

*To the Memory of my Professor  
Octav Onicescu*

## ELLIPTIC EIGENVALUE VARIATIONAL INEQUALITIES

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Since their inception by G. Stampacchia and G. Fichera in the mid-sixties, variational inequalities have become a rich source of inspiration both in pure and applied mathematics. Their real founder is H. Brézis, who in his thesis [4] provided a general operator setting for the investigation of solutions of stationary and evolution inequalities. Today variational inequalities are an indispensable tool in a large variety of models in science and engineering. In particular, eigenvalue problems for variational inequalities have been first approached by Cl. Do [18] in connection with the buckling of thin elastic plates subjected to unilateral conditions.

Nearly over three decades, the theory of mappings of monotone type have turned out to be an efficient operator support to investigate variational problems. A particular case of maximal monotone mappings easily handled in applications is the subdifferential of a convex function. Nontrivial implementations of subdifferentials in the study of nonlinear partial differential equations have been pointed out, beginning with H. Brézis [5]. These applications state the reason for extensions of the subdifferential concept beyond the class of convex functionals.

It is the purpose of the paper to draw together our contribution in the critical point theory [24] and the recent results about lower stationary points in the nonsmooth subdifferential analysis developed by M. Degiovanni and A. Marino [10]-[16], and their collaborators [6]-[8], [20], [21], [25].

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Let  $X$  be a real reflexive Banach space,  $X^*$  its dual space and  $(\cdot, \cdot)$  denotes the duality. Assume two Fréchet differentiable functions  $F, G : X \rightarrow \mathbb{R}$  and a proper l.s.c. function  $\varphi : X \rightarrow \mathbb{R} \cup \{+\infty\}$  are given. We look for *eigenfunctions*, i.e. eigenvalue - eigenfunction pairs  $(\lambda, u) \in \mathbb{R} \times X$ ,  $u \in D(\varphi)$  for the variational inequality

$$(F'u - \lambda G'u, v - u) + \varphi(v) - \varphi(u) \geq 0 \quad \forall v \in D(\varphi), \quad (\text{EVI})$$

where the constraints are determined by

$$D(\varphi) = \{ v \in X \mid \varphi(v) < +\infty \},$$

the effective domain of  $\varphi$ . The nondifferentiability is needed for Von Karman's theory of a thin plate. We also suppose that  $[\lambda, 0]$  is a solution for every  $\lambda \in \mathbb{R}$ . Such a pair is a *trivial* solution. A sufficient condition for this to occur is that  $F'(0) = G'(0) = 0$  and  $\varphi(0) \leq \varphi(v)$  for all  $v \in X$ . Below, we are interested in nontrivial solutions  $[\lambda, u]$  in  $X \times \mathbb{R}$  with  $u \neq 0$  and we will also investigate the bifurcation points  $(\lambda_0, 0)$  accumulating nontrivial solutions of (EVI).

As a special case,  $\varphi$  may be the indicator function  $I_K$  of a closed convex set  $K \subset X$ , i.e.  $I_K(v) = 0$  if  $v \in K$  and  $I_K = +\infty$  otherwise.

In the latter case, we consider EVIs of the form

$$(F'u, v - u) \geq \lambda (G'u, v - u) \quad \text{for all } v \in K. \quad (1)$$

Such eigenvalue problems often arise in elasticity theory, when the plates are constrained by rigid obstacles. With the inequality (1) we associate its linearization at the point  $u = 0$ , that is

$$(F''(0)u, v - u) \geq \lambda (G''(0)u, v - u) \quad \text{for all } v \in K_0, \quad (2)$$

where  $K_0$  is the convex closure of  $\bigcup_{t>0} tK$  in  $X$ . This is a *quadratic variational inequality* and the goal is the following basic result: EVI (2) has a greatest positive eigenvalue  $\lambda_0$ , and  $\lambda_0$  is, at the same time, the greatest bifurcation value of (1). The only known result concerns the case in which  $\lambda_0$  is the smallest positive eigenvalue or the greatest negative eigenvalue of (2) and the set  $K$  is a cone; this amounts to tackling only constraints of the form  $u \geq 0$  on some region and  $u \leq 0$  on some other

region. A detailed treatment of quadratic variational inequalities (1) as well as pertinent references may be found in [29], Chap.64. Another study of the local behavior of branches of solutions of inequalities (1), related to the so-called conical linearization, turning and transition points, is performed in a more recent approach [9].

In concrete situations, the constraint  $K$  is defined by a pointwise relation, such as  $\{u \in H \mid u \leq \varphi\}$  or  $\{u \in H \mid |Du| \leq \varphi\}$ , where  $\varphi$  is an assigned function. The lack of smoothness is noticeable by two facts. First when the searched critical points of  $F'$  are restricted to the sets  $K \cap \{v \mid (G'v, v) = \pm r^2\}$ , which are nonconvex. The second difficulty comes from the fact that the convex set  $K$  and the manifolds  $(G'v, v) = \pm r^2$  may be tangent in a sense precised below. If  $K$  is a convex cone, these sets are never tangent. A way of overcoming these difficulties is by means of critical point techniques for the lower subdifferential of non-smooth functionals and a generalization of the Lagrange multiplier rule.

### 1. Subdifferentials and deformation techniques

We assume for the time being that the convex l.s.c. function  $\varphi$  is *subdifferentiable*, i.e. the set of subgradients

$$\partial\varphi(u) = \{f \in X^* \mid \varphi(u) - (f, u) \leq \varphi(v) - (f, v) \quad \forall v \in X\}$$

is nonempty and, so, the mapping  $\partial\varphi : X \rightarrow 2^{X^*}$  is defined. In particular,  $\partial I_K(u) = \{f \in X^* \mid (f, u - v) \geq 0 \quad \forall v \in K\}$  is the outward normal cone to  $K$  at  $u$ . Denote  $J = \varphi + F$ . Then  $\partial J = \partial\varphi + F'$  and the (EVI) becomes the parametrized inclusion

$$\partial J(u) - \lambda G'u \ni 0.$$

A basic procedure for  $u$  to be a solution with a suitable choice of  $\lambda$  is that  $J(u) = \sup \{J(v) \mid G(v) = r\}$ , where  $r > 0$  is a prescribed constant. The level set  $\delta G_r = \{v \in X \mid G(v) = r\}$  is a *regular* manifold if  $(G'v, v) \neq 0$  for each  $v \in \delta G_r$ . The tangent space to  $\delta G_r$  coincides with  $N(G')$  and so it has the form

$$T_u = T_u(\delta G_r) = \{v \in X \mid (G'u, v) = 0\}.$$

We say that  $u \in \delta G_r$  is a *critical point* of  $J$  restricted to  $\delta G_r$  if there is a subgradient  $h \in \partial J(u)$  such that  $(h, v) = 0$  for all  $v \in T_u$ . Therewith  $c = J(u)$  is called a *critical value* of  $J$ . It is easily shown that any local extremum of  $J|_{\delta G_r}$  is a critical point [24].

Using the Lagrange multiplier rule, we can check that critical points of  $J$ , constrained to regular  $\delta G_r$ , are eigenfunctions of the inclusion

$$\partial J(u) - \lambda G'u \ni 0, \quad G(u) = r$$

for some real number  $\lambda$ . More precisely,  $\lambda^{-1}h - G'u = 0$  with  $\lambda = (h, w)$  where  $w = \frac{u}{(G'u, u)}$ .

The well clamped elastic plate with obstacles. Let  $\Omega$  be a bounded domain in  $\mathbb{R}^2$  (the rest position of the plate) with boundary  $\partial\Omega$  consisting of a finite number of arcs on each of which a tangent rotates continuously and let  $\phi \in C^2(\bar{\Omega})$  be the initial Airy stress function. With the l.s.c. obstacle  $\varphi: \Omega \rightarrow \mathbb{R}^+$  we associate the symmetric constraint

$$K = \{ v \in H_0^1(\Omega) \mid -\varphi \leq v \leq \varphi \text{ a.e.} \}.$$

The Von Karman's problem with the obstacle  $\varphi$  looks for triplets  $(\lambda, u, h) \in \mathbb{R} \times H_0^2(\Omega) \times H_0^2(\Omega)$  such that

$$\begin{cases} \int_{\Omega} \Delta u \Delta(v - u) dx dy \geq \int_{\Omega} [\lambda \phi + h, u](v - u) dx dy & \forall v \in K, \\ \Delta^2 h + [u, u] = 0 & \text{in } \mathcal{D}'(\Omega), \end{cases} \quad (3)$$

where  $[u, v] = u_{xx}v_{yy} - 2u_{xy}v_{xy} + u_{yy}v_{xx}$ . The function  $u$  represents the vertical deflection of the plate while the function  $h$  is the increment of the Airy stress function, so that  $\lambda\phi + h$  is the final Airy function corresponding to the deflection  $u$ . Assume  $u = 0$  is the position of the plate when there are no forces. The eigenvalue parameter represents a measure of the applied forces.

Now for every  $v \in H_0^2(\Omega)$  we set  $G(v) = \int_{\Omega} [\phi, v]v dx dy$  and

$$J(v) = \int_{\Omega} \left\{ \frac{1}{2} |\Delta v|^2 + \frac{1}{4} \left[ \Delta^{-2}[v, v], v \right] v \right\} dx dy, \quad (4)$$

where  $\Delta^{-2}: L^1(\Omega) \rightarrow H_0^1(\Omega)$ . Let  $u$  be a critical point of  $J$  on  $K \cap \delta G_r$ .

Then there exists  $\lambda \in \mathbb{R}$  such that  $(\lambda, u, h)$  with  $h = \Delta^{-2}[u, u]$  is an eigensolution of (3). We are interested in solutions bifurcated from the undeflected state at each eigenvalue of a corresponding linearized problem ([2], Sec. 4.3.B).

For another interpretation of the above eigenvalues, let

$$N_u = N_u(\delta G_r) = \left\{ y \in X^* \mid (y, z) = 0 \quad \forall z \in T_u \right\}$$

be the normal space to  $\delta G_r$  at  $u$  and define the mapping  $\Psi: \delta G_r \rightarrow X^*$  by setting  $\Psi u = N_u \cap \partial J(u)$ . Then  $(\lambda, u) \in \mathbb{R} \times D(\partial J)$  is an eigensolution of (1) if there is  $h \in \Psi u$  such that  $\lambda = (w, h)$  and  $h = G'u$ . Thereby, we derive a general form of eigenvalue corresponding to  $u$

$$\lambda(u) \in \left\{ \frac{(h, u)}{(G'u, u)} \mid h \in \Psi u \right\}.$$

Define now on  $\delta G_r$  the set-valued mapping

$$\Pi(u) = \left\{ f - \frac{(f, u)}{(G'u, u)} G'u \mid f \in \partial J(u) \right\}$$

and note that any critical point  $u$  of  $J$ , relative to  $\partial G_r$ , satisfies  $0 \in \Pi(u)$ . Since  $\Pi$  inherits the properties of  $\partial J$ , an element  $\Pi^\circ(u)$  of minimal norm exists and the following set-valued variant of the local Palais-Smale compactness condition for a real  $c \in \mathbb{R}$  makes sense:

(PS<sub>c</sub>) Every sequence  $\{u_n\} \subset \delta G_r$  with  $J(u_n) \rightarrow c$  and  $\Pi^\circ(u_n) \rightarrow 0$  as  $n \rightarrow \infty$  contains a convergent subsequence.

Let  $K$  be the set of all critical points of  $J|_{\delta G_r}$ ,  $K_c = \{u \in K \mid J(u) = c\}$  and let  $J^c = \{v \in \delta G_r \mid J(v) \leq c\}$  be the corresponding sublevel set of  $J$ .

The core of our minimax method is the extension of the:

**Deformation Lemma 1.** Let  $J \in C^1(\delta G_r, \mathbb{R})$  satisfy (PS<sub>c</sub>) and let  $\varepsilon_0 > 0$ . Then there exists  $\varepsilon \in (0, \varepsilon_0)$  and a deformation  $\eta \in C([0, 1] \times \delta G_r, \delta G_r)$  with the following basic properties:

- i)  $\eta(1, u) = u$  if  $J(u) \notin [c - \varepsilon_0, c + \varepsilon_0]$ ;
- ii)  $\eta(1, J^{c+\varepsilon}) \subset J^{c-\varepsilon}$  if  $K_c \neq \emptyset$ .

We construct the critical values of  $J$  in the form

$$c = \inf_{\mathcal{S} \in \mathcal{B}} \max_{u \in \mathcal{S}} J(u)$$

where  $\mathcal{B}$  is a nonempty class of subsets of  $\delta G_r$  which is invariant under the deformation  $\eta$ , i.e.  $\eta(1, \mathcal{J}) \in \mathcal{B}$  whenever  $\mathcal{J} \in \mathcal{B}$ . An usual routine proves that if  $c < \omega$  and  $(PS_c)$  holds, then  $J$  has at least one critical point on  $\delta G_r$  at the level  $c$ . A simpler and explicit form of such a kind of deformation technique is presented in [19].

In the following, a useful variant of the deformation lemma, for functions  $F$  with locally Lipschitz continuous derivatives in a Hilbert space  $H$ , i.e.  $F \in C^{1,1^-}(H; \mathbb{R})$ , takes account the change of topologies on the sets  $F^c$ , (see e.g. [22],[28]).

**Deformation Lemma 2.** Let  $F \in C^{1,1^-}(H; \mathbb{R})$  and  $a < b$  be such that  $[a, b]$  does not contain any critical point of  $F$  and  $F$  satisfies the Palais-Smale condition on  $F^{-1}([a, b])$ . Then there exists a continuous  $\eta: [0, 1] \times F^a \longrightarrow F^b$  such that

$$\begin{aligned} \eta(t, u) &= u \quad \text{if } (t, u) \in [0, 1] \times F^a; \\ \eta(0, u) &= u \quad \text{if } u \in F^b; \\ \eta(1; F^b) &= F^a; \\ \eta(1; F^{-1}[a, b]) &\subset F^{-1}\{a\} \end{aligned}$$

and  $f(\eta(\cdot, u))$  is decreasing on  $[0, 1]$ .

Let  $A \subset B$  be sets in  $X$ . We say that  $A$  is a *strong deformation retract* of  $B$  if there is a continuous map  $\eta: [0, 1] \times B \longrightarrow B$  such that  $\eta(t, u) = u$  for all  $(t, u) \in I \times A$ ,  $\eta(0, u) = u$  and  $\eta(1, u) \in A$  for all  $u \in B$ . Then, Lemma 2 proves that  $F^a$  is a strong deformation retract of  $F^b$ .

## 2. Nonsmooth variational bifurcation

Let  $H$  be a real Hilbert space,  $W$  an open subset of  $H$  containing the origin and  $F: W \longrightarrow \mathbb{R} \cup \{+\infty\}$  a function. The function  $F$  is said to be *lower subdifferentiable* at  $u \in D(F)$  if there exists  $\alpha \in H$  such that

$$\liminf_{v \rightarrow u} \frac{F(v) - F(u) - (\alpha, v - u)}{\|v - u\|} \geq 0$$

For every  $u \in D(F)$  we denote by  $\partial^- F(u)$  the (possibly empty) set of such

$\alpha$ 's and  $D(\partial^-F) = \{u \in D(F) \mid \partial^-F(u) \neq \emptyset\}$ . When  $F$  is l.s.c., then the domain  $D(\partial^-F)$  is dense in  $D(F)$ . Of course, if  $W = H$  and  $F$  is convex, the notion of  $\partial^-F$  coincides with the usual subdifferential in convex analysis.

Moreover, for two lower subdifferentiable functions  $F, G: W \rightarrow \mathbb{R} \cup \{+\infty\}$  and  $u \in D(\partial^-(F+G))$  we have

$$\partial^-F(u) + \partial^-G(u) \subseteq \partial^-(F+G)(u),$$

the equality holds if one of the functions is Fréchet differentiable.

It is easy to check that  $\partial^-F(u)$  is closed and convex in  $H$ . If  $\partial^-F(u) \neq \emptyset$ , the element of minimal norm of  $\partial^-F(u)$  is called *subgradient of  $F$  at  $u$*  and is denoted by  $\text{grad}^-F(u)$ . A point  $u \in D(\partial^-F)$  is called *critical from below* if  $0 \in \partial^-F(u)$ , actually  $\text{grad}^-F(u) = 0$ , and the corresponding real number  $c = F(u)$  is called a *critical value*.

Let  $V$  be a subset of  $H$  and  $I_V: H \rightarrow \mathbb{R} \cup \{+\infty\}$  its characteristic function. Here  $V$  plays the role of a constraint of the problem. For  $u \in V$  and  $v \in H$ , we say that  $v$  is an outward normal to  $V$  at  $u$  provided  $v \in \partial^-I_V(u)$ . If  $u \in D(\partial^-F) \cap V$ , we also say that  $u$  is a *lower critical point of  $F$  on  $V$* , provided  $0 \in \partial^-(F + I_V)(u)$ . This relation

$$\partial^-F(u) + \partial^-I_V(u) \ni 0$$

extends in a certain sense the Lagrange multipliers rule.

Some classes of functions are needed, for which relaxed subgradient inequalities hold. These classes include the functions of the form  $F + I_V$ . Thus, a lower semicontinuous  $F$  is said to be *q-convex of order  $r$* ,  $r \geq 0$ , if there is a continuous function  $q: D(F) \rightarrow \mathbb{R}^+$  such that

$$F(v) \geq F(u) + (\alpha, v - u) - q(u)(1 + \|\alpha\|^r)\|v - u\|^2, \quad \forall u, v \in D(F)$$

for some  $\alpha \in H$ . When  $r = 0$  the respective functions will be simply called *q-convex*. We agree that the previous inequality holds whenever  $\partial^-F(u) \neq \emptyset$ . The function  $F$  is convex on convex open parts of  $W$  where  $q(u) = 0$ . Moreover, if  $F: H \rightarrow \mathbb{R} \cup \{+\infty\}$  is convex and  $G: H \rightarrow \mathbb{R}$  is of class  $C^2$ , then  $F + G$  is q-convex for some suitable constant  $q$ .

Now if  $M$  is a hypersurface in  $H$  of class  $C^2$ , then for every  $u \in M$ , we have  $\partial_M^-I(u) = \{\lambda v(u) \mid \lambda \in \mathbb{R}\}$ , where  $v(u)$  is a normal vector to  $M$  at  $u$ .

Besides, let  $A$  and  $B$  be two sets in  $H$  and  $u \in A \cap B$ . Then  $A$  and  $B$  are said to be (outwardly) *tangent* at  $u$ , if there is  $\nu \neq 0$  such that

$$\nu \in \partial^- I_A(u) \quad \text{and} \quad -\nu \in \partial^- I_B(u).$$

If there are no such points in  $A \cap B$  we say that  $A$  and  $B$  are nontangent.

Suppose further that  $F$  is  $q$ -convex and  $M$  has a finite codimension. If  $u_0 \in D(F) \cap M$  such that  $D(F)$  and  $M$  are not tangent in  $u_0$ , then  $F + I_M$  is a  $q$ -function of order 1 and

$$\partial^-(F + I_M)(u_0) = \partial^- F(u_0) + \partial^- I_M(u_0).$$

More precisely, if for some  $u \in D(\partial F) \cap M$  and  $\alpha \in \partial^-(F + I_M)(u)$  there exists a unique  $\lambda \in \mathbb{R}$  such that  $\alpha - \lambda \nu(u) \in \partial^- F(u)$  and  $\lambda$  depends continuously on these  $u$  and  $\alpha$ .

An appropriate Palais-Smale condition at level  $c$  reads:

(PS<sub>c</sub>) For every sequence  $\{u_n\} \subset D(\partial^- F)$  with  $F(u_n) \rightarrow c$  and  $\text{grad}^- F(u_n) \rightarrow 0$  there exists a subsequence converging to an element of  $W$ .

In addition, we can prove the following [10]:

**Deformation Lemma 3.** Let  $F$  be a  $q$ -convex function and  $-\infty < a \leq b < +\infty$ . Suppose that  $[a, b]$  does not contain any critical point of  $F$ , the set  $F^c$  is closed in  $H$  for every  $c \in [a, b]$  and  $F$  satisfies the Palais-Smale condition on  $[a, b]$ . Then  $F^a$  is a strong deformation retract of  $F^b$ .

Now, we look for eigensolutions  $(\lambda, u) \in \mathbb{R} \times D(\partial^- F)$  of the inclusion

$$\partial^- F(u) - \lambda u \ni 0. \quad (5)$$

Later on, we assume that  $F(0) = 0$  and  $0 \in \partial^- F(0)$ . Then the pair  $(\lambda, 0)$  satisfies (5) for every  $\lambda \in \mathbb{R}$ . Moreover,  $(\lambda, 0)$  is a *bifurcation point* for (5) if there is a sequence  $\{(\lambda_n, u_n)\} \subset \mathbb{R} \times (W \setminus \{0\})$  of solutions of (5) such that

$$\lim_{n \rightarrow \infty} (\lambda_n, u_n) = (\lambda, 0).$$

Assume henceforth that  $F$  is a l.s.c. function only which does not satisfy the usual regularity assumptions. For such functions it is possible to define a generalized Hessian form and to show that certain eigenvalues of this are bifurcation values of (5). We can state the first result

Proposition 1. Let  $(\lambda, u)$  be a solution of (5) with  $u \neq 0$ . Then  $u$  is a lower critical point of  $F$  on the sphere  $S_r = \{v \in H \mid \|v\| = r\}$ , where  $r = \|u\|$ .

For the converse additional restrictions are required. First, we recall the concept of  $\Gamma$ -convergence or *epiconvergence*, [1].

Let  $\Psi_n: H \rightarrow \mathbb{R} \cup \{+\infty\}$  be a sequence of functions. We say that

$$\Psi_0 = \Gamma\text{-}\lim_{n \rightarrow \infty} \Psi_n$$

if the following facts hold:

j) for every  $u \in H$  and every sequence  $\{u_n\} \subset H$  converging to  $u$ ,

$$\Psi_0(u) \leq \liminf \Psi_n(u_n);$$

jj) for every  $u \in H$  there is a sequence  $\{w_n\} \subset H$  converging to  $u$ ,

such that

$$\Psi_0(u) = \lim \Psi_n(w_n).$$

In order to characterize the bifurcation points of (5), we introduce the functions  $F_\rho: H \rightarrow \mathbb{R} \cup \{+\infty\}$ , with  $\rho > 0$ , by  $F_\rho(u) = \rho^{-2} F(\rho u)$ , and we make the following assumptions on  $F$ :

A)  $F$  is l.s.c. and  $q$ -convex;

B) there is a function  $F_0: H \rightarrow \mathbb{R} \cup \{+\infty\}$  such that

$$F_0 = \Gamma\text{-}\lim_{\rho \rightarrow 0} F_\rho.$$

Among some nice properties of  $F_\rho$  is the weak-strong closure: let  $\{\rho_n\}$  be a sequence in  $]0, 1[$ ,  $\{u_n\}$  and  $\{\alpha_n\}$  two sequences in  $H$ . Let us suppose that  $\rho_n \rightarrow \rho \in [0, 1]$ ,  $u_n \rightarrow u$ ,  $\alpha_n \rightarrow \alpha$  and that  $\alpha_n \in \partial^- F(u_n)$ . Then

$$\alpha \in \partial^- F_\rho(u) \quad \text{and} \quad F_\rho(u) = \lim_n F_\rho(u_n).$$

The function  $F_0(u)$  replaces the role of the quadratic form  $\frac{1}{2} F''(0)(u, u)$  in the smooth case. In the following it will be convenient to consider also the "linearized" problem

$$\partial^- F_0(u) - \lambda u \ni 0 \quad \text{with} \quad (\lambda, u) \in \mathbb{R} \times H, \quad u \neq 0. \quad (6)$$

We note that if  $(\lambda, u)$  is an eigensolution of (6), then  $F_0(u) = \frac{\lambda}{2} (u, u)$ .

A first result concerns a necessary condition for having bifurcation:

**THEOREM 1.** Let us assume that A) - B) hold and for every sequence  $\{u_n\}$  in  $W \setminus \{0\}$  with

$$\lim_n u_n = 0 \quad \text{and} \quad \sup_n \|u_n\|^{-2} F(u_n) < +\infty, \quad (7)$$

the sequence  $\left\{ \frac{u_n}{\|u_n\|} \right\}$  has a convergent subsequence. If  $(\lambda, 0)$  is a bifurcation point for (5), then  $\lambda$  is an eigenvalue for  $\partial^- F_0$ .

Some counterexamples prove that the converse is not true in general. A sufficient condition requires the notion of essential critical value [16].

A real number  $c$  is said to be an *essential critical value* for  $F_0$  on  $S_1$  if there is  $\varepsilon > 0$  such that

- $\{v \in S_1 \mid F_0(v) \leq c - \varepsilon\}$  is not a deformation retract of  $\{v \in S_1 \mid F_0(v) \leq c + \varepsilon\}$ ;
- $c$  is the unique critical value of  $F_0$  on  $S_1$  in  $[c - \varepsilon, c + \varepsilon]$ .

We can prove the converse statement:

**THEOREM 2.** Assume that A) - B) and (7) hold. If  $\frac{\lambda}{2}$  is an essential critical value for  $F_0$  on  $S_1$  then  $\lambda$  is a bifurcation point for  $\partial^- F$ .

Additional information can be derived when  $F$  is an even function and the associated  $F_0$  fulfils the following condition

$$F_0(u+v) + F_0(u-v) = 2F_0(u) + 2F_0(v).$$

For every eigenvalue  $\lambda$  of  $\partial^- F_0$  the set  $\{u \mid (\lambda, u) \text{ satisfies (6)}\}$  is a linear space of finite dimension; this dimension is called the *multiplicity* of  $\lambda$ .

**COROLLARY 1.** For every eigenvalue  $\lambda$  of  $\partial^- F_0$  which has multiplicity  $k$ , there exist for  $\rho$  small,  $2k$  distinct eigensolutions  $(\lambda_i^{(\rho)}, u_i^{(\rho)})$ ,  $i = 1, \dots, k$  of (5) such that  $\|u_i^{(\rho)}\| = \rho$  and  $\lambda_i^{(\rho)} \rightarrow \lambda$  as  $\rho \rightarrow 0$  for  $i = 1, \dots, k$ .

We return now to the buckling of a thin elastic plate subjected to symmetric constraints.

Let  $K_0$  be the closure of the set  $\left[ \bigcup_{t>0} tK \right]$  in  $H_0^2(\Omega)$ .

Since  $-K = K$ ,  $K_0$  is a closed linear subspace in  $H_0^2(\Omega)$  and

$$\begin{cases} (\lambda, u) \in \mathbb{R} \times K_0, \\ (\Delta^2 u, v) = \lambda([\phi, u], v), \quad \forall v \in K_0 \end{cases} \quad (8)$$

can be regarded as a linearized problem associated with (3). Every eigenvalue of (8) is different from zero and we can prove [12]-[13]:

**THEOREM 3.** A real number  $\lambda$  is a bifurcation point for (3) if and only if  $\lambda$  is an eigenvalue of (8).

Let  $\partial G_r = \{ u \in H_0^1(\Omega) \mid \int_{\Omega} [\phi, u] u \, dx = r \}$ . Sufficient conditions of the nontangency of sets  $K$  and  $\partial G_r$  at any point  $u \in K \cap \partial G_r$  is one of the following:

- 1)  $\varphi = +\infty$  i.e. there is no obstacle;
- 2)  $\varphi \in H^2(\Omega)$  and  $r$  is sufficiently large;
- 3)  $\varphi \in H^2(\Omega)$  and  $[\phi, \varphi] \leq 0$ , in particular,  $\varphi$  is constant;
- 4) for some  $\varepsilon > 0$  we have  $\varphi \geq \varepsilon$  and there is a constant  $k > 0$  such that  $(\phi''(x, y)\zeta, \zeta) \leq -k|\zeta|^2$  whenever  $x, y \in \Omega$  and  $\zeta \in \mathbb{R}^2$ .

Let  $r \neq 0$  such that  $K \cap \partial G_r \neq \emptyset$ . If one of the above conditions of nontangency holds, there exist  $u \in K \cap \partial G_r$  which renders  $J$ , given by (4), stationary. Due to symmetry, there exist actually infinitely many solutions  $(\lambda_n, u_n, h_n)$  of (3) with  $u_n \in \partial G_r$  and  $\sup_n \lambda_n = +\infty$ .

### 3. Eigenvalue variational inequalities with symmetric constraint.

Let  $\Omega$  be a domain in  $\mathbb{R}^N$ ,  $g: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  a perturbation satisfying assumptions (G),  $\varphi: \Omega \rightarrow [0, +\infty]$  a l.s.c. obstacle and  $\rho > 0$  an assigned number.

Take  $H = L^2(\Omega)$  with the usual inner product. Set  $G(x, t) = \int_0^t g(x, \sigma) d\sigma$ ,  
 $K = \{ v \in H \mid |v| \leq \varphi \}$ ,  $K_g = \{ v \in K \cap H_0^1(\Omega) \mid G(\cdot, v) \in L^1(\Omega) \}$  and  
 $S_r = \{ v \in H \mid \|v\| = r \}$ . Our problem looks for eigensolutions  $(\lambda, u)$  such that

$$\begin{cases} \lambda \in \mathbb{R}, \quad u \in K_g \cap S_r, \quad g(x, u)(v - u) \in L^1(\Omega), \\ \int_{\Omega} [\nabla u \cdot \nabla(v - u) + g(x, u)(v - u)] dx \geq \lambda \int_{\Omega} u(v - u) dx \quad \forall v \in K, \end{cases} \quad (9)$$

where  $\nabla u = \left( \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N} \right)$ . When  $\varphi = +\infty$ , we have a nonlinear eigenvalue for the Laplace operator.

Assumptions (G) upon the nonlinear perturbation are:

(G.1)  $g(x,t)$  is measurable with respect to  $x$ , for all  $t \in \mathbb{R}$ , and  $g(x, \cdot)$  is of class  $C^1$ , for almost all  $x \in \Omega$ ;

(G.2)  $G(x,t) \leq a(x) + bt^2$ ,  $\forall (x,t) \in \Omega \times \mathbb{R}$ ,

for a suitable  $a \in L^2(\Omega)$  and a constant  $b > \lambda_1$  where  $\lambda_1$  is the first eigenvalue of  $(-\Delta, H_0^1)$ ;

(G.3) there is  $c \in \mathbb{R}$  such that  $g(x,t) \leq c$  for almost all  $x \in \Omega$  and for every  $t \in \mathbb{R}$ .

Let us consider the function  $F: H \rightarrow \mathbb{R} \cup \{+\infty\}$ , defined by

$$F(u) = \begin{cases} \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} G(x,u) dx & \text{if } u \in K_g, \\ +\infty & \text{otherwise.} \end{cases}$$

By assumptions (G),  $F$  is l.s.c. and  $D(F) = K_g$  while  $g(x,u)(u-v)$  and  $g(x,u)u$  are integrable for every  $u \in H$  and  $v \in K_g$ . The set  $K_g$  is convex and by a standard way we can check that  $F$  is bounded from below, [17].

Also, it is readily seen that  $K_g$  is convex and, for any  $\tau \in [0,1]$ , we have

$$G(x, \tau t_1 + (1-\tau)t_2) \geq \tau G(x, t_1) + (1-\tau)G(x, t_2) - \tau(1-\tau) \frac{c}{2} (t_2 - t_1)^2.$$

Then, for  $u, v \in D(F)$  and  $\tau \in [0,1]$ ,  $F$  will satisfy the opposite inequality

$$F((1-\tau)u + \tau v) \leq (1-\tau)F(u) + \tau F(v) + \tau(1-\tau) \frac{c}{2} \int_{\Omega} (u-v)^2 dx.$$

Since

$$G(x, u + \tau(v-u)) - G(x, u) \geq \tau[G(x, v) - G(x, u)] - \tau(1-\tau) \frac{c}{2} (v-u)^2.$$

we can infer

$$\lim_{\tau \rightarrow 0^+} \frac{F(u + \tau(v-u)) - F(u)}{\tau} \leq \int_{\Omega} \nabla u \nabla(v-u) dx - \int_{\Omega} g(x, u)(v-u) dx$$

and

$$F(v) \geq F(u) + \int_{\Omega} \nabla u \cdot \nabla(v - u) \, dx - \int_{\Omega} g(x, u)(v - u) \, dx - \frac{c}{2} \int_{\Omega} (v - u)^2 \, dx$$

whence  $g(x, u)(v - u) \in L^1(\Omega)$ . We easily deduce [8]:

a) If  $u \in D(\partial^- F)$  and  $\alpha \in \partial^-(u)$ , then  $F$  is  $\frac{c}{2}$ -convex, i.e.

$$F(v) \geq F(u) + \langle \alpha, v - u \rangle - \frac{c}{2} \|u - v\|^2 \quad \forall v \in D(F);$$

b) If  $u \in D(\partial^- F)$ , then  $\alpha \in \partial^- F(u)$  if and only if

$$\int_{\Omega} \nabla u \cdot \nabla(v - u) \, dx - \int_{\Omega} g(x, u)(v - u) \, dx \geq \int_{\Omega} \alpha(v - u) \, dx \quad \forall v \in D(F).$$

Furthermore,  $D(F) \cong H_0^1(\Omega) \cap L^\infty(\Omega)$  and  $\partial^- F(u) = \emptyset$  is equivalent to  $g(x, u) \in L^1(\Omega)$  and  $\Delta u - g(x, u)$ , taken in the sense of distributions, belongs to  $L^2(\Omega)$ .

Now, an additional constraint yields an eigensolution for the problem (9), when  $D(F) \cap S_r \neq \emptyset$ . Suppose further that  $D(F)$  and  $S_r$  are not tangent and define  $F_0 = F + I_{S_r}$ . Then  $F_0$  is  $\frac{c}{2}$ -convex of order 1 and

$$\partial^- F_0(u) = \partial^- F(u) + \{\lambda u \mid \lambda \in \mathbb{R}\} \quad \forall u \in D(\partial^- F).$$

Consequently, if  $u \in D(\partial^- F_0)$  then  $\alpha \in \partial^- F_0(u)$  if and only if

$$\int_{\Omega} [\nabla u \cdot \nabla(v - u) - g(x, u)(v - u)] \, dx + \lambda \int_{\Omega} u(v - u) \, dx \geq \int_{\Omega} \alpha(u - v) \, dx \quad \forall v \in K_g \cap S_r.$$

In particular, we have established:

**THEOREM 4.** Assume that hypotheses (G) are fulfilled while  $D(F)$  and  $S_r$  are not tangent. Then  $(\lambda, u) \in \mathbb{R} \times D(\partial^- F)$  is an eigensolution of the problem (9) if and only if  $0 \in \partial^- F(u)$ .

Moreover, let  $\varphi \in H^1(\Omega)$  and  $r > \|\varphi\|$  and suppose the oddness property

$$g(x, -t) = -g(x, t) \quad \forall x \in \Omega, \forall t \in \mathbb{R}.$$

Then there exist infinitely many eigensolutions  $(\lambda, \pm u)$  of the problem (9).

In our simple case, the nontangency condition can be specified for an obstacle  $\varphi \in C(\bar{\Omega}) \cap H^1(\Omega)$  in the following analytic way: there exists no open subset  $\Omega'$  of  $\Omega$  such that  $\varphi < 0$  on  $\Omega'$  and  $\varphi \in H_0^1(\Omega')$ . Then  $D(F)$  and  $S_r$  are not tangent whenever  $r > \|\varphi\|$ , (see, [20]).

We specify now some facts regarding bifurcating solutions of the problem (9). As above, let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  and  $\varphi : \Omega \rightarrow [0, +\infty]$  a l.s.c. obstacle. We consider now the convex set  $K = \{v \in H_0^1(\Omega) \mid u(x) \leq \varphi(x) \text{ a.e. in } \Omega\}$ .

Upon the perturbation  $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  the following hypotheses are imposed:

(g.1)  $g(x, t)$  is measurable with respect to  $x$ , for all  $t \in \mathbb{R}$ , and  $g(x, \cdot)$  is of class  $C^1$ , for almost all  $x \in \Omega$ ;

(g.2)  $g(x, 0) = 0$  and there is  $c \in \mathbb{R}$  such that  $|g_t(x, t)| \leq c$  in  $\Omega \times \mathbb{R}$ .

In this framework, a corresponding linearization of (9) is given by

$$\begin{cases} (\lambda, u) \in \mathbb{R} \times K_0, \\ \int_{\Omega} [\nabla u \cdot \nabla(u - v) - g_t(x, 0)u(v - u)] dx \geq \lambda \int_{\Omega} u(v - u) dx \quad \forall v \in K_0, \end{cases} \quad (10)$$

where  $K_0$  is the closure in  $H_0^1(\Omega)$  of the set  $\{v \in C_0^\infty(\Omega) \mid u \leq 0 \text{ in } \varphi^{-1}(0)\}$ .

It is easy to see that  $K_0$  is a closed convex cone and to prove

**THEOREM 5.** Let  $(\lambda, 0)$  be a bifurcation point of (9). Then there exists  $u \neq 0$  such that  $(\lambda, u)$  is an eigensolution of (10).

Moreover, the first eigenvalue of (10) has the form

$$\lambda = \inf \left\{ \int_{\Omega} [|\nabla u|^2 - g_t(x, 0)u^2] dx \mid u \in K_0 \cap S_1 \right\}.$$

For a more detailed study of the multiplicity of solutions of these variational inequalities and for a sharp estimate of the number of the solutions we refer the reader to [25].

#### 4. Eigenvalue variational inequalities with gradient constraint.

As in the previous section, we consider on a bounded domain  $\Omega$  in  $\mathbb{R}^N$ ,  $N \geq 3$ , a perturbation  $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  and a l.s.c. obstacle  $\varphi : (0, +\infty] \rightarrow \mathbb{R}$ . As constraint we take  $K = \{v \in H_0^1(\Omega) \mid |\nabla v(x)| \leq \varphi(x) \text{ a.e. in } \Omega\}$ . Denote  $q = \frac{N}{N-2}$ .

Upon the function  $g$  we impose the following assumptions:

(i)  $g(x, t)$  is measurable with respect to  $x$ , for all  $t \in \mathbb{R}$ , and  $g(x, \cdot)$  is of class  $C^1$ , for almost all  $x \in \Omega$ ;

(ii) there are  $a \in L^q$  and  $b, c \in \mathbb{R}$  such that

$$-c \leq g_t(x, t) \leq a(x) + b|t|^{2(q-1)} \quad \forall (x, t) \in \Omega \times \mathbb{R};$$

(iii)  $g(x, 0) = 0 \quad \forall x \in \Omega$ .

The main problem looks for nontrivial eigensolutions  $(\lambda, u)$  such that

$$\begin{cases} \lambda \in \mathbb{R}, & u \in \mathbb{K}, \\ \int_{\Omega} [\nabla u \cdot \nabla(v - u) + g(x, u)(v - u)] dx \geq \lambda \int_{\Omega} u(v - u) dx & \forall v \in \mathbb{K}. \end{cases} \quad (11)$$

To the problem (11) we associate its linearized problem called also the asymptotic to (11) at zero: find the pairs  $(\lambda, u)$  such that

$$\begin{cases} \lambda \in \mathbb{R}, & u \in H_0^1(\Omega), u \neq 0, \\ \int_{\Omega} \nabla u \cdot \nabla v dx + \int_{\Omega} g_t(x, 0)v dx = \lambda \int_{\Omega} uv dx & \forall v \in H_0^1(\Omega). \end{cases} \quad (12)$$

Assume hypotheses (i)-(iii) hold. Let  $(\lambda_n, u_n)$  be a sequence of eigensolutions of (11) such that  $(\lambda_n, u_n) \rightarrow (\lambda, u)$  in  $\mathbb{R} \times L^2(\Omega)$ . Then  $\lambda$  is an eigenvalue of  $-\Delta + g_t(\cdot, 0)I$  with respect to  $H_0^1(\Omega)$ , ([16]).

Let us introduce the functionals  $F, F_0: L^2(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$  defined by

$$F(u) = \begin{cases} \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} \left( \int_0^{u(x)} g(x, s) ds \right) dx & \text{for } u \in \mathbb{K}, \\ +\infty & \text{for } u \in L^2(\Omega) \setminus \mathbb{K}, \end{cases}$$

and

$$F_0(u) = \begin{cases} \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} g_t(x, 0) u^2 dx & \text{for } u \in H_0^1(\Omega), \\ +\infty & \text{for } u \in L^2(\Omega) \setminus H_0^1(\Omega). \end{cases}$$

We can easily check that:

- a) for  $(\lambda, u) \in \mathbb{R} \times \mathbb{K}$  :  $(\lambda, u)$  is a solution of (11)  $\Leftrightarrow \lambda u \in \partial^- F(u)$  ;
- b) for  $(\lambda, u) \in \mathbb{R} \times H_0^1$  :  $(\lambda, u)$  is a solution of (12)  $\Leftrightarrow \lambda u \in \partial^- F_0(u)$ .

In addition, one may show that all eigenvalues of the problem (12) are essential and therefore the reciprocity between the bifurcation points of (11) and the eigenvalues of (12) stated in Theorem 2 works.

### 5. Other related approaches

An equivariant lower critical point theory, involving Ljusternik-Schnirelman category and cohomological index as well as the stability under epiconvergence of homotopy type of lower level sets are presented in [13]. These techniques are used in [6] to study the existence and properties of

closed geodesics on so called  $p$ -convex sets. Extensions to parabolic variational inequalities on nonconvex constraints are recently treated [20].

Finally, we should notice S.I.Pokhozhaev's fibration method [26]:

Let us consider a differentiable function  $f: X \rightarrow \mathbb{R}$  on a real Banach space with differentiable norm on  $X \setminus \{0\}$ . Set  $x = rv$  with  $r \in \mathbb{R}$  and define on  $\mathbb{R}_+ \times X$  the function  $F(r,v) = f(rv)$  for  $(r,v) \in (\mathbb{R} \setminus \{0\}) \times S_1$ . If  $(r,v)$  is a critical point of  $F$  restricted to  $(0,+\infty) \times S_1$ , then  $u = rv$  is a critical point of the function  $f$ . The spherical fibration is given by the scalar equation  $f'_v(rv) = 0$  with respect  $r = r(v)$  for  $v \in S_1$ . If  $f$  is differentiable then the solution  $r = r(v)$  is of class  $C^1(S_1)$  for every  $v \in S_1$ . To each critical point  $v_c \in S_1$  with  $r_c = r(v_c) \neq 0$  of the function  $f(r(v)v)$ , restricted to  $S_1$ , there corresponds a critical point  $u_c = r_c v_c$  of the original function  $f$ .

The principle, described above in its simplest form, turned out to be efficient for studying the number of solutions or the absence of solutions of noncoercive semilinear equations.

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