

WEAKLY-ANOSOV DFFEOMORPHISMS  
AND  
EXPONENTIAL TRICHOTOMY

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INTRODUCTION. In this paper we introduce the notion of weakly-Anosov diffeomorphisms on a Riemannian manifold  $(M, g)$ , with a Riemannian metric  $g$ . This class of diffeomorphisms properly includes expansions, contractions, Anosov diffeomorphisms [4], and weak isometries [2]. They are easily generated by taking the products of weak isometries and Anosov diffeomorphisms. However, not every weakly-Anosov diffeomorphism is generated this way as it is shown in Example 1.4. Theorem 1.10 gives sufficient conditions under which a weakly-Anosov diffeomorphism is the product of a weak isometry and an Anosov diffeomorphism. It is also shown that a weakly Anosov diffeomorphism is Anosov if it is expansive [4].

In Section 2 we show that the periodic map associated with a nonlinear periodic differential equation is weakly Anosov if the variational system has an exponential trichotomy (Theorem 2.2). This generalizes the work of Palmer [6].

Throughout the paper, our manifolds will be smooth ( $= C^\infty$ ), connected, complete and Riemannian.

## 1. WEAKLY-ANOSOV DFFEOMORPHISMS

**Definition 1.1.** Let  $(M, g)$  be a smooth Riemannian  $m$ -dimensional manifold. A diffeomorphism  $f$  of  $M$  is said to be *weakly-Anosov* if the tangent bundle  $TM$  of  $M$  is the direct sum of three subbundles  $N, S, U$  (some of which may be trivial) of dimensions  $n, s, u$  respectively, such that:

- i.) Each of these subbundles is invariant under  $Tf$  (the derivative of  $f$ );
- ii.)  $N = \{v \in TM \mid \|Tf^n(v)\| \leq K\|v\|, \text{ for some } k \geq 1 \text{ and all } n \in \mathbb{Z}\}$
- iii.)  $S = \{v \in TM \mid \|Tf^n(v)\| \rightarrow 0 \text{ as } n \rightarrow \infty\}$
- iv.)  $U = \{v \in TM \mid \|Tf^n(v)\| \rightarrow 0 \text{ as } n \rightarrow -\infty\}$

For example, if  $\pi$  is the natural flow obtained by the suspension [4] of an Anosov diffeomorphism  $f : M \rightarrow M$ , then for every  $t \in \mathbb{R}$ , the diffeomorphism  $\pi^t : \tilde{M} \rightarrow \tilde{M}$  is weakly-Anosov, where  $\pi^t$  is the transition map [1,4].

**Remark 1.2.**

a.) In the case when  $M$  is compact, one can easily show that the above definition is independent of the choice of the metric. Thus, in this case, we

can choose a metric with respect to which the subbundles  $N$ ,  $S$ ,  $U$  are mutually orthogonal. For example, given  $g$ , we define the new metric  $\tilde{g}$  by:

$$\tilde{g}(X, Y) = \sum_{i=1}^3 g(X_i, Y_i),$$

where  $X = \sum_{i=1}^3 X_i$ ,  $Y = \sum_{i=1}^3 Y_i$ , with  $X_1, Y_1 \in N$ ,  $X_2, Y_2 \in S$ , and  $X_3, Y_3 \in U$ .

b.) If  $N = \phi$ ,  $M$  is compact and  $f$  is chain recurrent, then  $f$  is Anosov [2].

(c.) The case when both  $S$  and  $U$  are trivial has been studied by the authors in [3], where  $f$  was called a weak isometry, i.e., there exists a fixed number  $K \geq 1$  such that  $\|Tf^n(v)\| \leq Kv$ , for all  $n \in \mathbb{Z}$ .

**Proposition 1.3.** *The product of a weak isometry and an Anosov diffeomorphism is weakly-Anosov.*

PROOF: This follows directly from Definition 1.1. The converse of the above proposition is in general false. The diffeomorphisms  $\pi^t$  given by the suspension example mentioned earlier need not be such a product. The following is another example.

**Example 1.4.** Let  $L : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be the linear map given by the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

and take the action of  $Z^3$  on  $\mathbb{R}^3$  defined by

$$(n_1, n_2, n_3) \cdot (x, y, z) = (x + n_1, y + n_2, z + n_2).$$

Then, clearly,  $L$  induces a weakly-Anosov diffeomorphism  $f$  on the torus  $T^3 = \mathbb{R}^3/Z^3$ . However,  $f$  is not a product of a weak isometry and an Anosov diffeomorphism.

**Example 1.5.** It is important to notice that the existence of the two subbundles  $N$  and  $P = S \oplus U$  induces an almost product structure  $F$  on  $M$  defined by  $F(X+Y) = X - Y$ , for  $X \in N, Y \in P$ . Furthermore,  $F$  is a (1,1) tensor field with  $F^2 = I$ . On the other hand, if  $F$  is an almost product structure on a manifold  $M$  and  $f : M \rightarrow M$  is a diffeomorphism, then  $f$  is said to be compatible with  $F$  if it preserves the two distributions  $N$  and  $P$  induced by  $F$ .

We now give sufficient conditions for a weakly Anosov diffeomorphism to be a product of a weak isometry and an Anosov diffeomorphism.

**THEOREM 1.6.** *Let  $f$  be a weakly Anosov diffeomorphism on a simply connected Riemannian manifold  $(M, g)$ , and  $F$  be the induced almost product structure. If for all vector fields  $X, Y, Z$  on  $M$ ,  $g(X, Y) = g(FX, FY)$ , and  $\nabla_Z F = 0$ , then  $M$  is isometric to a Riemannian product  $M_1 \times M_2$  and  $f = \alpha_1 \times \alpha_2$  with  $\alpha_1 : M_1 \rightarrow M_1$  is a weak isometry, and  $\alpha_2 : M_2 \rightarrow M_2$  an Anosov diffeomorphism (where  $\nabla$  represents covariant differentiation with respect to the Levi-civita connection).*

**PROOF.** The fact that  $M$  is isometric to a Riemannian product  $M_1 \times M_2$  follows directly from the DeRham decomposition theorem. It remains to show that  $f = \alpha_1 \times \alpha_2$  with  $\alpha_i : M_i \rightarrow M_i$  a diffeomorphism,  $i = 1, 2$ . The rest follows from the fact that  $f$  is weakly-Anosov.

Since the distribution  $N$  is parallel and orthogonal to  $S \oplus U$ , both  $N$  and  $S \oplus U$  are integrable. Thus there exists a pair of complementary, parallel, and mutually orthogonal foliations  $F^1$  tangent to  $N$  and  $F^2$  tangent to  $S \oplus U$ . Hence for every point  $x \in M$  the leaves  $F_x^1$  and  $F_x^2$  through  $x$  can be identified with  $M_1$  and  $M_2$ , respectively.

Since  $N$  and  $S \oplus U$  are invariant under  $f$ , the two foliations  $F^1, F^2$  are locally invariant under  $f$ . Now we show that each of these foliations is preserved globally, i.e., leaves are mapped into leaves. The proof will be given only for  $F^1$ , since the proof for  $F^2$  is similar.

Let  $x \in M$  and  $y = f(x)$ . From the local invariance of  $F^1$  we know that there exists a neighborhood  $V$  of  $x$  such that  $f(F_x^1 \cap V) = F_y^1 \cap f(V)$ , where  $F_x^1, F_y^1$  are the leaves through  $x$  and  $y$ , respectively. Suppose that leaves are not preserved under  $f$ . So take a leaf  $F_o^1$  that is not mapped into a leaf by  $f$ . Since  $F_o^1$  is a connected submanifold of  $M$  we can find two points  $a, b \in F_o^1$  having two neighborhoods  $V_a$  and  $V_b$  such that:

$$i.) f(F_o^1 \cap V_a) = F_c^1 \cap f(V_a)$$

$$ii.) f(F_o^1 \cap V_b) = F_d^1 \cap f(V_b)$$

$$iii.) F_o^1 \cap V_a \cap V_b \neq \phi \text{ where } c = f(a), d = f(b).$$

Now for every  $x \in F_o^1 \cap V_a \cap V_b$ ,  $y = f(x)$  we have  $y \in F_c^1 \cap F_d^1$  which contradicts the fact that leaves are disjoint. This proves that leaves are globally preserved under  $f$ . Now we use this to define a splitting of  $f$  into a product  $F = \alpha_1 \times \alpha_2$ . Since  $M = M_1 \times M_2$  and  $F^1, F^2$  are the foliations given by this product structure, we take an arbitrary point  $(p_0, q_0) \in M$  and identify  $F_{(p_0, q_0)}^1$  and  $F_{(p_0, q_0)}^2$  with  $M_1$  and  $M_2$ , respectively. Define  $\alpha_1 : M_1 \rightarrow M_1$  by  $\alpha_1(p) = \pi_1 \circ f(p, q_0)$  and  $\alpha_2 : M_2 \rightarrow M_2$  by  $\alpha_2(q) = \pi_2 \circ f(p_0, q)$  where  $\pi_1, \pi_2$  are the projections on the first and second factors, respectively.

Now we use our previous argument that leaves are mapped into leaves to show that  $\alpha_1, \alpha_2$  are independent of the choice of  $(p_0, q_0)$ . If  $(a_0, b_0)$  is any other point of  $M$  then  $(p, b_0)$  and  $(p, q_0)$  belong to the same leaf of  $F^2$ . Hence  $f(p, q_0)$  and  $f(p, b_0)$  belong to the same leaf as well. This means that  $\pi_1 \circ f(p, q_0) = \pi_1 \circ f(p, b_0)$  and  $\alpha_1(p)$  is independent of the choice of  $p_0, f_0$ . Similar argument holds for  $\alpha_2$ .

The smoothness of  $\alpha_1, \alpha_2$  follows from that of  $f$  and the foliations. The metric properties of  $Tf|_{S \oplus U}$  and  $Tf|_N$  show that  $\alpha_2$  is an Anosov diffeomorphism and  $\alpha_1$  is a weak isometry. This completes the proof of the theorem.

If we drop the condition that  $M$  is simply connected, we obtain the following corollary.

**Corollary 1.7.** Let  $f$  be a weakly-Anosov diffeomorphism of  $(M, g)$  with  $g(FX, FY) = g(X, Y)$  and  $\nabla_Z F = 0$  for all vector fields  $X, Y, Z$  of  $M$ . Then  $M$  is isometric to the quotient space  $(M_1 \times M_2)/G$  where  $G$  is a group of isometries acting on  $M_1 \times M_2$  and is isomorphic to the fundamental group of  $M$ . Moreover, the diffeomorphism  $f$  is a factor of  $\alpha_1 \times \alpha_2$  where  $\alpha_1 : M_1 \rightarrow M_1$  is a weak isometry and  $\alpha_2 : M_2 \rightarrow M_2$  is an Anosov diffeomorphism.

Now, we give necessary and sufficient conditions for a weakly-Anosov diffeomorphism to be Anosov. But first, we need the following definition.

**Definition 1.8 [4].** A diffeomorphism  $f : M \rightarrow M$  is said to be expansive if there exists a uniform constant  $\epsilon > 0$  such that for every two distinct points  $x, y \in M$ , there is an integer  $n \in \mathbb{Z}$  such that  $d(f^n(x), f^n(y)) \geq \epsilon$ , where  $d$  is the distance (metric) function induced by  $g$ .

Let  $f : M \rightarrow M$  be a diffeomorphism. Then  $f$  (or the discrete dynamical system generated by  $f$ ) is said to be chain recurrent [4] if, for every  $x \in M$ ,  $\epsilon > 0$ ,  $r \in \mathbb{Z}^+$ , there exist two finite sets  $C_1 = \{x_i \in M \mid i = 1, \dots, k+1\}$  and  $C_2 = \{n_i \in \mathbb{Z} \mid n_i > r, i = 1, \dots, k\}$  such that  $x = x_1 = x_{k+1}$  and  $d(f^{n_i}(x_i), x_{i+1}) < \epsilon$ , for  $x_i \in C_1$  and  $n_i \in C_2$ .

**THEOREM 1.9.** *Let  $f$  be a weakly-Anosov and chain recurrent diffeomorphism and  $M$  is compact. Then  $f$  is Anosov if and only if it is expansive.*

**PROOF.** If  $f$  is Anosov then it is expansive [4]. Conversely, assume that  $f : (M, g) \rightarrow (M, g)$  is weakly-Anosov and the distribution  $N$  is not trivial and hence there is a non-zero tangent vector  $V \in N$ . Thus  $\|Tf^n(v)\| \leq K\|V\|$ , for all  $n \in \mathbb{Z}$ . Hence, for every  $\epsilon > 0$ , there exists a  $\delta > 0$  such that if  $u = \lambda V$ ,  $\lambda < \delta$ , then  $\|Tf^n(u)\| < \epsilon$  for every  $n \in \mathbb{Z}$ . If  $\gamma(t)$  is the geodesic initiated by the pair  $(\pi(u), u)$ ,  $\pi : TM \rightarrow M$  is the canonical projection, then there is an  $h$ ,  $0 < h < 1$ , such that  $\|Tf^n(\dot{\gamma}(t))\| < \epsilon$  for  $t < h$  and all  $n \in \mathbb{Z}$ . Now let  $x = \pi(u)$  and  $y = \gamma(h/2)$ , then  $x \neq y$  and for every  $n \in \mathbb{Z}$  we have

$$d(f^n(x), f^n(y)) \leq \int_0^{h/2} \|Tf^n(\dot{\gamma}(t))\| dt \leq \int_0^{h/2} \epsilon dt < \epsilon.$$

Thus  $f$  is not expansive, and we then have a contradiction. Thus  $N = \phi$ . Since  $f$  is chain recurrent, it follows that  $f$  is Anosov [2].

We give now a sufficient condition for a diffeomorphism to be weakly-Anosov.

**THEOREM 1.10.** *Let  $(M, g, F)$  be a compact almost product Riemannian manifold and  $f$  a diffeomorphism of  $M$  compatible with  $F$  such that*

- i.)  $f$  is chain recurrent,
- ii.) A tangent vector  $V$  is Lipschitz stable if and only if  $V \in N = \{X \in TM \mid FX = X\}$ .

*Then  $f$  is weakly-Anosov.*

**PROOF.** Recall that  $F$  induces two mutually orthogonal subbundles  $N$  and  $P$  of the tangent bundle  $TM$ .  $N$  is defined above in the statement of the theorem and  $P = \{X \in TM \mid FX = -X\}$ . Both  $N$  and  $P$  are invariant under  $f$  from the hypothesis. Now condition (ii) implies that the uniformly bounded subbundle of the vector bundle  $P$  is the zero section. Using chain recurrence and the results of Selgrade [8] and Sacker and Sell [7], we conclude that  $P$  splits into the sum of two subbundles  $S$  and  $U$ , where  $S$  is contracting under  $f$  and  $U$  is expanding. This shows that  $f$  is weakly-Anosov.

A point  $x \in M$  is said to be a nonwandering point of  $f$  if, for every neighborhood  $U$  of  $x$ , there exists a nonzero integer  $n$  such that  $f^n(U) \cap U \neq \emptyset$ . The set of all nonwandering points is called the nonwandering set of  $f$  and is denoted by  $\Omega(f)$  [4].

As in [1], let  $L(x)$  and  $J(x)$  denote the limit set and the prolongation limit set of  $x$ , respectively.

**THEOREM 1.11.** *If  $f : M \rightarrow M$  is a weak isometry and  $M$  is compact then  $\Omega(f) = M$ .*

**PROOF.** Since  $f$  is a weak isometry, then  $M$  is decomposed into the union of two disjoint invariant sets  $A$  and  $B$ , where  $A$  consists of points whose limit sets are empty; and  $B$  consists of points whose limit sets are equal to their orbit closures [2,3]. Since  $M$  is compact, then  $A$  must be empty. Hence, for every  $x \in M$  we have  $x \in L(x) \subset J(x)$ . This implies [1] that  $x$  is a nonwandering point. Hence  $\Omega(f) = M$ .

The above theorem says that, for one of the two important classes of weakly-Anosov diffeomorphisms, the nonwandering set is equal to the whole manifold. For the other important class, namely, Anosov diffeomorphisms, this is still a conjecture [5]. Since weakly-Anosov diffeomorphisms arise, at least locally, as products of Anosov diffeomorphisms and weak isometries, then in view of Theorem 1.11 above (and 1.13 below) it is reasonable to have the following conjecture.

**Conjecture 1.12.** *If  $f : M \rightarrow M$  is weakly-Anosov and  $M$  is compact, then  $\Omega(f) = M$ .*

We end this section with a theorem that leads us to believe that a positive answer for the conjecture on the nonwandering set of Anosov diffeomorphisms might lead to a positive answer to the corresponding conjecture in the general weakly-Anosov case.

**THEOREM 1.13.** *Let  $M_1$  and  $M_2$  be two compact manifolds,  $f_1$  a weak isometry on  $M_1$ , and  $f_2$  an Anosov diffeomorphism on  $M_2$ . If  $f = f_1 \times f_2$  on  $M = M_1 \times M_2$  and  $\Omega(f_2) = M_2$ , then  $\Omega(f) = M$ .*

**PROOF.** The set of periodic points of  $M_2$  is dense in  $\Omega(f_2) = M_2$ . Hence, it is enough to show that for  $a \in M_1$  and a periodic point  $b \in M_2$ ,  $(a, b)$  is a nonwandering point in  $M$ . Suppose that the period of  $b$  is  $n$ . Let  $U \times V$  be any neighborhood of  $(a, b)$ . Since  $f_1$  is a weak isometry then so is  $f_1^n$ , and hence  $\Omega(f_1^n) = M_1$ . Thus  $a$  is nonwandering with respect to  $f_1^n$ . This means that there exists a  $k \in \mathbb{Z}$ ,  $k \neq 0$ , such that  $f_1^{nk}(U) \cap U \neq \emptyset$ . But  $f_2^n(b) = b$  and hence  $f_2^{nk}(b) = b$  and  $b \in f_2^{nk}(V) \cap V$ . So if  $c \in f_1^{nk}(U) \cap U$ , then  $(c, b) \in (f_1^{nk}(U) \times f_2^{nk}(V)) \cap (U \times V) = f^{nk}(U \times V) \cap (U \times V)$ . This shows that  $f^{nk}(U \times V) \cap (U \times V) \neq \emptyset$ , and  $(a, b) \in \Omega(f)$ . This completes the proof of the theorem.

## 2. EXPONENTIAL TRICHOTOMY

Let  $G \subset \mathbf{R}^n$  be open and  $f : \mathbf{R} \times G \rightarrow \mathbf{R}^n$  a continuous function with period  $T$  in  $t$  and continuous partial derivative  $f_x(t, x)$ . Let  $x(t, \xi)$  be the solution of

$$\dot{x}(t) = f(t, x(t)) \quad (1)$$

with  $x(0, \xi) = \xi$ . Assume that the open set  $\tilde{G} = \{\xi \in G \mid \text{the minimum interval of existence of } x(t, \xi) \text{ contains } [0, T]\} \neq \emptyset$ . Then the map  $F : \tilde{G} \rightarrow \mathbf{R}^n$  defined by  $F(\xi) = x(T, \xi)$  is a  $C^1$ -diffeomorphism called the period map [4].

Consider now the variational system associated with the periodic solution  $x(t, \xi)$  with period  $T$ ,

$$y'(t) = f_x(t, x(t, \xi))y(t). \quad (2)$$

Then clearly (2) is also a periodic system.

Let  $Y(t, \xi)$  be the fundamental matrix of (2). Then by Floquet theory,  $Y(t, \xi) = Q(t)e^{tR}$ , where  $Q(t)$  is a  $T$ -periodic matrix, and  $R$  is a constant matrix. The fundamental matrix of  $\dot{z}(t) = Rz(t)$  is given by  $Q^{-1}(t)Y(t, \xi)$ . So let us associate with (2), the family

$$\dot{z}(t) = Rz(t). \quad (3)$$

Now

$$\begin{aligned} \dot{F}(\xi) &= \frac{\partial}{\partial \xi} x(T, \xi) = \frac{\partial}{\partial \xi} x(t+T, \xi) \Big|_{t=0} \\ &= Y(t+T, \xi) \Big|_{t=0} = Y(t, \xi)e^{TR} \Big|_{t=0}. \end{aligned}$$

It follows that

$$\dot{F}(\xi) = Y(T, \xi) = e^{TR}. \quad (4)$$

**Definition 2.1 [7].** Equation (2) is said to have an exponential trichotomy if for each  $\xi$  there are three mutually orthogonal projections  $P_1(\xi)$ ,  $P_2(\xi)$ ,  $P_3(\xi)$  such that

$$\begin{aligned} \|Y(t, \xi)P_1(\xi)Y^{-1}(s, \xi)\| &\leq Ke^{-\alpha(t-s)}, \quad t \geq s \\ \|Y(t, \xi)P_2(\xi)Y^{-1}(s, \xi)\| &\leq Ke^{-\alpha(s-t)}, \quad s \geq t \\ \|Y(t, \xi)P_3(\xi)Y^{-1}(s, \xi)\| &\leq K \quad \text{for } t, s \in \mathbf{R}. \end{aligned}$$

It may be easily verified that (2) has an exponential trichotomy if (3) has an exponential trichotomy. Furthermore, one may choose the same projections  $P_1(\xi)$ ,  $P_2(\xi)$ ,  $P_3(\xi)$  for both families (2) and (3).

**Definition 2.2 [4].** Let  $\xi$  be a fixed point of  $F$ . Then we have the following sets:

$$(i) \ W^s(\xi) = \{y \in M \mid \lim_{n \rightarrow \infty} F^n(y) = \xi\}$$

$$(ii) \ W^u(\xi) = \{y \in M \mid \lim_{n \rightarrow -\infty} F^n(y) = \xi\}$$

$$(iii) \ W^n(\xi) = \{y \in M \mid d(F^n(y), \xi) \leq Kd(y, \xi), \text{ for all } n \in \mathbf{Z}, \text{ and some constant } K\}.$$

**LEMMA 2.3.** Let  $\xi$  be a fixed point of  $F$ . Then the following statements are valid.

$$(i) \ \eta \in W^s(\xi) \text{ if } \lim_{t \rightarrow -\infty} \|x(t, \xi) - x(t, \eta)\| = 0$$

$$(ii) \ \eta \in W^u(\xi) \text{ if } \lim_{t \rightarrow -\infty} \|x(t, \xi) - x(t, \eta)\| = 0$$

$$(iii) \ \eta \in W^n(\xi) \text{ if } \|x(t, \xi) - x(t, \eta)\| \leq K\|\xi - \eta\|, \text{ for all } t \in \mathbf{R}.$$

**PROOF.** We prove here (iii). Let  $\eta \in W^n(\xi)$ . Then  $d(F^n(\eta), \xi) \leq Kd(\eta, \xi)$ , for all  $n \in \mathbf{Z}$ . For each  $t \in \mathbf{R}$ , there exists  $n \in \mathbf{Z}$  such that  $nT \leq t \leq (n+1)T$ . Let  $t = nT + \tau$ ,  $0 \leq \tau \leq T$ . Then

$$\begin{aligned} \|x(t, \eta) - x(t, \xi)\| &= \|x(\tau, x(nT, \eta)) - x(\tau, x(nT, \xi))\| \\ &= \|x(\tau, F^n(\eta)) - x(\tau, F^n(\xi))\| = \|x(\tau, F^n(\eta)) - x(\tau, \xi)\| \\ &\leq L\|F^n(\eta) - \xi\| \\ &\leq L\tilde{K}\|\eta - \xi\| \\ &= K\|\eta - \xi\| \end{aligned}$$

The converse is similarly proved.

**Definition 2.4.** A fixed point  $\xi$  of  $F$  is said to be weakly hyperbolic if the eigenvalues of  $F'(\xi)$  that lie on the unit circle are simple.

A point  $\eta$  is said to be a transversal weakly-homoclinic point of  $F$  with respect to a weakly hyperbolic fixed point  $\xi$  of  $F$  if  $\eta \in W^s(\xi) \cap W^u(\xi) \cap W^n(\xi)$ , and  $T_\eta M = T_\eta W^s(\xi) \oplus T_\eta W^u(\xi) \oplus T_\eta W^n(\xi)$ .

**PROPOSITION 2.5.** The following statements are valid:

(i) The point  $\xi$  is a weakly hyperbolic fixed point of  $F$  if and only if  $x(t, \xi)$  is a  $T$ -periodic solution of (1) and (2) possesses an exponential trichotomy.

(ii)  $\eta \in O$  is a transversal weakly homoclinic point with respect to a weakly hyperbolic fixed point  $\xi$  if  $\|x(t, \eta) - x(t, \xi)\| \rightarrow 0$  as  $|t| \rightarrow \infty$ ,  $\|x(t, \eta) - x(t, \xi)\| \leq K\|\eta - \xi\|$ , for all  $t \in \mathbf{R}$ , and (2) possesses an exponential trichotomy.

PROOF. (i) Assume that  $\xi$  is a weakly hyperbolic fixed point of  $F$ . Then  $F'(\xi)$  has  $s$  eigenvalues whose real parts are less than 1,  $u$  eigenvalues whose real parts are greater than 1, and  $n$  simple eigenvalues whose real parts are equal 1, with  $u + s + n = m$ . These eigenvalues  $\lambda$  are the Floquet multipliers of (2). Since the eigenvalues  $\rho$  of  $R$  are such that  $\lambda = e^{T\rho}$ , it follows that there are  $s$  eigenvalues of  $R$  with real parts less than zero,  $u$  eigenvalues of  $R$  with real parts greater than zero, and  $n$  simple eigenvalues of  $R$  which are pure imaginary. Let  $E_1, E_2, E_3$  be the corresponding generalized eigenspaces. Then  $\mathbf{R}^m = E_1 \oplus E_2 \oplus E_3$ . Let  $P_1(\xi), P_2(\xi), P_3(\xi)$  be the corresponding projections on  $\mathbf{R}^m$ , that is  $P_i(\xi)\mathbf{R}^m = E_i, 1 \leq i \leq 3$ . Then the family (3) has an exponential trichotomy and consequently (2) possesses an exponential trichotomy.

The converse is obtained by reversing the above steps.

(ii) This follows from Lemma 2.3.

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