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COMPLEMENTARY FUNCTIONS IN THE SENSE OF YOUNG

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1. Introduction

If φ is a nondecreasing right continuous function on $[0,\infty]$ into $[0,\infty]$, then the right inverse of φ is the function ψ defined on $[0,\infty]$ by

$$\psi(s) = \inf\{t : \varphi(t) > s\} .$$

It is known ([2], Theorem 1.16) that ψ is, in turn, a nondecreasing right continuous function and its right inverse is φ. If φ is continuous then its right inverse in the inverse function.

The functions of Φ and Ψ defined on $[0,\infty]$ by

$$\Phi(u) = \int_0^u \varphi(t)dt \text{ and}$$
$$\Psi(v) = \int_0^v \psi(s)ds$$

are said to be complementary in Young's sense.

Complementary functions are basic in Orlicz space theory. However only a few explicit examples of such functions are known. The most frequently mentioned is that obtained by taking

$$\varphi(t) = \varphi_p(t) = t^{p-1} , \; p > 1$$

whose inverse is given by:

$$\psi(\tau) = \psi_q(\tau) = \tau^{q-1} , \quad \text{where}$$
$$\frac{1}{p} + \frac{1}{q} = 1 .$$

The complementary functions generated by φ_p are Φ_p and Ψ_q given by:

$$\Phi_p(u) = \frac{u^p}{p} \text{ and } \Psi_q(v) = \frac{v^q}{q} .$$

The Orlicz space associated to these functions is the well known L^p space.

In this note some other explicit examples of complementary functions are given and inequalities analogous to those found for Φ_p and Ψ_q are proven.

2. The Functions $\Phi_{p,M}$ and their Complementary Functions

The functions $\Phi_{p,M}$ appeared first in [1], [3] as interesting module-generating functions. It has been shown that in order to have a kind of completness property with respect to the module the generating functions must fulfill some conditions proper to $\Phi_{p,M}$. One of them is the Δ_2 - Birnhaum and Orlicz condition, which is essential in Orlicz spaces theory.

The function $\varphi_{p,M}$ defined on $[0,\infty]$ by

$$\varphi_{p,M}(t) = \left(\frac{M}{2^p}\right)^n t^{p-1} \ , \ 2^n \le t < 2^{n+1} \ , \ n \in \mathbf{Z} \ , \ p > 1 \ , \ M \ge 2^p$$
$$\varphi_{p,M}(0) = 0 \ , \ \varphi_{p,M}(\infty) = \infty$$

generalizes φ_p in the sense that $\varphi_{p,M}$ is nondecreasing and right continuous function, but $\varphi_{p,M}$ is discontinuous at $t = 2^n$, $n \in \mathbf{Z}$ if $M > 2^p$ and it reduces to φ_p if $M = 2^p$.

A direct computation shows that the right inverse of $\varphi_{p,M}$ is the function $\psi_{q,M}$:

$$\psi_{q,M}(\tau) = \begin{cases} (\frac{2^p}{M})^{n(q-1)}\tau^{q-1} \ , & \left(\frac{M}{2}\right)^n \le \tau < \left(\frac{M}{2}\right)^n \cdot 2^{p-1} \\ 2^{n+1} & , \left(\frac{M}{2}\right)^n \cdot 2^{p-1} \le \tau < (\frac{M}{2})^{n+1}, \end{cases} \ n \in \mathbf{Z} \ , \ \text{where } \frac{1}{p} + \frac{1}{q} = 1 \ \text{and}$$
$$\psi_{q,M}(0) = 0 \ , \ \psi_{q,M}(\infty) = \infty \ .$$

The complementary functions generated by them are $\frac{1}{p}\Phi_{p,M}$ and $\frac{1}{q}\Psi_{q,M}$, where :

$$\Phi_{p,M}(u) = p\int_0^u \varphi_{p,M}(t)dt = p\left[\sum_{k=-\infty}^{n}\int_{2^{k-1}}^{2^k}\varphi_{p,M}(t)dt + \int_{2^k}^{u}\varphi_{p,M}(t)dt\right] \ , \ 2^n \le u < 2^{n+1} \ .$$

That gives:

$$\Phi_{p,M}(u) = M^n\left[\left(\frac{u}{2^n}\right)^p - \frac{M-2^p}{M-1}\right] \ , \ 2^n \le u < 2^{n+1} \ , \ n \in \mathbf{Z} \ ,$$
$$\Phi_{p,M}(0) = 0 \ , \ \Phi_{p,M}(\infty) = \infty \ .$$

Analogously:

$$\Psi_{q,M}(v) = \begin{cases} M^n[(\frac{2}{M})^{qn}v^q + \frac{q-1}{M-1}(M-2^p)] \ , & (\frac{M}{2})^n \le v < (\frac{M}{2})^n \cdot 2^{p-1} \\ 2^{n+1}qv - \frac{q-1}{M-1}M^{n+1}(2^p-1) \ , & (\frac{M}{2})^n \cdot 2^{p-1} \le v < (\frac{M}{2})^{n+1} \ , \end{cases} \ n \in \mathbf{Z} \ ,$$
$$\Psi_{q,M}(0) = 0 \ , \ \Psi_{q,M}(\infty) = \infty \ .$$

It is known [4] that $\Phi_{p,M}$ verifies a strong Δ_2–condition : $\Phi_{p,M}(2n) = M\Phi_{p,M}(n)$ for any $u \geq 0$.

On the other hand, if

$$\left(\frac{M}{2}\right)^n \leq \frac{M}{2}v < \left(\frac{M}{2}\right)^n \cdot 2^{p-1}\,, \quad \text{then}$$
$$\left(\frac{M}{2}\right)^{n-1} \leq v < \left(\frac{M}{2}\right)^{n-1} \cdot 2^{p-1}\,, \text{ therefore}$$

$$\begin{aligned}\Psi_{q,M}\left(\frac{M}{2}v\right) &= M^n\left[\left(\frac{2}{M}\right)^{qn} \cdot \frac{M^q}{2^q}v^q + \frac{q-1}{M-1}(M-2^p)\right]\\ &= M \cdot M^{n-1}\left[\left(\frac{2}{M}\right)^{q(n-1)} \cdot v^q + \frac{q-1}{M-1}(M-2^p)\right]\\ &= M\Psi_{q,M}(v)\end{aligned}$$

Similarly, if

$$\left(\frac{M}{2}\right)^n \cdot 2^{p-1} \leq \frac{M}{2}v < \left(\frac{M}{2}\right)^{n+1}\,, \text{ then}$$
$$\left(\frac{M}{2}\right)^{n-1} \cdot 2^{p-1} \leq v < \left(\frac{M}{2}\right)^{n}\,, \text{ therefore}$$

$$\begin{aligned}\Psi_{q,M}\left(\frac{M}{2}v\right) &= 2^{n+1} \cdot q \cdot \frac{M}{2}v - \frac{q-1}{M-1} \cdot M^{n+1}(2^p-1)\\ &= M[2^n qv - \frac{q-1}{M-1} \cdot M^n(2^p-1)]\\ &= M \cdot \Psi_{q,M}(v)\end{aligned}$$

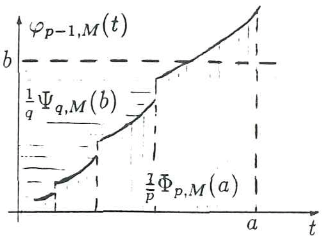
It is obvious that such an equality is true also if $v = 0$ or $v = +\infty$, therefore for any v , $0 \leq v \leq \infty$

$$\Psi_{q,M}\left(\frac{M}{2}v\right) = M\Psi_{q,M}(v)\,.$$

This equality generalizes the strong Δ_2–condition in the sense that it reduces to it if $M = 4$, i.e.

$$\Psi_{q,4}(2v) = 4\Psi_{q,4}(v)\,.$$

3. Young's Inequality



From the diagram below it is obvious that for any $a \geq 0$, $b \geq 0$ the inequality

$$ab \leq \frac{1}{p}\Phi_{p,M}(a) + \frac{1}{q}\Psi_{q,M}(b)$$

is true and equality is achieved if and only if

$$b = \varphi_{p-1,M}(a) .$$

Taking $a = \Phi_{p,M}^{-1}(t)$ and $b = \Psi_{q,M}^{-1}(t)$, we obtain that for any $t \geq 0$,

$$\Phi_{p,M}^{-1}(t) \cdot \Psi_{q,M}^{-1}(t) \leq t .$$

The two inverse functions are given by:

$$\Phi_{p,M}^{-1}(t) = 2^n \left[\frac{t}{M^n} + \frac{M-2^p}{M-1}\right]^{\frac{1}{p}} , \; M^n K \leq t < M^{n+1}K , \; K = \frac{2^p - 1}{M-1} ,$$

$$\Psi_{q,M}^{-1}(\tau) = \begin{cases} \left(\frac{M}{2}\right)^n \left[\frac{\tau}{M^n} - \frac{q-1}{M-1}(M-2^p)\right]^{\frac{1}{q}} , & M^n(1+L) \leq \tau < M^n(2^p + L) \\ \frac{1}{2^{n+1}q}\left[\tau + \frac{q-1}{M-1}M^{n+1}(2^p - 1)\right] , & M^n(2^p+L) \leq \tau < M^{n+1}(1+L), \end{cases}$$

where $L = \frac{q-1}{M-1}(M - 2^p)$, $\Phi_{p,M}^{-1}(0) = \Psi_{q,M}^{-1}(0) = 0$, $\Phi_{p,M}^{-1}(\infty)$ $= \Psi_{q,M}^{-1}(\infty) = \infty$.

4. A Hölder Type of Inequality

Let $(x,\mathcal{A},\mu)$ be a measure space and f , g be nonnegative measurable functions. The Hölder inequality $\int fgd\mu \leq (\int f^p d\mu)^{\frac{1}{p}}(\int g^q d\mu)^{\frac{1}{q}}$ can be interpreted in terms of the fucntions Φ_p and Ψ_q as

$$\int fgd\mu \leq \Phi_p^{-1}\left(\int \Phi_p(f)d\mu\right)\cdot \Psi_q^{-1}\left(\int \Psi_q(g)d\mu\right) .$$

In this section it will be shown that $\Phi_{p,M}$ and $\Psi_{q,M}$ verify a smiliar inequality.

Let's remark first that

$$\Phi_{p,M}\left(\frac{a}{\Phi_{p,M}^{-1}(b)}\right) = M^k\Phi_{p,M}(a) \text{ if } \Phi_{p,M}^{-1}(b) = 2^{-k} \text{ , or}$$

$$b = \Phi_{p,M}(2^{-k}) = \frac{2^p-1}{M-1}M^{-k} \quad \text{, therefore:}$$

$$M^k\Phi_{p,M}(a) \leq \Phi_{p,M}\left(\frac{a}{\Phi_{p,M}^{-1}(b)}\right) < M^{k+1}\Phi_{p,M}(a) \text{ if } \frac{2^p-1}{M-1}M^{-k-1} < b \leq \frac{2^p-1}{M-1}M^{-k} .$$

Analogously

$$\Psi_{q,M}\left(\frac{a}{\Psi_{q,M}^{-1}(b)}\right) = M^j\Psi_{q,M}(a) \text{ if } \Psi_{q,M}^{-1}(b) = \left(\frac{M}{2}\right)^{-j} \text{ , or}$$

$$b = \Psi_{q,M}\left(\left(\frac{M}{2}\right)^{-j}\right) = \frac{q(M-2^p)+(2^p-1)}{M-1}M^{-j} ,$$

therefore:

$$M^j\Psi_{q,M}(a) \leq \Psi_{q,M}\left(\frac{a}{\Psi_{q,M}^{-1}(b)}\right) < M^{j+1}\Psi_{q,M}(a) \text{ if}$$

$$\frac{q(M-2^p)+(2^p-1)}{M-1}M^{-j-1} < b \leq \frac{q(M-2^p)+(2^p-1)}{M-1}M^{-j} .$$

Let $a = \frac{f(t)}{\Phi_{p,M}^{-1}(\int \Phi_{p,M}(f)d\mu)}$, $b = \frac{g(t)}{\Psi_{q,M}^{-1}(\int \Psi_{q,M}(g)d\mu)}$.

The two integrals are well defined since $\Phi_{p,M}$ and $\Psi_{q,M}$ are continuous functions. If $\int \Phi_{p,M}(f)d\mu = 0$, then $a = +\infty$ and vice-versa. A similar remark is true for b .

The Young's inequality becomes:

$$\frac{f(t)g(t)}{\Phi_{p,M}^{-1}(\int \Phi_{p,M}(f)d\mu)\Psi_{q,M}^{-1}(\int \Psi_{q,M}(g)d\mu)}$$
$$\leq \frac{1}{p}\Phi_{p,M}\left(\frac{f(t)}{\Phi_{p,M}^{-1}(\int \Phi_{p,M}(t)d\mu)}\right) + \frac{1}{q}\Psi_{q,M}\left(\frac{g(t)}{\Psi_{q,M}^{-1}(\int \Psi_{q,M}(g)d\mu)}\right)$$
$$\leq \frac{1}{p}M^{k+1}\Phi_{p,M}(f(t)) + \frac{1}{q}M^{j+1}\Psi_{q,M}(g(t)) \text{ if}$$
$$\frac{2^p-1}{M-1}M^{-k-1} < \int \Phi_{p,M}(t)d\mu \leq \frac{2^p-1}{M-1}M^{-k} \text{ and}$$
$$M^{-j-1}\frac{q(M-2^p)+(2^p-1)}{M-1} < \int \Psi_{q,M}(g)d\mu \leq M^{-j}\frac{q(M-2^p)+(2^p-1)}{M-1}.$$

It results that:

$$\int \frac{fg}{\Phi_{p,M}^{-1}(\int \Phi_{p,M}(f)d\mu)\,\Psi_{q,M}^{-1}(\int \Psi_{q,M}(f)d\mu)}d\mu$$
$$\leq \frac{1}{p}M^{k+1}\int \Phi_{p,M}(t)d\mu + \frac{1}{q}M^{j+1}\int \Psi_{q,M}(g)d\mu$$
$$\leq \frac{1}{p}M^{k+1}\cdot\frac{2^p-1}{M-1}M^{-k} + \frac{1}{q}M^{j+1}\cdot\frac{q(M-2^p)+(2^p-1)}{M-1}\cdot M^{-j}$$
$$= \frac{M}{M-1}[\frac{1}{p}(2^p-1) + \frac{1}{q}\cdot q(M-2^p) + \frac{1}{q}(2^p-1)] = M$$

therefore the following Hölder type of inequality is true.

$$\int fg d\mu \leq M\Phi_{p,M}^{-1}\left(\int \Phi(f)d\mu\right)\Psi_{q,M}^{-1}\left(\int \Psi_{q,M}(g)d\mu\right).$$

5. The Complementary Functions Generated by $\Phi_{p-1,M}$

For $p > 1$ and $M \geq 2^{p-1}$ the function $\Phi_{p-1,M}$ is continuous and increasing in $[0,\infty]$. A direct computation shows that

$$\Phi_{p-1,M}^{-1}(\tau) = 2^n\left[\frac{\tau}{M^n} + \frac{M-2^{p-1}}{M-1}\right]^{q-1}, \quad M^nK' \leq \tau < M^{n+1}K',$$

$$\text{where } K' = \frac{2^{p-1}-1}{M-1}.$$

The complementary functions generated by $\Phi_{p-1,M}$ are $\frac{1}{p}\Omega_{p,M}$ and $\frac{1}{q}\Theta_{q,M}$, where

$$\Omega_{p,M}(x) = p\int_0^x \Phi_{p-1,M}(t)dt = (2M)^n \left[\left(\frac{x}{2^n}\right)^p - (M-2^{p-1})\left(\frac{p}{M-1}\cdot\frac{x}{2^n} - \frac{p-1}{2M-1}\right)\right],$$
$$\text{if } 2^n \leq x < 2^{n+1} \text{ , } n \in Z$$

$$\Omega_{p,M}(0) = 0$$

$$\Theta_{q,M}(y) = q\int_0^y \Phi^{-1}_{p-1,M}(\tau)d\tau = (2M)^n\left[\left(\frac{y}{M^n} + \frac{M-2^{p-1}}{M-1}\right)^q - \frac{M-2^{p-1}}{M-\frac{1}{2}}\right],$$
$$\text{if } M^nK' \leq y < M^{n+1}K' \text{ ,} n \in Z$$

$$\Theta_{q,M}(0) = 0 \ .$$

The Young's inequality takes the form

$$ab \leq \frac{1}{p}\Omega_{p,M}(a) + \frac{1}{q}\Theta_{q,M}(b) \text{ for any } a \geq 0 \text{ , } b \geq 0 \ .$$

Let $H = \Omega_{p,M}(1) = 1 - (M - q^{p-1})(\frac{p}{M-1} - \frac{p-1}{2M-1})$.

Then $(2M)^{-k-1}H < \int \Omega_{p,M}(f)d\mu \leq (2M)^{-k}H$ implies

$$2^{-k-1} < \Omega^{-1}\left(\int \Omega_{p,M}(f)d\mu\right) \leq 2^{-k} \ .$$

Analogously $(2M)^{-j-1}\frac{2^p-1}{2M-1} < \int \Theta_{q,M}(g)d\mu \leq (2M)^{-j}\frac{2^p-1}{2M-1}$ implies

$$M^{-j-1}K' < \Theta^{-1}_{q,M}\left(\int \Theta_{q,M}(g)d\mu\right) \leq M^{-j}K' \ .$$

Then replacing in the Young's inequality above

$$a = \frac{f(t)}{\Omega^{-1}_{p,M}(\int \Omega_{p,M}(t)d\mu)} \text{ , } b = \frac{g(t)}{\Theta^{-1}_{q,M}(\int \Theta_{q,M}(g)d\mu)}$$

the inequality becomes:

$$\frac{f(t)g(t)}{\Omega^{-1}_{p,M}\left(\int \Omega_{p,M}(f)d\mu\right)\Theta^{-1}_{q,M}\left(\int \Theta_{q,M}(g)d\mu\right)}$$
$$\leq \frac{1}{p}\Omega_{p,M}\left(\frac{f(t)}{\Omega^{-1}_{p,M}\left(\int \Omega_{p,M}(f)d\mu\right)}\right) + \frac{1}{q}\Theta_{q,M}\left(\frac{g(t)}{\Theta^{-1}_{q,M}\left(\int \Theta_{q,M}(g)d\mu\right)}\right)$$
$$\leq \frac{1}{p}\Omega_{p,M}\left(2^{k+1}f(t)\right) + \frac{1}{q}\Theta_{q,M}\left(\frac{M^{j+1}}{K'}g(t)\right)$$
$$\leq \frac{1}{p}(2M)^{k+1}\Omega_{p,M}(f(t)) + \frac{1}{p}(2M)^{j+2}\Theta_{q,M}(g(t)) \ ,$$

since $\frac{1}{K'} = \frac{M-1}{2^p-1} < M$.

The integral of the last term is

$$\frac{1}{p}(2M)^{k+1} \int \Omega_{p,M}(f)d\mu + \frac{1}{q}(2M)^{j+2} \int \Theta_{q,M}(g)d\mu$$
$$\leq \frac{1}{p}(2M)^{k+1} \cdot (2M)^{-k}H + \frac{1}{q}(2M)^{j+2} \cdot (2M)^{-j}\frac{2^p-1}{2M-1}$$
$$\leq (2M)^2 \left[\frac{1}{p} - \frac{M-2^{p-1}}{M-1} - \frac{M-2^{p-1}}{2M-1} + \frac{1}{p} \cdot \frac{M-2^{p-1}}{2M-1} + \frac{1}{q} \cdot \frac{2^p-1}{2M-1}\right] \leq (2M)^2 .$$

Therefore the following Hölder type of inequality is true:

$$\int fg d\mu \leq (2M)^2 \Omega_{p,M}^{-1} \left(\int \Omega_{p,M}(f)d\mu\right) \Theta_{q,M}^{-1} \left(\int \Theta_{q,M}(g)d\mu\right) .$$

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