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ON HARDY TYPE INEQUALITY INVOLVING YOUNG FUNCTIONS

B.G.Pachpatte

ABSTRACT

The aim of the present note is to establish a new integral inequality involving Young functions which claim its origin to a certain variant of Hardy's inequality given by Isumi and Izumi.

1. INTRODUCTION

The integral inequality established by G.H. Hardy in [2] has, over the years, attracted considerable attention in the literature and a large number of papers have been appeared which deals with alternative proofs, various extensions and generalizations of this inequality, see [1-10] and the references given therein. With a view of proving an inequality sharper than the inequality established by Askey and Wainger [1], in [4, Theorem 3], Izumi and Izumi proved the following interesting variant of the Hardy's inequality given in [2].

Let $p > 1$ *and* $s > -1$ *and let* f *be a nonnegative, non-increasing and integrable function on* (o,π). *If* $x^s f^p(x)$ *is integrable, then we have*

$$(1)\quad \int_0^\pi x^s G^p(x)\,dx \leq A \int_0^\pi x^s \left(f\left(\frac{x}{2}\right) - f(x)\right)^p dx + A\left(\int_{\frac{\pi}{2}}^{\pi} f(x)\,dx\right)^p ,$$

where $G(x) = \int_{\frac{x}{2}}^{x} \frac{f(t)}{t}\,dt$ *and* A *is a constant depending only on* p *and* s.

The main purpose of this note is to establish a new integral inequality involving Young functions which claim its origin to the inequality (1) given by Izumi and Izumi [4,Theorem 3]. An interesting feature of the inequality is that it provides a new estimate on this type of inequality.

2. STATEMENT OF RESULT

In what follows, we shall say that F is a Young function if it admits the representation of the form

$$(2)\qquad F(u) = \int_0^u f(x)\,dx , \qquad u \geq 0 ,$$

where f is a real-valued function defined on $[0,\infty)$ such that (i) $f(0) = 0$; (ii) $f(x) > 0$ for $x > 0$; (iii) $f(x)$ is right continuous at any point $x \geq 0$; (iv) $f(x)$ is nondecreasing on $(0,\infty)$; (v) $\lim_{x\to\infty} f(x) = \infty$. We shall denote by YF the class of Young functions and write $F \in YF$ if F is a Young function. Young function F is said to satisfy the Δ_2-condition (we shall write $F \in \Delta_2$) if there exist constants $k > 0$ and $T \geq 0$ such that

$$F(2t) \leq kF(t)\,, \qquad t \geq T\,. \tag{3}$$

It is known that [5, Lemma 3.2.2, p.128], the Young function F is continuous, nonnegative, strictly increasing and convex on $[0,\infty)$. Moreover, it is easy to observe that Young function F satisfy

$$F(u) \leq uf(u) \leq F(2u)\,, \qquad u \geq 0\,. \tag{4}$$

For a discussion of the general theory of Young functions including the elementary facts quoted above the reader is referred to [5, Chapter 3].

Our main result is established in the following theorem.

THEOREM. *Let* $F \in YF \cap \Delta_2$ *and* $m > -1$, $p > 1$ *be constants. Let* $g(x)$ *be nonnegative, nonincreasing and integrable function on* $(0,\pi)$. *If* $G(x)$ *is defined by*

$$G(x) = \int_{\frac{x}{2}}^{x} \frac{g(t)}{t}\,dt\,, \qquad x \in (0,\pi)\,, \tag{5}$$

then

$$\int_0^\pi x^m F^p(G(x))\,dx \leq MF^p(G(\pi)) + N\int_0^\pi x^m \left\{ kF(g(\tfrac{x}{2})) - 2F(\tfrac{1}{2}\,g(x)) \right\}^p dx, \tag{6}$$

where $k > 0$ *is a constant as defined in* (3) *and*

$$M = \frac{2^{(m+2)(p-1)}\,\pi^{m+1}\,k^{p(p-1)}}{(m+1)^p}\,, \tag{7}$$

$$(8) \qquad N = 2^{p-1}\left(\frac{p}{m+1}\right)^p .$$

REMARK. We note that in (6) we cannot have $k \le 2$ since $F \in \Delta_2$, see [5, p.137]. In the special case when $f(t) = t^{r-1}$, $r > 1$, we have $F(t) = \frac{t^r}{r}$ and since F satisfies Δ_2-condition, we can take $k = 2^r$, the inequality obtained in (6) reduces to the inequality

$$(9) \qquad \int_0^\pi x^m G^{rp}(x)dx \le M_0 G^{rp}(\pi)$$
$$+ N\int_0^\pi x^m \left\{ 2^r g^r\left(\frac{x}{2}\right) - \left(\frac{1}{2}\right)^{r-1} g^r(x) \right\}^p dx,$$

where M_0 is defined by the right side of (7) by taking $k = 2^r$ and N is as defined in (8). The inequality (9) leads us to the interesting and new inequality in which the bound obtained on the right side is different from the bound given by Izumi and Izumi in (1).

3. PROOF OF THE THEOREM

Integrating by parts we have

$$(10) \qquad \int_0^\pi x^m F^p(G(x))dx = \left[\frac{x^{m+1}}{m+1}F^p(G(x))\right]_0^\pi - \int_0^\pi \frac{x^{m+1}}{m+1} pF^{p-1}(G(x))f(G(x))$$
$$\cdot \frac{1}{x}\left(g(x) - g\left(\frac{x}{2}\right)\right) dx.$$

From (10) we observe that

$$(11)\quad \int_0^\pi x^m F^p(G(x))dx = \frac{\pi^{m+1}}{m+1} F^p(G(\pi)) + \frac{p}{m+1}\int_0^\pi \left[(x^m)^{-(\frac{p-1}{p})} x^m f(G(x))\left(g(\tfrac{x}{2}) - g(x)\right)\right] \cdot \left[(x^m)^{(\frac{p-1}{p})} F^{p-1}(G(x))\right]dx .$$

Now applying Hölder's inequality with indices p, $\frac{p}{p-1}$ on the right side of (11) and dividing both sides of the resulting inequality by $\left\{\int_0^\pi x^m F^p(G(x))dx\right\}^{\frac{p-1}{p}}$ and then raising both sides to the pth power we get

$$(12)\quad \int_0^\pi x^m F^p(G(x))dx \leq \left[\frac{\frac{\pi^{m+1}}{m+1} F^p(G(\pi))}{\left\{\int_0^\pi x^m F^p(G(x))dx\right\}^{\frac{p-1}{p}}} + \frac{p}{m+1}\left\{\int_0^\pi x^m\left\{f(G(x))(g(\tfrac{x}{2})-g(x))\right\}^p dx\right\}^{\frac{1}{p}}\right]^p .$$

Using the elementary inequality (see, [7,p.338]).

$$(c_1 + c_2)^r \leq d_r(c_1^r + c_2^r),$$

where $c_1, c_2 \geq 0$ are real and $d_r = 2^{r-1}$, $(r > 1)$ and $d_r = 1$ $(0 \leq r \leq 1)$ on the right side of (12) we have

$$(13)\quad \int_0^\pi x^m F^p(G(x))dx \leq 2^{p-1}\left[\frac{\left(\frac{\pi^{m+1}}{m+1}\right)^p F^{p^2}(G(\pi))}{\left\{\int_0^\pi x^m F^p(G(x))dx\right\}^{p-1}} + \left(\frac{p}{m+1}\right)^p \int_0^\pi x^m \left\{f(G(x))(g(\tfrac{x}{2}) - g(x))\right\}^p dx\right].$$

From (5) we observe that

$$G(x) = \int_{\frac{x}{2}}^{x} \frac{g(t)}{t}\, dt \geq \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\frac{x}{2}} \frac{g(t)}{t}\, dt \geq \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \frac{g(t)}{t}\, dt \geq \int_{\frac{3\pi}{4}}^{\pi} \frac{g(t)}{t}\, dt,$$

for all x in $(\frac{\pi}{2},\pi)$ and hence

$$G(\pi) = \int_{\frac{\pi}{2}}^{\pi} \frac{g(t)}{t}\, dt = \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \frac{g(t)}{t}\, dt + \int_{\frac{3\pi}{4}}^{\pi} \frac{g(t)}{t}\, dt$$

$$\leq G(x) + G(x) = 2G(x),$$

for all such x. Since F is increasing and $F \in \Delta_2$, we observe that

$$(14) \qquad F(G(\pi) \leq F(2G(x)) \leq kF(G(x)) ,$$

where k is the constant as defined in (3). Now, in view of (14) we observe that

$$(15) \quad \int_0^{\pi} x^m F^p(G(x))dx \geq (\frac{\pi}{2})^m \int_{\frac{\pi}{2}}^{\pi} F^p(G(x))dx$$

$$\geq (\frac{\pi}{2})^{m+1} (\frac{1}{k})^p F^p(G(\pi)).$$

Using (15) on the right side of (13) we have

$$(16)\quad \int_0^\pi x^m F^p(G(x))dx \leq MF^p(G(\pi)) + N\int_0^\pi x^m \left\{ f(G(x))(g(\tfrac{x}{2})-g(x)) \right\}^p dx,$$

where M and N are constants defined by (7) and (8). Since $g(x)$ is nonnegative, nonincreasing, it is easy to observe from (5) that

$$(17)\quad \frac{1}{2}\, g(x) \leq G(x) \leq g(\frac{x}{2}),\ x\in(o,\pi).$$

From (17) and the nondecreasing character of $f(x)$ on (o,π) we have

$$(18)\quad f(\frac{1}{2}\, g(x)\ ) \leq f(G(x)) \leq f(g(\frac{x}{2})),\quad x\in(o,\pi).$$

From (18) we observe that

$$(19)\quad g(\frac{x}{2})f(G(x)) \leq g(\frac{x}{2})f(g(\frac{x}{2})),$$

and

$$(20)\quad g(x)f(\frac{1}{2}\, g(x)) \leq g(x)f(G(x)),$$

for $x\in(o,\pi)$. From (4) and the fact that $F\in\Delta_2$ we observe that

$$(21)\quad g(\frac{x}{2})f(g(\frac{x}{2})) \leq F(2g(\frac{x}{2})) \leq kF(g(\frac{x}{2})),$$

and

$$(22)\quad F(\frac{1}{2}g(x)) \leq \frac{1}{2}g(x)\, f(\frac{1}{2}g(x)),$$

for $x \in (0,\pi)$. Now, using (19) - (22) on the right side of (16) we have

$$(23) \qquad \int_0^\pi x^m F^p(G(x))\,dx \le MF^p(G(\pi)) + N\int_0^\pi x^m \left\{ kF\left(g\left(\frac{x}{2}\right)\right) - 2F\left(\frac{1}{2}\, g(x)\right) \right\}^p dx.$$

This completes the proof of the Theorem.

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