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SECOND ORDER FUNCTIONAL-DIFFERENTIAL EQUATIONS INVOLVING ABSTRACT VOLTERRA OPERATORS

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We shall investigate the second order functional-differential equation

$$\ddot{x}(t)=(Fx)(t),\quad t\in[0,T),\quad 0<T\leq\infty, \tag{1}$$

in which F denotes an abstract Volterra operator acting from the space $\mathcal{C}([0,T),R^n)$ into the space $L^1_{loc}([0,T),R^n)$. The initial conditions attached to equation (1) have the classical form

$$x(0)=x^0,\quad \dot{x}(0)=\dot{x}^0, \tag{2}$$

where x^0, $\dot{x}^0\in R^n$.

The space $\mathcal{C}([0,T),\ R^n)$ denotes the linear space of continuous functions from $[0,T)$ into R^n, endowed with the topology of uniform convergence on every compact subinterval of $[0,T)$, while $L^1_{loc}([0,T),R^n)$ has the usual meaning.

Let us point out, first, that (1) could be reduced to the first order system

$$\dot{\xi}(t)=(\mathcal{F}\xi)(t),\quad t\in[0,T), \tag{3}$$

in which $\xi=\mathrm{col}(x,y)$, with $y=\dot{x}$. For such systems, several existence results are known for the initial value problem (see [1],[3]). More references can be formed in these books.

In reducing (1) to a system of the form (3), the derivative $\dot{x}$ must be placed in a function space, a feature that complicates somewhat the treatment of the existence problem. Since (1) does not contain explicitly $\dot{x}(t)$, one can avoid involving directly this function in the statements. It will be clear that the solution $x(t)$ of the problem (1),(2) is continuously differentiable, its first derivative being locally absolutely continuous.

It is an elementary fact that the problem (1),(2) is equivalent to the integral equation system

$$x(t)=x^0+\dot{x}^0t+\int_0^t du\int_0^u (Fx)(s)ds. \tag{4}$$

More precisely, a Carathéodory solution of (1), (2) is a continuous solution (even continuously differentiable) of (4). Vice-versa, a continuous solution of (4) on some interval $[0,\tau)\subset[0,T)$ will necessarily be continuously differentiable and its derivative will be locally absolutely continuous, while conditions (1), (2) are valid.

It would be simpler to limit our considerations to the case when F is linear and continuous from $\mathcal{C}([0,T),R^n)$ into $L^1_{loc}([0,T),R^n)$. Nevertheless, we prefer to envisage a slightly more general situation, whose discussion does not require much more details than the linear case.

Let us denote

$$(F\theta)(t)=f(t)\in L^1_{loc}([0,T),R^n), \tag{5}$$

where θ is the zero element in R^n (and in $\mathcal{C}([0,T),R^n)$). The inclusion in (5) is obviously the consequence of

$$F:\mathcal{C}([0,T),R^n)\to L^1_{loc}([0,T),R^n). \tag{6}$$

The condition we shall impose on F is slightly more general that its continuity, and actually reduces to the continuity of F when F is linear. Namely,

$$|Fx_t - Fy_t|_{L^1} \le \lambda(t)|x_t - y_t|_{\mathcal{C}}, \tag{7}$$

with $0<t<T$, and x_t meaning the restriction of x(usually given on $[0,T)$) to the interval $[0,t]$. We have not changed the notation for F because F is Volterra and Fx_t makes sense for every $x\in\mathcal{C}$ (which does not happen for non-Volterra operators). The function $\lambda:[0,T)\to(0,\infty)$ is assumed to be monotone nondecreasing. This last property does not represent a restriction of the generality. Indeed, if we only assume that (7) takes place for some positive $\lambda(t)$, for each $t\in(0,T)$, then it can be easily shown that λ must be nondecreasing (see [1], Ch. 3).

The equation (4) leads to the following process of successive approximations:

$$x^{m+1}(t) = x^0 + \dot{x}^0 t + \int_0^t du \int_0^u (Fx^m)(s)ds. \tag{8}$$

One can start with x^0 from (2), or choose any $x^0(t)\in\mathcal{C}([0,T),R^n)$ as initial approximation. It can be seen, taking (7) into account, that the following recurrent inequality is verified for every $m \geq 1$:

$$(9) \qquad \|x^{m+1}(t)-x^m(t)\| \leq \int_0^t \lambda(s) \sup_{0\leq u\leq s} \|x^m(u)-x^{m-1}(u)\| ds.$$

Since the right hand side of (9) is nondecreasing in t, there results from (9) that

$$(10) \qquad \sup_{0\leq s\leq t} \|x^{m+1}(s)-x^m(s)\| \leq \int_0^t \lambda(s) \sup_{0\leq u\leq s} \|x^m(u)-x^{m-1}(u)\| ds,$$

which immediately leads to the estimate

$$(11) \qquad \sup \|x^{m+1}(t)-x^m(t)\| \leq \frac{K}{m!}\left(\int_0^t \lambda(s) ds\right)^m, \quad 0\leq t\leq t_1,$$

with $K \geq \|x^1(t)-x^0\|$ on $[0,t_1]$, $t_1 < T$ being arbitrarily fixed.

From (11) one obtains the uniform convergence of the successive approximations, defined by (8), on each interval $[0,t_1] \subset [0,T)$. This is exactly the convergence in the space $\mathcal{C}([0,T),R^n)$. Letting $m \to \infty$ in (8), one obtains the integral equation (4). The process is valid because of (7).

The uniqueness of the solution of the integral equation (4) can be obtained using the classical pattern.

The conclusion of the above discussion can be summarized in the following result:

Theorem. Consider the problem (1), (2), with the abstract Volterra operator F satisfying (6) and (7). Then, there exists a unique solution of (1), (2) on $[0,T)$, continuously differentiable and whose first derivative is locally absolutely continuous.

Remark 1. In [1], we have proven the existence and uniqueness of solution under the main assumption that F is acting from the space $L^2_{loc}([0,T),R^n)$ into itself. The result stated above does not represent a consequence of the similar result in [1]. Let us also point out that only the linear case has been treated in [1].

The results in [3], concerning functional differential equations with abstract Volterra operators, are either of local type or involve hypotheses of a different nature than those used above.

Remark 2. It is not necessary to deal only with the finite-dimensional case R^n. The procedure of successive approximations can be applied in the case of infinite-dimensional spaces, such as Hilbert spaces or even reflexive Banach spaces. The reflexivity must be imposed in order to secure the well known and basic property relating absolute continuity and integrability: in a reflexive Banach space, a function which is absolutely continuous is the integral of its derivative.

Remark 3. A local existence theorem can be obtained for (1), (2), in various functions spaces. We reduce first this problem to the equation (3), with the initial condition $\xi(0)=\mathrm{col}(x^0,\dot{x}^0)$. See [1;Ch.3] for details and precise conditions.

As an application of the Theorem given above, let us consider the infinite delay second order linear equation

$$\dot{x}(t)=\sum_{j=0}^{\infty} A_j(t)x(t-t_j)+\int_0^t B(t,s)x(s)ds+f(t). \tag{12}$$

This equation is mentioned, for instance, in [2]. We assume that $t_j \geq 0$, the case when $\{t_j\}$ is unbounded being possible. The matrices $A_j(t)$ are supposed to be locally integrable on $[0,\infty)=R_+$, while B is measurable in (t,s), $0\leq s\leq t<\infty$, such that

$$\int_0^t \|B(t,s)\|ds \in L^1_{loc}(R_+,R). \tag{13}$$

One must also admit that

$$\sum_{j=0}^{\infty} \|A_j(t)\| \in L^1_{loc}(R_+,R), \tag{14}$$

which together with (13) guarantee the property (6) for the operator appearing in the right hand side of (12), whenever $f \in L^1_{loc}(R_+,R)$.

Since delays are involved in (12), one must assign convenient initial conditions. For sake of simplicity, we will assume that, besides (2), one must have $x(s)\equiv \dot{x}(s)\equiv 0$ for $s<0$.

The condition (7) can be checked without difficulty, taking

$$\lambda(t)=\sum_{j=0}^{\infty} \|A_j(t)\|+\int_0^t \|B(t,s)\|ds. \tag{15}$$

While $\lambda(t)$ is not necessarily monotone, the estimates (11) remain valid.

Consequently, equation (12) has a unique solution $x(t)$, satisfying (12) a.e. on R_+, and such that initial conditions (2) are verified, together with $x(s)=\dot{x}(s)\equiv 0$ for $s<0$.

Of course, it is possible to assign more general initial conditions on the negative half-axis, non-necessarily the zero conditions. This will change only f(t) in (12), and the conditions must be such that $f \in L^1_{loc}(R_+, R^n)$. Such considerations can be found in [1; Ch.3].

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