with $0 \leq -y(t) \leq -a$ for $t \in [0, \infty)$.

Remark. If $a = 0$ and (2.15) holds then $y \equiv 0$ satisfies the boundary value problem (3.1).

The results of Theorem's 3.1 and 3.2 can be used to discuss the more general situation.

Theorem 3.3 *Let $f : [0, \infty) \times \Re \to \Re$ be continuous with conditions (2.3), (2.4), (2.6), (2.7), (2.9), (2.10), (2.14) and (3.2) satisfied. In addition assume*

$$f(t, y) > 0 \text{ for } -M < y \leq M \text{ and for all } t \in [0, \infty) \tag{3.8}$$

and

$$\left\{ \begin{array}{l} \text{if } a > -M \text{ then for all } u \text{ with } u + M \geq \beta > 0 \text{ and } u \leq a \text{ and } x \geq c > 0 \\ \text{there exists a constant } K > 0 \text{ (which may depend on } \beta) \text{ such that} \\ f(x, u) \geq K, \text{ whereas if } a < -M \text{ assume for all } u \text{ with } u + M \leq \beta < 0 \text{ and} \\ u \geq a \text{ and } x \geq c > 0 \text{ that there exists a constant } K > 0 \text{ such that } f(t, u) \leq -K. \end{array} \right. \tag{3.9}$$

Then the boundary value problem

$$\begin{cases} \frac{1}{p}(py')' = \frac{f(t,y)}{w}, \ 0 < t < \infty, \\ y(0) = a \neq -M, \ \lim_{t\to\infty} y(t) = -M \end{cases} \tag{3.10}$$

has at least one solution $y \in BC^2[0,\infty)$. If $a > -M$ then $-M \leq y(t) \leq a$, $t \in [0,\infty)$ whereas if $a < -M$ then $a \leq y(t) \leq -M$, $t \in [0,\infty)$.

PROOF: Notice assumption (2.6) and (3.8) imply $f(t,-M) = 0$ for $t \in [0,\infty)$. We will show that the boundary value problem

$$\begin{cases} \frac{1}{p}(pw')' = \frac{f(t,w-M)}{w} \equiv \frac{G(t,w)}{w}, \ 0 < t < \infty, \\ w(0) = a + M, \ \lim_{t\to\infty} w(t) = 0 \end{cases} \tag{3.11}$$

has at least one solution $w \in BC^2[0,\infty)$. Then $y(t) = w(t) - M$ satisfies (3.10). To show (3.11) has a solution in $BC^2[0,\infty)$ we will use Theorem 3.1 or 3.2. First we claim that (2.15) is true for G i.e. we will show that $vG(t,v) = vf(t,v-M) > 0$ for all $t \in [0,\infty)$ and $|v| > 0$. To see this notice (3.8) implies

$$f(t,v-M) > 0 \text{ for } 0 < v \leq 2M \text{ and } t \in [0,\infty) \tag{3.12}$$

and (2.6) yields

$$(v-M)f(t,v-M) > 0 \text{ for } v > 2M \text{ or } v < 0 \text{ and for all } t \in [0,\infty). \tag{3.13}$$

Now if $0 < v \leq 2M$ then (3.12) implies $vG(t,v) > 0$ for $t \in [0,\infty)$. On the other hand for $v > 2M$ then (3.13) implies $vG(t,v) = (v-M)f(t,v-M) + Mf(t,v-M) > 0$ for $t \in [0,\infty)$. Finally for $v < 0$, (3.13) implies $vG(t,v) = vf(t,v-M) > 0$ for $t \in [0,\infty)$. Consequently (2.15) is true for G. If $a > -M$ then all the conditions of Theorem 3.1 are satisfied for our function G. Thus (3.11) has a solution $w \in BC^2[0,\infty)$ with $0 \leq w(t) \leq a + M$, $t \in [0,\infty)$ and so (3.10) has a solution $y \in BC^2[0,\infty)$ with $-M \leq y(t) \leq a$, $t \in [0,\infty)$. On the other hand if $a < -M$ then all the conditions of Theorem 3.2 are satisfied and so (3.11) has a solution $w \in BC^2[0,\infty)$ with $a + M \leq w(t) \leq 0$, $t \in [0,\infty)$ and so (3.10) has a solution $y \in BC^2[0,\infty)$ with $a \leq y(t) \leq -M$, $t \in [0,\infty)$.□

Remark. If $a = -M$ and (2.15), (3.8) hold, then $y \equiv -M$ satisfies the boundary value problem (3.1).

In addition we have the following.

Theorem 3.4 *Let $f : [0,\infty) \times \Re \to \Re$ be continuous with conditions (2.3), (2.4), (2.6), (2.7), (2.9), (2.10), (2.14) and (3.2) satisfied. In addition assume*

$$f(t,y) < 0 \textit{ for } -M \leq y < M \textit{ and for all } t \in [0,\infty) \tag{3.14}$$

and

$$\begin{cases} \textit{if } a < M \textit{ then for all } u \textit{ with } u - M \leq \beta < 0 \textit{ and } u \geq a \textit{ and } x \geq c > 0 \\ \textit{there exists a constant } K > 0 \textit{ (which may depend on } \beta) \textit{ such that} \\ f(x,u) \leq -K, \textit{ whereas if } a > M \textit{ assume for all } u \textit{ with } u - M \geq \beta > 0 \textit{ and} \\ u \leq a \textit{ and } x \geq c > 0 \textit{ that there exists a constant } K > 0 \textit{ such that } f(t,u) \geq K. \end{cases} \tag{3.15}$$

Then the boundary value problem

$$\begin{cases} \frac{1}{p}(py')' = \frac{f(t,y)}{w}, \ 0 < t < \infty, \\ y(0) = a \neq M, \ \lim_{t\to\infty} y(t) = M \end{cases} \tag{3.16}$$

has at least one solution $y \in BC^2[0,\infty)$. *If* $a < M$ *then* $a \le y(t) \le M$, $t \in [0,\infty)$ *whereas if* $a > M$ *then* $M \le y(t) \le a$, $t \in [0,\infty)$.

PROOF: Essentially the same proof as in Theorem 3.3 implies

$$\begin{cases} \frac{1}{p}(pw')' = \frac{f(t,w+M)}{w}, \ 0 < t < \infty, \\ w(0) = a - M, \ \lim_{t\to\infty} w(t) = 0 \end{cases}$$

has at least one solution $w \in BC^2[0,\infty)$. Then $y(t) = w(t) + M$ satisfies (3.16).□

Remark. If $a = M$ and (2.15), (3.14) hold, then $y \equiv M$ satisfies (3.1).

Examples.

(i). Consider the generalised Emden-Fowler equation

$$\begin{cases} \frac{1}{t^j}(t^j y')' = ct^r y^q, \ 0 < t < \infty, \\ y(0) = a > 0, \ \lim_{t\to\infty} y(t) = 0 \end{cases} \tag{3.17}$$

with $0 \le j < 1$, $c > 0$, $q > 1$, $r > -2$, $2j + r \le 0$ and $r + j + 1 > 0$. Then (3.17) has a solution $y \in BC^2[0,\infty)$ with $y \ge 0$ on $[0,\infty)$.

To see this look at

$$\begin{cases} \frac{1}{t^j}(t^j y')' = ct^r y|y|^{q-1}, \ 0 < t < \infty, \\ y(0) = a > 0, \ \lim_{t\to\infty} y(t) = 0 \end{cases} \tag{3.18}$$

and with $p(t) = t^j$, $f(t,y) = cy|y|^{q-1}$ and $\psi(t) = t^{-r}$ we see that (2.3), (2.4), (2.7), (2.9), (2.10), (2.14) and (2.15) are satisfied. In addition (3.2) holds since $r + j + 1 > 0$ and (3.3) is true with $K = c\beta^q$. Consequently Theorem 3.1 implies (3.18) has a solution $y \in BC^2[0,\infty)$ with $y \ge 0$ on $[0,\infty)$ and so y is a solution to (3.17).

(ii). Consider the boundary value problem

$$\begin{cases} \frac{1}{t^j}(t^j y')' = t^r(y-1), \ 0 < t < \infty, \\ y(0) = a, \ \lim_{t\to\infty} y(t) = 1 \end{cases} \tag{3.19}$$

with $0 \le j < 1$, $r > -2$, $2j + r \le 0$ and $r + j + 1 > 0$.

Then with $p(t) = t^j$, $f(t,y) = y - 1$ and $\psi(t) = t^{-r}$ we see that all the conditions of Theorem 3.4 are satisfied (here $M = 1$) and so (3.19) has at least one solution in $BC^2[0,\infty)$.

(iii). Consider

$$\begin{cases} \frac{1}{t^j}(t^j y')' = t^r(y^3 + 1), \ 0 < t < \infty, \\ y(0) = a, \ \lim_{t\to\infty} y(t) = -1 \end{cases} \tag{3.20}$$

with $0 \le j < 1$, $r > -2$, $2j + r \le 0$ and $r + j + 1 > 0$. Then Theorem 3.3 implies (3.20) has at least one solution in $BC^2[0,\infty)$.

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