LIBERTAS MATHEMATICA, VOL. 12, 1992

FINITELY SUBADDITIVE OUTER MEASURES

Carmen Vlad

Let X be an abstract set and $\mathcal{L}$ a lattice of subsets of X, such that $\emptyset, X$ belong to $\mathcal{L}$. We denote by $\mathcal{A}(\mathcal{L})$ the algebra generated by $\mathcal{L}$ and by $M(\mathcal{L})$ the set of finite valued bounded finitely additive measures on $\mathcal{A}(\mathcal{L})$.

Definition 1. $\mu \in M(\mathcal{L})$ is a finitely subadditive outer measure if:

a) μ is nondecreasing;

b) $\mu\left(\bigcup_{i=1}^{n} E_i\right) \le \sum_{i=1}^{n} \mu(E_i)$, for any $E_1, E_2, \dots E_n \in X$;

c) $\mu(\emptyset) = 0$.

Definition 2. Let S_ν be the set of ν-measurable sets, with ν a finitely subadditive outer measure. ν is called cover regular if for any $A \subset X, \exists E \in S_\nu$ such that $A \subset E$ and $\nu(A) = \nu(E)$.

Let ν be a finitely subadditive outer measure. We say that a set E is measurable with respect to ν if for any $A \subset X$:

$$\nu(A) = \nu(A \cap E) + \nu(A \cap E').$$

Theorem 1. Let ν be a finitely subadditive outer measure and suppose $\nu(X)$ finite. Then $E \in S_\nu$ iff $\nu(X) = \nu(E) + \nu(E')$.

Proof. Let $A = X$ in the definition of ν-measurability. Conversely, ν is regular, hence for any set A, $\exists B \in S_\nu$, $A \subset B$ and $\nu(A) = \nu(B)$. B is ν-measurable, hence $\nu(E) = \nu(E \cap B) + \nu(E \cap B')$ and $\nu(E') = \nu(E' \cap B) + \nu(E' \cap B')$. Then $\nu(X) = \nu(E) + \nu(E') = \nu(E \cap B) + \nu(E \cap B') + \nu(E' \cap B) + \nu(E' \cap B') \ge$ $\nu[(E \cap B) \cup (E' \cap B)] + \nu[(E \cap B') \cup (E' \cap B')] = \nu(B) + \nu(B') = \nu(X)$. Hence:

$$\nu(B \cap E) + \nu(B \cap E') + \nu(E \cap B') + \nu(E' \cap B') = \nu(B) + \nu(B') \tag{1}$$

Also: $\nu(E \cap B') + \nu(E' \cap B') \ge \nu(B')$ implies

$$-\nu(E \cap B') - \nu(E' \cap B') \le -\nu(B') \tag{2}$$

By adding relations (1) and (2) we get

$$\nu(B\bigcap E)+\nu(B\bigcap E')\leq\nu(B).$$

Since $A\subset B$, we have $A\bigcap E\subset B\bigcap E$ and $A\bigcap E'\subset B\bigcap E'$, therefore $\nu(A\bigcap E)+\nu(A\bigcap E')\leq\nu(B\bigcap E')\leq\nu(B)=\nu(A)$, i.e.

$$\nu(A\bigcap E)+\nu(A\bigcap E')\leq\nu(A).$$

On the other hand, by the definition of ν as an outer measure, we have for $A=(A\bigcap E)\bigcup(A\bigcap)E'$:

$$\nu(A)=\nu[(A\bigcap E)\bigcup(A\bigcap E')]\leq\nu(A\bigcap E)+\nu(A\bigcap E').$$

Theorem 2. Let ν be a finitely subadditive outer measure and suppose $\nu(X)$ is finite. Then S_ν the set of all ν-measurable sets is an algebra.

Proof. Let E_1, $E_2\in S_\nu$. For all $A\subset X$:

$$\nu(A)=\nu(A\bigcap E_1)+\nu(A\bigcap E_1'),$$

and

$$\nu(A\bigcap E_1')=\nu[(A\bigcap E_1')\bigcap E_2]+\nu[(A\bigcap E_1'\bigcap E_2']=\nu[(A\bigcap E_1')\bigcap E_2]+\nu[A\bigcap(E_2\bigcup E_2)'].$$

But: $[(A\bigcap E_1')\bigcap E_2\bigcup(A\bigcap E_1)=[(A\bigcap E_1'\bigcup(A\bigcap E_1))]\bigcap[E_2\bigcup(A\bigcap E_1)]=$
$=[A\bigcap(E_1'\bigcup E_1)]\bigcap[E_2\bigcup(A\bigcap E_1)]=A\bigcap[E_2\bigcup(A\bigcap E_1)]=(A\bigcap E_2)\bigcup(A\bigcap E_1)=$
$=A\bigcap(E_1\bigcup E_2)$.

Hence, $\nu\{[(A\bigcap E_1')\bigcap E_2]\bigcup(A\bigcap E_1)\}=\nu[A\bigcap(E_1\bigcup E_2)]$.

By making substitutions in the first relation we get:

$\nu(A)=\nu(A\bigcap E_1)+\nu[(A\bigcap E_1')\bigcap E_2]+\nu[A\bigcap(E_1\bigcup E_2)']\geq\nu\{A\bigcap E_1)\bigcup[(A\bigcap E_1')\bigcap E_2]+$
$+\nu[A\bigcap(E_1\bigcup E_2)']=\nu[A\bigcap(E_1\bigcup E_2)]+\nu(A\bigcap(E_1\bigcup E_2)']$.

Hence, $E_1\bigcup E_2\in S_\nu$. By induction, suppose that any union of n ν-measurable sets is ν-measurable. Then, since $\bigcup_{k=1}^{n+1}E_k=\left(\bigcup_{k=1}^{n}E_k\right)\bigcup E_{n+1}$, it follows that $\bigcup_{k=1}^{n+1}E_k$ is also ν-measurable, by the above proof. Therefore, any finite union of ν-measurable sets is ν-measurable.

Now, to show that if $E\in S_\nu$ then $E'\in S_\nu$. Since $A\bigcap E=A\bigcap(E')'$ we have for all $A \in X$:

$$\nu(A) = \nu(A \cap E) + \nu(A \cap E') = \nu(A \cap (E')') + \nu(A \cap E').$$

Theorem 3. Let ν be a finitely subadditive outer measure and suppose $\nu(X)$ is finite. The restriction $\nu | S_\nu = \mu$ is a finitely additive measure.

Proof. Let E_1, $E_2 \in S_\nu$, $A \in X$ and $E_1 \cap E_2 = \emptyset$. $E_1 \in S_\nu$ implies $\mu(A) = \mu(A \cap E_1) + \mu(A \cap E_1')$. By replacing A by $A \cap E_1$ and $A \cap E_1'$ we get:

$$\mu(A \cap E_1) = \mu(A \cap E_1 \cap E_2) + \mu(A \cap E_1 \cap E_2'); \quad \mu(A \cap E_1') = \mu(A \cap E_1' \cap E_2) + \mu(A \cap E_1' \cap E_2').$$

Hence,

$$\mu(A) = \mu(A \cap E_1 \cap E_2) + \mu(A \cap E_1 \cap E_2') + \mu(A \cap E_1' \cap E_2) + \mu(A \cap E_1' \cap E_2'). \tag{3}$$

Now, replace here A by $A \cap (E_1 \cup E_2)$; we get the following sets:

$$[A \cap (E_1 \cup E_2)] \cap E_1 \cap E_2 = A \cap E_1 \cap E_2; \quad [A \cap (E_1 \cup E_2)] \cap E_1 \cap E_2' = A \cap E_1 \cap E_2'$$
$$[A \cap (E_1 \cup E_2)] \cap E_1' \cap E_2 = A \cap E_1' \cap E_2; \quad [A \cap (E_1 \cup E_2)] \cap E_1' \cap E_2' = \emptyset.$$

Therefore we get from (3):

$$\mu[A \cap (E_1 \cup E_2)] = \mu(A \cap E_1 \cap E_2) + \mu(A \cap E_1 \cap E_2') + \mu(A \cap E_1' \cap E_2) = \mu(A \cap E_1) + \mu(A \cap E_2)$$

In general, for E_1, E_2, ... E_n disjoint sets of S_ν we get by induction:

$$\mu\left(A \cap \bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu(A \cap E_i). \tag{4}$$

If now we replace in (4) A by $E = \bigcup_{i=1}^{n} E_i \in S_\nu$, we get $\mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu(E_i)$, which shows that $\mu = \nu | S_\nu$ is finitely additive measure.

Definition 3. Let $\mu \in M(\mathcal{L})$ and define $\mu'(E) = \inf \mu(L')$, $E \subset L'$, $L \in \mathcal{L}$, $E \subset X$.

Theorem 4. For μ' defined as above we have: $\mu \le \mu'$ on $\mathcal{L}$ and $\mu = \mu'$ on $\mathcal{L}'$.

Proof. Let $A \in \mathcal{L}$ and $\mu'(A) = \inf \mu(L')$, $A \subset L'$, $L \in \mathcal{L}$. $A \subset L'$ implies $\mu(A) \le \mu(L')$ for all $A \subset L'$, $L \in \mathcal{L}$. Then $\mu(A) \le \mu'(A)$ i.e. $\mu \le \mu'$ on $\mathcal{L}$.

For $B \in \mathcal{L}'$, $B \subseteq B$, hence $\mu'(B) = \mu(B)$ i.e. $\mu = \mu'$ on $\mathcal{L}'$.

Theorem 5. μ' is finitely subadditive outer measure.

Proof. a) $\mu(L') \geq 0$ all $L \in \mathcal{L}$, hence $\mu'(E) \geq 0$ all $E \subset X$.

b) Let $E_1 \subset E_2 \subset X$; then, clearly, $\mu'(E_1) \leq \mu'(E_2)$.

c) Let $E_i \subset X$, $i = 1, 2, \ldots N$.

$$\mu'(E_i) = \inf\{\mu(L_i') \mid E_i \subset L_i',\ L_i \in \mathcal{L}\}.$$

Therefore $\mu'(E_i) + \varepsilon/N \geq \mu(L_i')$; $E_i \subset L_i'$ implies $\bigcup_{i=1}^{N} E_i \subset \bigcup_{i=1}^{N} L_i'$ and so we have

$$\mu'\left(\bigcup_{i=1}^{N} E_i\right) \leq \mu'\left(\bigcup_{i=1}^{N} L_i'\right) = \mu\left(\bigcup_{i=1}^{N} L_i'\right) \leq \sum_{i=1}^{N} \mu(L_i') \leq \sum_{i=1}^{N} [\mu'(E_i) + \varepsilon/N] = \sum_{i=1}^{N} \mu'(E_i) + \varepsilon,$$

with ε arbitrary small, then $\mu'\left(\bigcup_{i=1}^{N} E_i\right) \leq \sum_{i=1}^{N} \mu'(E_i)$.

d) $\mu'(\emptyset) = \inf\{\mu(L'), \emptyset \subset L', L \in \mathcal{L}\} = 0$.

Theorem 6. Define $\mathcal{L}_\mu = \{L \in \mathcal{L} \mid \mu(L) = \mu'(L)]$. Then $\mathcal{L}_\mu$ is a lattice and $\emptyset$, $X \in \mathcal{L}_\mu$.

Proof. $\mu(\emptyset) = \mu'(\emptyset) = 0$, therefore $\emptyset \in \mathcal{L}_\mu$
$\mu'(X) = \mu(X)$, therefore $X \in \mathcal{L}_\mu$.

Let L_1, $L_2 \in \mathcal{L}_\mu$. Then $\mu(L_1) = \mu'(L_1)$ and $\mu(L_2) = \mu'(L_2)$

$$\mu(L_1 \cup L_2) \leq \mu'(L_1 \cup L_2) \leq \mu'(L_1) + \mu'(L_2) = \mu(L_1) + \mu(L_2) = \mu(L_1 \cup L_2).$$

Hence $\mu'(L_1 \cup L_2) = \mu(L_1 \cup L_2)$ i.e. $L_1 \cup L_2 \in \mathcal{L}_\mu$,

Now, there exist $\tilde{L}_1' \supset L_1$ and $\tilde{L}_2' \supset L_2$, $\tilde{L}_1$, $\tilde{L}_2 \in \mathcal{L}$ such that :

$$\mu(\tilde{L}_1') < \mu'(L_1) + \varepsilon/2 = \mu(L_1) + \varepsilon/2; \quad \mu(\tilde{L}_2') < \mu'(L_2) + \varepsilon/2 = \mu(L_2) + \varepsilon/2.$$

Then $\mu'(L_1 \cap L_2) \leq \mu(\tilde{L}_1' \cap \tilde{L}_2') = \mu(\tilde{L}_1') + \mu(\tilde{L}_2') - \mu(L_1 \cup L_2) = \mu(L_1 \cap L_2) + \varepsilon$, with ε arbitrary small, therefore $\mu'(L_1 \cap L_2) \leq \mu(L_1 \cap L_2)$. Since in general, $\mu(L_1 \cap L_2) \leq \mu'(L_1 \cap L_2)$; it follows that $\mu(L_1 \cap L_2) = \mu'(L_1 \cap L_2)$, hence $L_1 \cap L_2 \in \mathcal{L}_\mu$.

Let $S_{\mu'}$ be the set of all μ'-measurable sets, where e is measurable with respect to μ' if for any $A \in X$:

$$\mu'(A) = \mu'(A \cap E) + \mu'(A \cap E').$$

Theorem 7. $E \subset S_{\mu'}$ iff

$$\mu'(A') \geq \mu'(A' \bigcap E) + \mu'(A' \bigcap E') \quad \text{for all } A \in \mathcal{L}.$$

Proof. a) If $E \subset S_{\mu'}$, then the relation clearly holds. b) Let B be arbitrary in X, $B \subset A'$, $A \in \mathcal{L}$. We have:

$$\mu(A') = \mu'(A') \geq \mu'(A' \bigcap E) + \mu'(A' \bigcap E') \geq \mu'(B \bigcap E) + \mu'(B \bigcap E').$$

Since this is true for any $B \subset A'$, $A \in \mathcal{L}$

$$\mu'(B) \geq \mu'(B \bigcap E) + \mu'(B \bigcap E').$$

By the definition of μ' as an outer measure, we have for $B = (B \bigcap E) \bigcup (B \bigcap E')$: $\mu'(B) \leq \mu'(B \bigcap E) + \mu'(B \bigcap E')$. Therefore, $\mu'(B) = \mu'(B \bigcap E) + \mu'(B \bigcap E')$, B arbitrary in X.

Theorem 8. $\mathcal{L}_\mu \subset S_{\mu'}$.

Proof. Must show for $L \in \mathcal{L}_\mu$ and any $A' \in \mathcal{L}'$

$$\mu'(A') \geq \mu'(A' \bigcap L) + \mu'(A' \bigcap L').$$

There exist $\tilde{L}' \supset L$, $\tilde{L} \in \mathcal{L}$ such that $\mu(\tilde{L}') < \mu'(L) + \varepsilon/2 = \mu(L) + \varepsilon/2$. Now, $\mu'(A' \bigcap L') = \mu(A' \bigcap L')$ since $\mu = \mu'$ on $\mathcal{L}'$ and $\mu'(A' \bigcap L) \leq \mu(A' \bigcap L') = \mu(A') + \mu(\tilde{L}') - \mu(A' \bigcup \tilde{L}') \leq \mu(A') + \mu(L) + \varepsilon/2 - \mu(A' \bigcup L) = \mu(A' \bigcap L) = \varepsilon/2$. Then $\mu'(A' \bigcap L) = \mu(A' \bigcap L)$ and $\mu'(A' \bigcap L) + \mu'(A' \bigcap L') = \mu(A' \bigcap L) + \mu(A' \bigcap L') + \mu(A') = \mu'(A')$, i.e. $\mathcal{L}_\mu \subset S_{\mu'}$.

Theorem 9. Consider μ_1, $\mu_2 \in M(\mathcal{L})$ such that $\mu_1 \leq \mu_2$ on $\mathcal{L}$, $\mu_1(X) = \mu_2(X)$. Then $\mathcal{L}_{\mu_1} \subset \mathcal{L}_{\mu_2}$.

Proof. $\mu_1 \leq \mu_2$ on $\mathcal{L}$ then $\mu_2 \leq \mu_1$ on $\mathcal{L}'$, hence $\mu_2' \leq \mu_1'$ on all sets. Then on $\mathcal{L}$ we have: $\mu_1 \leq \mu_2 \leq \mu_2' \leq \mu_1'$. Therefore, if $L \in \mathcal{L}_{\mu_1}$ i.e. $\mu_1(L) = \mu_1'(L)$, then $\mu_2(L) = \mu_2'(L)$.

Theorem 10. $S_{\mu'} \bigcap \mathcal{L} = \mathcal{L}_\mu$.

Proof. $\mathcal{L}_\mu \subset S_{\mu'}$ by Theorem 6. Clearly, $\mathcal{L}_\mu \subset \mathcal{L}$; hence, $\mathcal{L}_\mu \subset S_{\mu'} \bigcap \mathcal{L}$. Conversely, let $L \in S_{\mu'} \bigcap \mathcal{L}$. Then let $L \in \mathcal{L}$ and $\mu'(A) = \mu'(A \bigcap L) + \mu'(A \bigcap L')$ all $A \in \mathcal{L}'$, μ finitely additive; therefore $\mu(A) = \mu(A \bigcap L) + \mu(A \bigcap L')$. Let $A = X$:

$\mu(X) = \mu(L) + \mu(L')$ and $\mu'(X) = \mu'(L) + \mu'(L')$ and then $\mu(L) + \mu(L') = \mu'(L) + \mu'(L')$. $\mu = \mu'$ on $\mathcal{L}'$ implies $\mu(L) = \mu'(L)$ i.e. $L \in \mathcal{L}_{\mu}$.

Definition 4. A measure $\mu \in M(\mathcal{L})$ is called $\mathcal{L}$-regular if for any $A \in \mathcal{A}(\mathcal{L})$, $\mu(A) = \sup\{\mu(L) \mid L \subset A,\ L \in \mathcal{L}\}$.

Theorem 11. Let $\mu \in M(\mathcal{L})$. Then $\mu = \mu'$ on $\mathcal{L}$ iff $\mu \in M_R(\mathcal{L})$.

Proof. $\mu = \mu'$ on $\mathcal{L}$ iff for any $L \in \mathcal{L}$, $\mu(L) = \inf\{\mu(L') \mid L \subset L',\ \tilde{L} \in \mathcal{L}\}$ iff μ is $\mathcal{L}$-regular.

REFERENCES

[1] A. D. Alexandroff, Additive set functions in abstract spaces, Mat. Sb. (N.S.) 8, 50 (1940), pp. 307-348.

[2] G. Bachman & A. Sultan, Regular lattice measures: mappings and spaces, Pacific Journal Math. 67, 2 (1976), pp. 291-321.

[3] L. Gillman & M. Jerrison, Rings of continuous functions, Van Nostrand, Princeton, NJ (1960).

[4] C. Vlad, Remarks on semiseparation of lattices, International Journal of Mathematics and Math. Sciences, Univ. of Central Florida, 14, 2 (1991), pp. 319-324.