LIBERTAS MATHEMATICA, VOL. 12, 1992

A REMARK ON LOGARITHMIC SOBOLEV CONSTANT

Andrzej Korzeniowski

Let $(E, \mathfrak{F}, m)$ be a probability space and $\{P_t, t > 0\}$ a strongly continuous, contractive, symmetric on $L^2(m)$ Markov semigroup for which m is an invariant measure. More precisely, let X_t be the corresponding Markov process with transition probability function $P(t,x,B) \equiv P(X_t \in B | X_0 = x)$ for $B \subset E$. Then $P_t f(x) = \int f(y) P(t,x,dy)$ for $f \in L^2(m)$ and $m(B) = \int P(t,x,B) m(dx)$.

The *Logarithmic Sobolev inequality* discovered by Gross [2] says

(LS) $$\int f^2 \log f^2 dm \le \alpha D(f,f) + \|f\|^2_{L^2(m)} \log \|f\|^2_{L^2(m)}$$

for some constant $\alpha > 0$, where $D(f,f)$ is the Dirichlet form ($= \langle f, Af \rangle$) for smooth f and $-A$ is the infinitesimal generator of the semigroup $\{P_t, t > 0\}$.

It is known, see Stroock [8], that (LS) is equivalent to the *Hypercontractivity property,* discovered by Nelson [5] in his work on constructive field theory,

(HC) $$\|P_t\|_{L^p(m) \to L^q(m)} = 1, \quad p < q < \infty \quad \text{for} \quad e^{\frac{4t}{\alpha}} \ge \frac{q-1}{p-1} .$$

Subsequently Rothaus [6] has shown that logarithmic Sobolev constant α is related to *spectral gap* μ as follows

(SG) $$\alpha \ge \frac{2}{\mu}, \qquad \mu = \inf\{D(f,f) \mid \|f\|_{L^2(m)} = 1, \int f dm = 0\}$$

where μ is the first positive eigenvalue of A .

Since initially many known classical examples where found to have $\alpha = \frac{2}{\mu}$ the question about whether it must always be the case was posed. The answer turned out to be negative due to an example of $\alpha > \frac{2}{\mu}$ given by Korzeniowski and Stroock [3]. For related spectral gap estimates involving both discrete and continuous Markov semigroup see Chen [1] .

The purpose of this note is to shed more light about what causes spectral gap inequality (SG) to be strict. To this end, based on the analysis given in [4], we state the following

THEOREM. Let P_t be *Hypercontractive* , α be the constant given in (LS) and μ be as in (SG) .

Then

$$\alpha = \frac{2}{\mu} \tag{1}$$

implies that each of the two conditions below must hold

(i) $\langle \varphi^3 \rangle = 0$

(ii) $\langle \varphi^4 \rangle \leq 9$

where ϕ is the eigenfunction corresponding to μ and $\langle \psi \rangle \equiv \int \psi dm$.

Equivalently, if either (i) or (ii) fails then

$$\alpha > \frac{2}{\mu} . \tag{2}$$

$$P_t\varphi = e^{-t\mu}\varphi, \quad P_t a = a, \quad \langle\varphi\rangle = 0, \quad \langle\varphi^2\rangle = 1$$

we get

$$2a^2(3\cdot 3^{-\frac{1}{2}\alpha\mu} - 1) + 4a3^{-\frac{3}{4}\alpha\mu}\langle\varphi^3\rangle + 3^{-\alpha\mu}\langle\varphi^4\rangle - 1 \quad \le \quad 0$$

for every $a \in R$, which implies $\alpha\mu \ge 2$, i.e. (SG) holds. Consequently $\alpha\mu = 2$ forces both (i) and (ii) to hold which concludes the proof.

In the following examples the one parameter semigroups P_t^λ given by means of their generators

$$-H_\lambda, \quad -J_\lambda, \quad -L_\lambda \text{ with } D \equiv \tfrac{d}{dx} \text{ illustrate the THEOREM .}$$

Example 1. (Hermite). For $\lambda > 0$ consider

$$H_\lambda = -\lambda D^2 + xD, \quad m_\lambda(dx) = \tfrac{1}{\sqrt{2\pi\lambda}} e^{-\frac{x^2}{2\lambda}} dx, \quad E = R.$$

Then

$$\varphi = \tfrac{x}{\sqrt{\lambda}}, \quad \langle\varphi\rangle = \langle\varphi^3\rangle = 0, \quad \langle\varphi^2\rangle = 1, \quad \langle\varphi^4\rangle = 3.$$

Since $\mu_\lambda = \mu = 1$ therefore (SG) gives $\alpha\mu \ge 2$. On the other hand by Nelson-Gross result $\alpha = \frac{2}{\mu}$ and (1) holds. Notice that both (i) and (ii) also (necessarily) hold.

Example 2. (Jacobi). For $\lambda > -\frac{1}{2}$ consider

$$J_\lambda = -(1-x^2)D^2 + (2\lambda+1)xD, \quad m_\lambda(dx) = \tfrac{\Gamma(\lambda+1)}{\sqrt{\pi\Gamma(\lambda+\frac{1}{2})}}(1-x^2)^{\lambda-\frac{1}{2}}dx, \quad E = [-1,1].$$

Then

$$\varphi = x\sqrt{2\lambda+2}, \quad \langle\varphi\rangle = \langle\varphi^3\rangle = 0, \quad \langle\varphi^2\rangle = 1, \quad \langle\varphi^4\rangle = 3\tfrac{\lambda+1}{\lambda+2}.$$

Since $\mu_\lambda = 2\lambda + 1$ then by (SG) $\alpha \geq \frac{2}{\mu_\lambda} = \frac{2}{2\lambda+1}$ and by Weissler-Mueller [9] result $\alpha = \frac{2}{\mu_\lambda}$

Here again (1) ,(i) and (ii) hold.

Example 3. (Laguerre). For $\alpha > 0$ consider

$$L_\lambda = -xD^2 + (x-\lambda)D, \quad m(dx) = \tfrac{x^{\lambda-1}}{\Gamma(\lambda)} e^{-x} dx, \quad E = R^+ .$$

Then

$$\varphi = \tfrac{x-\lambda}{\sqrt{\lambda}}, \quad \langle \varphi \rangle = 0, \quad \langle \varphi^2 \rangle = 1, \quad \langle \varphi^3 \rangle = \tfrac{3\lambda^2-1}{\sqrt{\lambda}}, \quad \langle \varphi^4 \rangle = 3 + \tfrac{6}{\lambda} .$$

Since $\mu_\lambda = \mu = 1$ and (i) fails, therfore $\vdash$ THEOREM $\alpha > \frac{2}{\mu} = 2$. Incidently for $\lambda < 1$ (ii) also fails. In the special case $\lambda = 1$ Korzeniowski-Stroock [3] have shown that $\alpha = 4$.

To complete the analysis of this example note that

$$(3) \qquad \left\| P_t^\lambda f - \langle f \rangle \right\|_{L^2(m_\lambda)} \leq e^{-t} \| f \|_{L^2(m_\lambda)}$$

and

$$(4) \qquad \left\| P_t^\lambda f \right\|_{L^4(m_\lambda)}^4 \leq \| f \|_{L^2(m_\lambda)}^4 \iint K_\lambda^4(x,y,t) m_\lambda(dx) m_\lambda(dy) < \infty \quad \text{for large t}$$

where

$$K_\lambda(x,y,t) = (1-e^{-t})^{-\lambda} \exp[-(x+y)\tfrac{e^{-t}}{1-e^{-t}}]\, \psi(xy\, \tfrac{e^{-t}}{(1-e^{-t})^2})$$

with

$$\psi_\lambda(x) = \sum_n \frac{x^n}{n!(\lambda+1)\ldots(\lambda+n-1)}$$

is the integral kernel of the semigroup P_t^λ for each $\lambda > 0$.

Conditions (3) and (4) imply Hypercontractivity (HP) as shown by Simon [7] .

<u>Remark.</u> It was conjectured in [3] that $\alpha > \frac{2}{\mu}$ may be attributed to degeneracy of L_λ at $x = 0$ but in view of the above it rather has to do with $\left\langle \varphi^3 \right\rangle \neq 0$.

References

[1] Chen, M. F. , Exponential L^2-Convergence and L^2-Spectral Gap for Markov Processes, Acta Mathematica Sinica, New Series, 1991,Vol.7, No.1, 19-37.

[2] Gross, L. , Logarithmic Sobolev inequalities, Amer. J. Math., 97 (1975), 1061-1083.

[3] Korzeniowski, A. and Stroock, D. W. , An example in the Theory of Hypercontractive Semigroups,Proc. Amer. Math. Soc. ,94 (1985), 87-90.

[4] Korzeniowski, A., On logarithmic Sobolev constant for diffusion semigroups, Journal Funct. Anal.,71(1987), 363-370.

[5] Nelson, E., Probability theory and Euclidean field theory, Lecture Notes in Physics, vol 25, Springer [illegible] ag, Berlin and New York, 1974, 94-124.

[6] Rothaus, O. S., Logarithmic Sobolev inequalities and the spectrum of Sturm-Liouville Operators, Journal Funct Anal. 39(1980), 42-56.

[7] Simon, B., Schrödinger Semigroups, Bull. AMS, 7(1982),447-526.

[8] Stroock, D. W., Notes on Hypercontractivity,(private communication).

[9] Weissler, F. and Mueller, C. E., Hypercontractivity for the heat semigroup for ultraspherical polynomials and on the n-sphere, Journal Funct. Anal. 48(1982), 252-283.