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TOTAL COLLAPSE DYNAMICS FOR PARTICLE SYSTEMS

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Abstract : We study solutions leading to a simultaneous total collapse in the n-body problem with generalized attraction law given by the inverse $(\alpha+1)$-power of the distance, $\alpha>0$. We generalize a result due to Weierstrass regarding the angular momentum constant and prove further that, for $\alpha=2$, the collision instant can be computed explicitly as a function of the initial position and velocity vectors. Finally we show that a total collapse in infinite time is impossible.

1. Introduction

Consider the motion of n particles (bodies, point masses) in the three dimensional Euclidean space $\mathbb{R}^3$, under a generalized Newtonian law of attraction given by the inverse $(\alpha+1)$-power of the distance, $\alpha>0$. The equations of motion are :

$$(1)\qquad \begin{aligned} \dot{q}_i &= m_i^{-1}p_i, \quad i = \overline{1,n} \\ \dot{p}_i &= \partial_i U(q), \quad i = \overline{1,n}, \end{aligned}$$

where $q_i = (q_i^1, q_i^2, q_i^3)$, $i = \overline{1,n}$ are the position vectors, $p_i = m_i\dot{q}_i$, $i = \overline{1,n}$, denote the *momenta*, $m_i > 0$, $i = \overline{1,n}$, are the masses and

$$U : \mathbb{R}^{3n} \setminus \Delta \to \mathbb{R}_+ ,$$

is the *force function* of the system ($-U$ being the *potential energy*) given by the relation :

$$U(q) = \sum_{1\le i<j\le n} m_i m_j |q_i - q_j|^{-\alpha}, \quad \alpha > 0,$$

where $q = (q_1,\dots,q_n)$, $|\ .\ |$ is the Euclidean norm and

$$\Delta = \bigcup_{1 \le i < i \le j} \{q | q_i = q_j\}$$

represents the *collision set*. Throughout this note we use a single dot over a variable to denote its first derivative and two dots to represent its second derivative, both with respect to the time t.

Standard results of the differential equations theory ensure, for given initial conditions $(q,p)(0)$, the existence and uniqueness of an analytic solution of the Eqs.(1), defined on some interval (t^-,t^+), $t^- < 0 < t^+$. Since the Eqs.(1) are time reversible we may obviously consider our study on $[0,t^+)$. The solution can be analytically extended to a maximal interval $[0,t^*)$, $0 < t^+ \le t^* \le +\infty$. In case $t^* = +\infty$ the solution is called *regular* and for t^* finite it is said to experience a *singularity* at t^*. From the physical point of view this singularity corresponds to a *collision* or to a *pseudocollision* (i.e. a motion becoming unbounded in finite time). For $\alpha=1$ the existence of pseudocollisions has been conjectured by Painléve [P] almost a century ago but a proof of it is only recently available [X].

Without loss of generality we further restrict the Eqs.(1) to the invariant set Q×P, where

$$Q = \{q| \sum_{i=1}^{n} m_i q_i = 0\}, \quad P = \{p| \sum_{i=1}^{n} q_i = 0\}.$$

The invariance of this set is obvious by the *integrals of center of mass and momenta*, namely

$$\sum_{i=1}^{n} m_i q_i(t) = at + b$$

$$\sum_{i=1}^{n} p_i = a$$

where a, b, are constant vectors. From the physical point of view, the restriction of the equations of motion to the above invariant set means that the origin of the coordinate system is situated in the center of mass of the n-particles system.

Besides the six integrals, the Eqs.(1) posses other four integrals, algebraic with respect to momenta and time (the remaining 6n - 10 being transcendent, as it has been proved by Bruns [B] and Poincaré [W]. These are the *angular momentum integrals* :

(2) $$\sum_{i=1}^{n} q_i \times p_i = c,$$

where c is a constant vector, × denoting the cross product, and the *energy integral*

(3) $$T(p) - U(q) = h,$$

where h is the *energy constant* and $T : \mathbb{R}^{3n} \to [0,\infty)$,

$$T(p) = \sum_{i=1}^{n} m_i^{-1} |p_i|^2,$$

defines the *kinetic energy* of the particle system.

Also define the *inertial momentum* as to be $J : \mathbb{R}^{3n} \to [0,\infty)$,

$$J(q) = (1/2) \sum_{i=1}^{n} m_i^{-1} |q_i|^2,$$

which is involved in the *Lagrange-Jacobi relation* :

(4) $$\ddot{J}(q(t)) = (2 - \alpha)U(q(t)) + 2h.$$

The case $\alpha=1$ describes the classical n-body problem with the Newtonian (gravitational) attraction law. Our first and last result are generalizations of some properties, well known in the Newtonian case. Such extensions are motivated by the fact that some physical processes are described by generalized force laws. For example, the attraction between certain molecules follows the inverse 7^{th}- or 9^{th}-power law (by Van der Waals). We show in Theorem 2 that for a total collapse solution the angular momentum constant vanishes, extending thus to all α with $0 < \alpha \leq 2$, a result known to Weierstrass and first proved, in the three body problem, by Sundman (see [W]). Theorem 4 is a result with crude cosmogonical interpretations for the classical gravitational case and appears, for $\alpha=1$ in the book of Wintner [W]. It states that there are no total collapse solutions in infinite time.

A surprising result is Theorem 3, by which, for the inverse 3^{rd}-power attraction law, the total collapse instant t^* can be computed explicitely as a function of the initial data (position and velocity vectors). This is a consequence of the special form of the Lagrange-Jacobi identity for $\alpha=2$.

We close this introductory section by noting the elementary character of the proofs.

2. The angular momentum constant

In this section we generalize a theorem due to Weierstrass and give a very simple proof of it. For this we use an asymptotic result due to Pollard and Saari [PS], [S1] :

Theorem 1. *Let $(q,p)\in Q\times P$ be an analytic solution of the Eqs.(1), defined on the interval $[0,t^*)$ and leading to a collision at t^* instant. Then, for $t \to t^*$, we have the asymptotic relations :*

$$J(q(t)) \sim A(t^* - t)^{4/(\alpha+2)},$$

$$U(q(t)) \sim [4A/(\alpha + 2)^2](t^* - t)^{-2\alpha/(\alpha+2)},$$

where A is a positive constant.

We are able to prove now the result of this section. A proof for the case $\alpha=1$, based on a different idea, can be found in [SM].

Theorem 2. *Let $0 < \alpha \le 2$ and consider an analytic solution $(q,p)\in Q\times P$ of the Eqs.(1), defined on $[0,t^*)$ and leading to a total collapse of the n bodies when $t \to t^*$. Then $c = 0$, where c is the angular momentum constant defined in (2).*

Proof. We will use the following classical inequality due to Sundman [Su].

$$4J(q(t))T(p(t)) - \dot{J}(q(t)) \geq |c|^2.$$

By the energy integral (3) and Th.1 we obtain the asymptotic relation

$$4J(q(t))T(p(t)) \sim [16 A^2/(\alpha + 2)^2](t^* - t)^{2(2-\alpha)/(\alpha+2)}.$$

Using Th.1 again and Lagrange-Jacobi relation (4), it follows that

$$\ddot{J}(q(t)) \sim [4A(2 - \alpha)/(\alpha + 2)^2](t^* - t)^{-2\alpha/(\alpha+2)}$$

and by integration it yields

$$\dot{J}(q(t)) \sim [4A/(\alpha + 2)](t^* - t)^{(2-\alpha)/(\alpha+2)}.$$

Since a necessary and sufficient condition for the total collision is $J(q(t)) \to 0$ for $t \to t^*$, it follows from the inequality of Sundman that, for $t \to t^*$, $0 \geq |c|^2$ and the theorem is proved.

Remark. While preparing this paper I have found that a similar result, by different considerations, has been recently obtained by Saari [S2].

3. The inverse 3^{rd}-power force law

The case $\alpha=2$ of the inverse 3^{rd}-power of the distance has a surprising property. The total collision instant can be expressed as a function of $(q,p)(0)$. In other words, having the initial conditions which lead to a simultaneous total collapse, we already know when this collision takes place. This is the main result of our note.

In order to eliminate possible confusions we consider here the initial time instant at t_0 instead of 0.

Theorem 3. *Let* $\alpha=2$ *and* $(q,p)\in Q\times P$ *be an analytic solution of the Eqs. (1), defined on* $[0,t^*)$, *leading to a simultaneous total collision when* $t \to t^*$. *Then we have :*

(i) for $h < 0$:

$$t^* = [2ht_0 - \dot{J}(q(t_0)) - (\dot{J}^2(q(t_0)) - 4hJ(q(t_0)))^{1/2}]/2h$$

(ii) for $h = 0$, *in case such solutions exist* :

$$t^* = (\dot{J}(q(t_0)) - \dot{J}(q(t_0))t_0)/\dot{J}(q(t_0))$$

(iii) for $h > 0$, *in case such solutions exist* :

$$t^* = [2ht_0 - \dot{J}(q(t_0)) + (\dot{J}^2(q(t_0)) - 4hJ(q(t_0)))^{1/2}]/2h.$$

The above expressions do not depend on the choice of t_0.

Proof. Observe first that

(5) $$J(q(t)) > 0,\ \forall\ t\in[t_0,t^*).$$

Since $J(q(t)) \to 0$ for $t \to t^*$, it follows that in a neighborhood of t^*, $\dot{J}(q(t)) < 0$ and since we will further see that the expressions do not depend on the choice of t_0, we may consider a t_0 such that

(6) $$\dot{J}(q(t_0)) < 0,\ \forall\ t\in[t_0,t^*).$$

The Lagrange-Jacobi relation becomes, for $\alpha=2$:

(7) $$\ddot{J}(q(t)) = 2h.$$

In case $h < 0$ it means that J is decreasing and convex down on the interval $[t_0,t^*)$. Integrating relation (7) between t_0 and $t > t_0$, we obtain

$$\dot{J}(q(t)) = 2ht - 2ht_0 + \dot{J}(q(t_0)),$$

and integrating again it follows

(8) $$J(q(t)) = ht^2 + (\dot{J}(q(t_0)) - 2ht_0)t + (ht_0 - \dot{J}(q(t_0)))t_0 + J(q(t_0)).$$

Let $t \to t^*$ in (8) and by solving the equation we obtain two values for t^* :

(9) $$t^*_{1,2} = [2ht_0 - \dot{J}(q(t_0)) \pm (\dot{J}^2(q(t_0)) - 4hJ(q(t_0)))^{1/2}]/2h.$$

Since we must have $t^* > t_0$, the only t^* fulfilling this condition is that with the minus sign in (9) and (i) is thus proved.

Observe that for $h > 0$, only the t^* in (9) with the positive sign fulfills the condition $t^* > t_0$, and (iii) is proved, too. A problem arises anyway. The existence of t^* in (i) is obvious but, for $h > 0$, we don't know whether

$$(10) \qquad \dot{J}^2(q(t_0)) - 4hJ(q(t_0)) \geq 0.$$

Using the energy integral (3) we have

$$4J(q(t_0))h = (\sum_{i=1}^{n} m_i |q_i(t_0)|^2)(\sum_{j=1}^{n} m_j |\dot{q}_j(t_0)|^2 - 2m_i m_j |q_i(t_0) - q_j(t_0)|^2),$$

so, relation (10) which we study may be written as

$$(11) \qquad [\sum_{i=1}^{n} m_i q_i^T(t_0)\dot{q}_i(t_0)]^2 - (\sum_{i=1}^{n} m_i |q_i(t_0)|^2)(\sum_{j=1}^{n} m_j |\dot{q}_j(t_0)|^2 +$$

$$+ \sum_{i=1}^{n} m_i |q_i(t_0)|^2)(\sum_{1\leq i<j\leq n} m_i m_j |q_i(t_0) - q_j(t_0)|^{-2} \geq 0.$$

By the identity of Lagrange the difference of the first two terms is

$$R(q(t)) = - \sum_{1\leq i<j\leq n} (m_i m_j)^{1/2}(q_i^T(t_0)\dot{q}_j(t_0) - q_j^T(t_0)\dot{q}_i(t_0))^2 \leq 0.$$

Observing that

$$\sum_{i=1}^{n} m_i |q_i|^2 = (\sum_{i=1}^{n} m_i)^{-1} \sum_{1\leq i<j\leq n} m_i m_j |q_i - q_j|^2,$$

and using the inequality of Cauchy, the last term of the equality (11) can be minorated by

$$(\sum_{i=1}^{n} m_i)^{-1} \sum_{1\leq i<j\leq n} m_i m_j = k \text{ (constant)}.$$

Therefore

$$(12) \qquad R(q(t_0)) + k \geq 0$$

would be a sufficient condition to decide the truth of inequality (10).

It is easy to show that there are initial conditions leading to a total collapse and which verify relation (12), for example by considering the n bodies at the vertices of a regular polygon with zero initial velocities. Since this is a *central configuration* (see, e.g. [W]) the particles tend to the simultaneous total collision like in the Newtonian case. But such a solution has a negative energy constant as we can see from the integral of energy with $T(q(t_0)) = 0$. This example ensures the existence of total collapse solutions for $h < 0$. Unfortunately we don't know anything about such motion for $h \geq 0$. Anyway, under the assumption of existence the collision instant can be computed. By (8) and for $h = 0$, it is obvious that

$$t^* = (J(q(t_0)) - \dot{J}(q(t_0))t_0)/\dot{J}(q(t_0)),$$

and since $\dot{J}(q(t_0)) < 0$, we have $t^* > t_0$.

It remains to prove that the above formulae do not depend on the choice of t_0.

For $h = 0$, $\dot{J}(q(t)) = \dot{J}(q(t_0))$ = constant, and since $\dot{J}(q(t)) < 0$ near t^*, it follows that $\dot{J}(q(t_0)) < 0$. Thus

$$\dot{J}(q(t))t - J(q(t)) = \dot{J}(q(t_0))t - \dot{J}(q(t_0))(t - t_0) - J(q(t_0)) =$$
$$= \dot{J}(q(t_0))t_0 - J(q(t_0)) = \text{constant}$$

and the independence of t_0 is proved in this case.

For $h \neq 0$, observe first that, using relation (7),

$$(d/dt)[\dot{J}^2(q(t_0)) - 4hJ(q(t_0))] = 2\dot{J}(q(t))[\ddot{J}(q(t)) - 2h] = 0.$$

Remark also that

$$(d/dt)[2ht - \dot{J}(q(t))\ ] = 2h - \ddot{J}(q(t)) = 0,$$

and the independence on t_0 follows for $h \neq 0$, too. Theorem 3 is now completely proved.

4. Total collapse in infinite time.

For the sake of completeness we study in this last section the possibility of a total collapse in infinite time and we will see that the answer is negative as expected from the gravitational case (see [W]).

Theorem 4. *For any* $\alpha > 0$ *there is no solution of the Eqs.(1) leading to a simultaneous total collapse for* $t \to +\infty$.

Proof. Let's suppose that there is a solution of the Eqs.(1) leading to the total collapse for $t \to +\infty$. Then

$$J(q(t)) \to 0 \quad \text{and} \quad U(q(t)) \to +\infty \quad \text{for } t \to +\infty.$$

We will consider the three possibilities : (1) $0 < \alpha < 2$, (2) $\alpha = 2$ and (3) $\alpha > 2$.

In case (1), $2 - \alpha > 0$ and by Lagrange-Jacobi identity it follows that

$$\ddot{J}(q(t)) \to +\infty \quad \text{for } t \to +\infty.$$

Therefore there is some $t_1 \in [0,\infty)$ and a positive constant k_1 such that

$$\ddot{J}(q(t)) > k_1, \quad \forall\, t \in [t_1,\infty).$$

By integrating this inequality between t_1 and t, $t > t_1$, we obtain

$$\dot{J}(q(t)) > k_1(t - t_1) + \dot{J}(q(t_1)), \ \forall\, t \in [t_1,\infty).$$

Integrating again it follows that

$$J(q(t)) > (1/2)k_1 t^2 + (\dot{J}(q(t_1)) - k_1 t_1)t - (1/2)k_1 t_1^2 - \dot{J}(q(t_1))t_1 + J(q(t_1)), \ \forall\, t \in [t_1,\infty).$$

Since $k_1 > 0$ it yields $J(q(t)) \to +\infty$ for $t \to +\infty$, a contradiction.

In case (2), $\ddot{J}(q(t)) = 2h$, $\forall\, t \in [0,\infty)$. Integrating twice between 0 and t, $t > 0$, we have

$$J(q(t)) = 2ht^2 + \dot{J}(q(0))t + J(q(0)), \ \forall\, t \in [0,\infty).$$

For $h \neq 0$,

$$|J(q(t))| \to +\infty \quad \text{for } t \to +\infty, \tag{13}$$

a contradiction.

For $h = 0$, if $\dot{J}(q(0)) \neq 0$, relation (13) follows again and if $\dot{J}(q(0)) = 0$, since $J(q(0)) \neq 0$, we obtain $J(q(t)) = \text{constant} > 0$ and $J(q(t))$ can not tend to 0 for $t \to +\infty$.

In case (3) the same Lagrange-Jacobi relation implies

$$\ddot{J}(q(t)) \to -\infty \quad \text{for } t \to +\infty.$$

This means there exist $t_2 \in [0,\infty)$ and $k_2 < 0$, such that

$$\ddot{J}(q(t)) < t_2, \quad \forall\, t \in [t_2,\infty).$$

By integration we obtain

$$J(q(t)) < (1/2)k_2 t^2 + (\dot{J}(q(t_2)) - k_2 t_2)t - (1/2)k_2 t_2^2 - \dot{J}(q(t_2))t_2 + J(q(t_2)), \ \forall\, t \in [t_2,\infty)$$

and since $k_2 < 0$ we would obtain $J(q(t)) \to -\infty$, $t \to +\infty$, leading again to a contradiction. The theorem is completely proved.

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