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A REMARK ON NONLINEAR CONTRACTIONS

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Let (X,d) be a complete metric space and $F : X \to X$ be a map which p–th iteration F^p is a nonlinear contraction, i.e. there is an increasing and continuous function $a : R_+ \to R_+$, $R_+ = [0,+\infty)$, such that $a(t) < t$, $t > 0$, $a(0) = 0$, and

(1) $$d(F^p x, F^p y) \le a(d(x,y)), \quad x,y \in X.$$

Here $F^p = F(F^{p-1})$, $F^0 = I$, I – the identity map in X.

It is well known that these assumptions guarantee that there is an unique fixed point of the map F, say x^*, and that the sequence $\{F^n x_0\}$, $n = 0,1,\ldots$, converges to x^* for every $x_0 \in X$.

Our question is whether the map F itself is a nonlinear contraction with respect to some metric in X.

Observe first: if F is continuous, and nondecreasing, continuous function $b : R_+ \to R_+$ has the properties:

$$b(0) = 0, \qquad b(t_1 + t_2) \le b(t_1) + b(t_2)$$

then

(2) $$d_b(x,y) \overset{df}{=} d(x,y) + b\,(d(Fx,Fy)) + \cdots + b^{p-1}(d(F^{p-1}x, F^{p-1}y))$$

is a metric in X and (X,d_1) is a complete metric space. It is clear that the metric d_b is stronger than d and that they define the same topology if there is a continuous function $h : R_+ \to R_+$, $h(0) = 0$ such that

$$d(Fx,Fy) \le h(d(x,y)).$$

We see that there is plenty of metrics in X (at least as much as functions b

with the properties mentioned). This gives us a chance to find one with respect to which F will be nonlinear contraction.

Indeed, assume that there is an increasing and continuous solution $\varphi : R_+ \to R_+$ of the equation

(3) $$\varphi^p(t) = a(t), \qquad t \in R_+$$

which has the properties:

(i) $$\varphi(t) < t, \qquad t \in R_+,$$

(ii) $$\varphi(t_1) + \varphi(t_2) \leq \varphi(t_1 + t_2), \qquad t_1,t_2 \in R_+.$$

Here as before φ^p denotes the p–th iteration of the function φ.

Having this assumed we use the metric d_b for $b(t) = \varphi^{-1}(t)$ (φ^{-1} – means the inverse function to φ).

Now we are in the position to show that the map F is a nonlinear contraction with respect to the metric d_b with the contraction function φ. Indeed we have

$$d_b(Fx,Fy) = d(Fx,Fy) + b(d(F^2x,F^2y)) + \cdots + b^{p-2}(d(F^{p-1}x,F^{p-1}y))$$
$$+ b^{p-1}(d(F^px,F^py)) \leq$$
$$\leq d(Fx,Fy) + b(d(F^2x,F^2y)) + \cdots + b^{p-2}(d(F^{p-1}x,F^{p-1}y)) +$$
$$+ b^{p-1}(a(d(x,y))) =$$
$$= \varphi[b(d(Fx,Fy) + b(d(F^2x,F^2y)) + \cdots + B^{p-2}(d(F^{p-1}x,F^{p-1}y)) + b^{p-1}(a(d(x,y))))]$$
$$\leq \varphi[b(d(Fx,Fy)) + b^2(d(F^2x,F^2y)) + \cdots b^{p-1}(d(F^{p-1}x,F^{p-1}y)) + b^p(a(d(s,y)))] =$$
$$= \varphi[d(x,y) + b(d(Fx,Fy)) + \cdots + b^{p-1}(d(F^{p-1}x,F^{p-1}y))] =$$
$$= \varphi(d_b(x,y)),$$

what means that our claim holds true.

Finally we may conclude

Theorem. If (1) holds and there is a solution φ of (3) with the properties (i), (ii) then each iteration of F is a nonlinear contraction with respect to the metric d_b, $b(t) = \varphi^{-1}(t)$, defined by (2) and it has an unique fixed point x^*.

This result and the theorem of Bessaga [1] guarantee that there is a metric ρ in X such that F is also a clasical contraction with respect to ρ.

However the proof of Bessaga's result is non constructive, the choice axiom is used.

There is an example when the solution φ can be found immediately, it is the case when $a(t) = \alpha \cdot t$ for some $\alpha \in [0,1)$. In this case $\varphi(t) = \sqrt[4]{\alpha}.t$ is a solution of (3) with the properties required.

In the general case there is the question whether for a given function a with the properties mentioned there is a solution φ of (3) with the properties required. (see [2]).

References:

[1] C. Bessaga, On the converse of the Banach fixed point principle, Colloq. Math. 7, 1959, pp. 41–43.

[2] M. Kwapisz, Problem 187, Aequationes Mathematicae, vol. 19, 1979, p. 299.

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