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# Banach categories

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**Abstract**

The aim of this paper is to present the Banach categories, which offer a general frame for many theories of functional analysis (see also R. Ahrens [1]).

**Notations**

We denote by $\mathbb{K}$ the field of real or complex numbers; they will be the scalars of our Banach spaces. Let $E, F$ be two Banach spaces. We denote by $\mathcal{L}(E,F)$ the Banach spaces of continuous linear maps of $E$ into $F$, by $E'$ and $E''$ the dual and the bidual of $E$ respectively, and for every $u \in \mathcal{L}(E,F)$ by $u'$ and $u''$ and transposed and the bitransposed of $u$ respectively. $u \in \mathcal{L}(E,F)$ is called an *isomorphismus* if it is bijective; if in addition $u$ preserves the norms then $u$ is called an *isometry*.

## 1. Banach systems

A *Banach system* is a class $\Omega$ and a map $\mathcal{A}$ defines on $\Omega^2$ so that $\mathcal{A}(E,F)$ is a Banach space for all $E, F \in \Omega$ and

$$(E,F) \neq (G,H) \Rightarrow \mathcal{A}(E,F) \cap \mathcal{A}(G,H) = \emptyset$$

for all $E, F, G, H \in \Omega$. We will use the expressions "the Banach system $(\Omega, \mathcal{A})$" or "the Banach system $\mathcal{A}$ over $\Omega$" or simply "the Banach system $\mathcal{A}$". The elements of $\Omega$ are called *the objects* of the Banach system and the elements of $\mathcal{A}(E,F)$ (for $E, F \in \Omega$) are called *the morphisms* of the Banach system. We set

$$E \xrightarrow{x} F \;:\Leftrightarrow\; E \underset{\mathcal{A}}{\xrightarrow{x}} F \;:\Leftrightarrow\; x \in \mathcal{A}(E,F)$$

for all $E, F \in \Omega$.

A Banach space may be considered as a Banach system with one object.

Let $(\Omega, \mathcal{A})$ be a Banach system. The map

$$(E,F) \mapsto \mathcal{A}(F,E)'$$

defined over $\Omega^2$ is a Banach system. We call it *the dual of* $\mathcal{A}$ and denote it by $\mathcal{A}'$. The dual Banach system of $\mathcal{A}'$ is called *the bidual of* $\mathcal{A}$ and is denoted by $\mathcal{A}''$. The dual Banach system of $\mathcal{A}''$ is called *the tridual of* $\mathcal{A}$ and is denoted by $\mathcal{A}'''$. A Banach system $\mathcal{B}$ is called a *dual Banach system* if there is a Banach system $\mathcal{A}$ so that $\mathcal{B} = \mathcal{A}'$.

Let $(\Omega, \mathcal{A})$ and $(\Omega, \mathcal{B})$ be two Banach systems. A *functor of* $\mathcal{A}$ *into* $\mathcal{B}$ is a map $f$ defined on $\Omega^2$ so that

$$f(E,F) \in \mathcal{L}(\mathcal{A}(E,F), \mathcal{B}(E,F))$$

for all $E, F \in \Omega$. The functor $f$ is called *isometric (isomorphic)* if $f(E,F)$ is an isometry (isomorphy) for all $E, F \in \Omega$. If $\mathcal{A}(E,F)$ is a Banach subspace of $\mathcal{B}(E,F)$ and $i(E,F)$ denotes the inclusion map

$$\mathcal{A}(E,F) \to \mathcal{B}(E,F)$$

for all $E, F \in \Omega$ then $\mathcal{A}$ is called a *Banach subsystem of* $\mathcal{B}$ and $i$ is called *the inclusion functor of* $\mathcal{A}$ *into* $\mathcal{B}$. We denote this sometimes by $\mathcal{A} \subset \mathcal{B}$. The map

$$(E,F) \mapsto \mathcal{A}(F,E)^\circ := \{x' \in \mathcal{B}'(E,F) | x' = 0 \text{ on } \mathcal{A}(F,E)\}$$

defined on $\Omega^2$ is a Banach subsystem of $\mathcal{B}'$. It is denoted by $\mathcal{A}^\circ$.

If $f$ is a functor of $\mathcal{A}$ into $\mathcal{B}$ then the map

$$(E,F) \mapsto f(F,E)'$$

defined on $\Omega^2$ is a functor of $\mathcal{B}'$ into $\mathcal{A}'$. It is called *the transposed functor of* $f$ and is denoted by $f'$. The transposed functor of $f'$ is called *the bitransposed functor of* $f$ and is denoted by $f''$.

Let $(\Omega, \mathcal{A})$ be a Banach system. The functor of $\mathcal{A}$ into $\mathcal{A}$ which maps every $(E,F) \in \Omega^2$ into the identity map of $\mathcal{A}(E,F)$ is called *the identity functor of* $\mathcal{A}$. The functor of $\mathcal{A}$ into $\mathcal{A}''$ which maps every $(E,F) \in \Omega^2$ into the evaluation

$$\mathcal{A}(E,F) \to \mathcal{A}''(E,F)$$

of $\mathcal{A}(E,F)$ is called *the evaluation functor of* $\mathcal{A}$.

Let $(\Omega,\mathcal{A}),(\Omega,\mathcal{B})$, and $(\Omega,\mathcal{C})$ be Banach systems, $f$ be a functor of $\mathcal{A}$ into $\mathcal{B}$, and $g$ be a functor of $\mathcal{B}$ into $\mathcal{C}$. The map

$$(E,F)\mapsto g(E,F)\circ f(E,F)$$

defined on $\Omega^2$ is a functor of $\mathcal{A}$ into $\mathcal{C}$; it is called *the composition of the functors* $f$ *and* $g$ and it is denoted by $g\circ f$.

Let $(\Omega,\mathcal{A})$ and $(\Omega,\mathcal{B})$ be two Banach systems and $f$ be a functor of $\mathcal{A}$ into $\mathcal{B}$. If $i$ and $j$ denote the evaluation functors of $\mathcal{A}$ and $\mathcal{B}$ resepctively then

$$j\circ f = f''\circ i\,.$$

The map defined on $\Omega^2$ which sends every $(E,F)\in\Omega^2$ into the image of $f(E,F)$ is a Banach subsystem of $\mathcal{B}$. It is denoted by $f(\mathcal{A})$.

Let $(\Omega,\mathcal{A})$ be a Banach system. A *projection of* $\mathcal{A}$ is a functor $p$ of $\mathcal{A}$ into $\mathcal{A}$ so that

$$p\circ p = p\,.$$

Let $(\Omega,\mathcal{A})$ be a Banach system and $(\Omega,\mathcal{B})$ be a Banach subsystem of $(\Omega,\mathcal{A})$. We set

$$\mathcal{A}/\mathcal{B}(E,F)=\mathcal{A}(E,F)/\mathcal{B}(E,F)$$

and denote by $q(E,F)$ the quotient map

$$\mathcal{A}(E,F)\mapsto\mathcal{A}/\mathcal{B}(E,F)$$

for all $E,F\in\Omega$. Then $(\Omega,\mathcal{A}/\mathcal{B})$ is a Banach system and $q$ is a functor of $\mathcal{A}$ into $\mathcal{A}/\mathcal{B}$; $\mathcal{A}/\mathcal{B}$ is called *the quotient Banach system of* $\mathcal{A}$ *through* $\mathcal{B}$ and $q$ is called *the quotient functor*.

Let $(\Omega,\mathcal{A})$ and $(\Omega,\mathcal{B})$ be Banach systems and $f$ be a functor of $\mathcal{A}$ into $\mathcal{B}$. We denote by $\mathrm{Ker}f$ the map

$$(E,F)\mapsto \mathrm{Ker}f(E,F):=\{x\in\mathcal{A}(E,F)\,|\,f(E,F)x=0\}\,;$$

it is a Banach subsystem of $\mathcal{A}$.

Let $(\Omega, \mathcal{A})$ be a Banach system and $(\Omega, \mathcal{B}), (\Omega, \mathcal{C})$ be Banach subsystems of $\mathcal{A}$. If $\mathcal{A}(E,F)$ is the direct sum of $\mathcal{B}(E,F)$ and $\mathcal{C}(E,F)$, i.e.

$$\mathcal{A}(E,F) = \mathcal{B}(E,F) \oplus \mathcal{C}(E,F)$$

for all $E, F \in \Omega$ then we say that $\mathcal{A}$ is *the direct sum of* $\mathcal{B}$ *and* $\mathcal{C}$ and denote this by

$$\mathcal{A} = \mathcal{B} \oplus \mathcal{C}.$$

**Proposition 1** ([3] Th éorème 15) *Let* $(\Omega, \mathcal{A})$ *be a Banach system,* $i$ *and* $j$ *be the evaluation functor of* $\mathcal{A}$ *and* $\mathcal{A}'$ *respectively, and*

$$p := j \circ i'.$$

*We have:*

*a)* $i' \circ j$ *is the identity functor of* $\mathcal{A}'$.

*b)* $p$ *is a projection of* $\mathcal{A}'''$, $p(\mathcal{A}''') = j(\mathcal{A}')$, *and* $\|p_{E,F}\| \leq 1$ *for all* $E, F \in \Omega$.

*c)* $\operatorname{Ker} p = i(\mathcal{A})^\circ$.

*d)* $\mathcal{A}''' = j(\mathcal{A}') \oplus i(\mathcal{A})^\circ$.

*e) If* $q$ *and* $r$ *are the quotient functors of* $\mathcal{A}''$ *into* $\mathcal{A}''/i(\mathcal{A})$ *and* $\mathcal{A}'''$ *into* $\mathcal{A}'''/j(\mathcal{A}')$ *then the functor* $r \circ q'$ *is isometric.*

a) Let $E, F \in \Omega$ and

$$E \xrightarrow[\mathcal{A}]{x} F, \quad F \xrightarrow[\mathcal{A}']{x'} E.$$

We have

$$\langle x, i' \circ j(x') \rangle = \langle ix, jx' \rangle = \langle ix, x' \rangle = \langle x, x' \rangle$$

so

$$i' \circ j(x') = x'$$

and $i' \circ j$ is the identity functor of $\mathcal{A}'$.

b) By a)

$$p \circ j = j \circ i' \circ j = j\,,$$

$$p \circ p = p \circ j \circ i' = j \circ i' = p\,,$$

so $p$ is a projection of $\mathcal{A}'''$ and from

$$j(\mathcal{A}') = p \circ j(\mathcal{A}') \subset p(\mathcal{A}''') = j \circ i'(\mathcal{A}''') \subset j(\mathcal{A}')$$

we get

$$p(\mathcal{A}''') = j(\mathcal{A}')\,.$$

We have

$$\|p_{EF}\| \leq \|j_{EF}\|\,\|i'_{EF}\| \leq 1$$

for all $E, F \in \Omega$.

c) By a)

$$i' \circ p = i' \circ j \circ i' = i'$$

so

$$\text{Ker } i' \supset \text{Ker } p = \text{Ker}(j \circ i') \supset \text{Ker } i'\,,$$

$$\text{Ker } p = \text{Ker } i' = i(\mathcal{A})^{\circ}\,.$$

d) follows from b), c), and Murray's theorem.

e) We have

$$q'((\mathcal{A}''/i(\mathcal{A}))') = i(\mathcal{A})^{\circ}$$

and the assertion follows from d). □

**Proposition 2** *Let $E$ be a Banach space, $j : E \to E''$ be the evaluation, and $p$ be a projection of $E''$ on $j(E)$. We set*

$$F := Ker\ p \quad , \quad F^{\circ} := \bigcap_{x'' \in F} Ker\ x''\,,$$

$$u : E \to (F^{\circ})'\,,\ x \mapsto (jx)|F^{\circ}\,.$$

*We have:*

*a)* $u \in \mathcal{L}(E,(F^\circ)'), \|u\| \leq 1$.

*b)* *For every* $y' \in (F^\circ)'$ *there is an* $x \in E$ *so that*

$$ux = y', \quad \|x\| \leq \|p\|\|y'\|.$$

*c)* $u$ *is an isomorphism iff* $F$ *is closed in the topology of pointwise convergence of* $E''$.

*d)* $u$ *is an isometry iff* $F$ *is closed in the topology of pointwise convergence of* $E''$ *and* $\|p\| \leq 1$.

a) $u$ is linear and

$$\|ux\| = \|(jx)|F^\circ\| \leq \|jx\| = \|x\|$$

for every $x \in E$.

b) By Hahn-Banach theorem there is an $x'' \in E''$ so that

$$x''|F^\circ = y', \quad \|x''\| = \|y'\|.$$

Let $x \in E$ with

$$jx = px''.$$

Then

$$x'' - px'' \in F$$

so

$$\langle ux, x'\rangle = \langle jx, x'\rangle = \langle px'', x'\rangle = \langle x'', x'\rangle = \langle y', x'\rangle$$

for every $x' \in F^\circ$ and

$$ux = y'.$$

Moreover

$$\|x\| = \|jx\| = \|px''\| \leq \|p\|\|x''\| = \|p\|\|y'\|.$$

c) Assume $u$ is an isomorphism and let $x'' \in E'', x''|F^\circ = 0$. There is an $x \in E$ so that

$$jx = px''.$$

Since

$$x'' - px'' \in F$$

we get

$$\langle ux, x'\rangle = \langle jx, x'\rangle = \langle px'', x'\rangle = \langle x'', x'\rangle = 0$$

for every $x' \in F^{\circ}$. Hence

$$ux = 0 , \quad x = 0 , \quad x'' \in F$$

and $F$ is a closed set in the topology of pointwise convergence of $E''$.

Assume now $F$ is a closed set in the topology of pointwise convergence of $E''$. Let $x \in Ker\ u$. Then

$$j\ x \in F$$

so

$$j\ x = 0 , \quad x = 0 .$$

Hence $u$ is injective. By b) $u$ is bijective. By the principle of the inverse operator $u$ is am isomorphism.

d) Assume $u$ is an isometry and let $u'' \in E''$. Then

$$px'' - x'' \in F , \quad \|(px'' - x'')|F^{\circ}\| = 0 .$$

Let $x \in E$ so that

$$jx = px'' .$$

We have

$$\|px''\| = \|jx\| = \|x\| = \|ux\| = \|(jx)|F^{\circ}\| = \|(px'')|F^{\circ}\| \leq$$
$$\leq \|(px'' - x'')|F^{\circ}\| \leq \|x''\|$$

so $\|p\| \leq 1$. By c) $F$ is closed in the topology of pointwise convergence on $E''$.

The convex implication follows from a), b), and c). □

**Corollary 3** ([3] Théorème 17′) *Let $(\Omega, \mathcal{A})$ be a Banach system and $j$ be the evaluation functor of it. The following assertions are equivalent:*

*a) $\mathcal{A}$ is a dual Banach system;*

*b) there is a projection $p$ of $\mathcal{A}''$ with $p(\mathcal{A}'') = j(\mathcal{A})$ so that $\|p_{EF}\| \leq 1$ and $(Ker\ p)(E, F)$ is a closed set in the topology of pointwise convergence of $\mathcal{A}''(E, F)$ for all $E, F \in \Omega$;*

c) *there is a projection* $p$ *of* $\mathcal{A}''$ *with* $p(\mathcal{A}'') = j(\mathcal{A})$ *so that* $(\text{Ker } p)(E,F)$ *is a closed set in the topology of pointwise convergence of* $\mathcal{A}''(E,F)$ *for all* $E, F \in \Omega$.

a $\Rightarrow$ b follows form Proposition 1 b), c).

b $\Rightarrow$ c is trivial.

c $\Rightarrow$ a follows from Proposition 2 c). □

**Remark.** Let $\mu$ be an atomfree Radon measure on a Hausdorff space, $\mu \neq 0$, and let $j : L^1(\mu) \to L^1(\mu)''$ be the evaluation map. Then the unit ball of $L^1(\mu)$ has no extreme points, so $L^1(\mu)$ is not a dual space. Nevertheless there is a projection $p$ of $L^1(\mu)''$ on $L^1(\mu)$ with $p(L^1(\mu)'') = j(L^1(\mu))$, so that $\|p\| \leq 1$. This example shows that we cannot drop the hypothesis "$(\text{Ker } p)(E,F)$ is a closed set for the topology of pointwise convergence of $\mathcal{A}''(E,F)$ for all $E, F \in \Omega$" in b).

## 2. Banach categories

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be Banach systems over the same class $\Omega$. An $(\mathcal{A}, \mathcal{B}, \mathcal{C})$*-multiplication* is a map $\varphi$ defined on $\Omega^3$ so that the following holds for all $E, F, G \in \Omega$:

$\varphi(E, F, G)$ is a bilinear map

$$\mathcal{A}(E,F) \times \mathcal{B}(F,G) \to \mathcal{C}(E,G), (x,y) \mapsto yx$$

so that

$$\|yx\| \leq \|x\| \, \|y\|$$

for all

$$E \underset{\mathcal{A}}{\overset{x}{\to}} F \underset{\mathcal{B}}{\overset{y}{\to}} G.$$

A *left (right) multiplication on* $\mathcal{A}$ *over* $\mathcal{B}$ is an $(\mathcal{A}, \mathcal{B}, \mathcal{A})$-multiplication (a $(\mathcal{B}, \mathcal{A}, \mathcal{A})$-multiplication). A left and a right multiplication on $\mathcal{A}$ are called *compatible* if

$$(ax)b = a(xb)$$

for all $E, F, G, H \in \Omega$ and

$$E \overset{b}{\to} F \underset{\mathcal{A}}{\overset{x}{\to}} G \overset{a}{\to} H.$$

An *inner multiplication on* $\mathcal{A}$ is an $(\mathcal{A},\mathcal{A},\mathcal{A})$–multiplication so that

$$(xy)z = x(yz)$$

for all $E,F,G,H \in \Omega$ and

$$E \xrightarrow{z} F \xrightarrow{y} G \xrightarrow{x} H \, .$$

A left and a right multiplication on $\mathcal{A}$ over $\mathcal{B}$ are called *simultaneous compatible with an inner multiplication on* $\mathcal{A}$ if

$$(xa)y = x(ay)$$

for all $E,F,G,H \in \Omega$ and

$$E \underset{\mathcal{A}}{\xrightarrow{y}} F \underset{\mathcal{B}}{\xrightarrow{a}} G \underset{\mathcal{A}}{\xrightarrow{x}} H \, .$$

A *Banach category* is a Banach system endowed with an inner multiplication.

If $\Omega$ is a class of Banach space then the Banach system

$$(E,F) \mapsto \mathcal{L}(E,F)$$

over $\Omega$ endowed with the usual composition of maps as inner multiplication is a Banach category. We denote it by $\mathcal{L}$.

Let $E$ be a Banach algebra. We may consider $E$ as a Banach category with a unique object and the multiplcation of $E$ as the inner multiplication of this category.

Let $\Omega$ be a class and $\Lambda$ be a Banach category over $\Omega$. A *left (right)* $\Lambda$*–modulus* is a Banach system $\mathcal{A}$ over $\Omega$ endowed with a left (right) multiplication over $\Lambda$ so that

$$(ab)x = a(bx) \qquad\qquad ((xa)b = x(ab))$$

for all $E,F,G,H \in \Omega$ and

$$E \underset{\mathcal{A}}{\xrightarrow{x}} F \underset{\Lambda}{\xrightarrow{b}} G \underset{\Lambda}{\xrightarrow{a}} H \qquad\qquad \left(E \underset{\Lambda}{\xrightarrow{b}} F \underset{\Lambda}{\xrightarrow{a}} G \underset{\mathcal{A}}{\xrightarrow{x}}\right) .$$

**Proposition 4** *Let* $(\Omega,\Lambda)$ *be a Banach category,* $\mathcal{A}$ *be a left* $\Lambda$*–modulus, and* $H$ *be a Banach space. We set*

$$\mathcal{A}_H(E,F) := \mathcal{L}(\mathcal{A}(F,E),H)$$

*for all* $E, F \in \Omega$ *and*

$$ua : \mathcal{A}(G,E) \longrightarrow H, y \mapsto u[ax]$$

*for all* $E, F, G \in \Omega$ *and*

$$E \underset{\Lambda}{\overset{a}{\longrightarrow}} F \underset{\mathcal{A}_H}{\overset{u}{\longrightarrow}} G\,.$$

*Then the Banach system* $A_H$ *endowed with the above multiplication is a right* $\Lambda$-*modulus. The assertion still holds if we interchange left and right.*

It is easy to see that the defined maps form a multiplication. Let $E, F, G, I \in \Omega$ and

$$E \underset{\Lambda}{\overset{a}{\longrightarrow}} F \underset{\Lambda}{\overset{b}{\longrightarrow}} G \underset{\mathcal{A}_H}{\overset{u}{\longrightarrow}} I\,.$$

We have

$$(u(ba))[x] = u[(ba)x] = u[b(ax)] = (ub)[ax] = ((ub)a)[x]$$

for every

$$I \underset{\mathcal{A}}{\overset{x}{\longrightarrow}} E$$

so

$$u(ba) = (ub)a\,. \quad \square$$

Let $(\Omega, \Lambda), (\Omega, \Delta)$ be two Banach categories. A $(\Lambda, \Delta)$-*modulus* is a Banach system over $\Omega$ endowed with the structures of a left $\Lambda$- modulus and of a right $\Delta$-modulus so that the left and the right multiplications are compatible.

Let $\Lambda$ be a Banach category. A $\Lambda$-*modulus* is a $(\Lambda, \Lambda)$-modulus. A $\Lambda$-*category* is a $\Lambda$-modulus $\mathcal{A}$ endowed with an inner multiplication so that each left multiplication on $\mathcal{A}$ is compatible with each right multiplication on $\mathcal{A}$ and the left and the right multiplication on $\mathcal{A}$ over $\Lambda$ are simultaneously compatible with the inner multiplication on $\mathcal{A}$. $\Lambda$ is a $\Lambda$-category.

**Corollary 5** *Let* $(\Omega, \Lambda), (\Omega, \Delta)$ *be two Banach categories and* $\mathcal{A}$ *be a* $(\Lambda, \Delta)$-*modulus. Then* $\mathcal{A}'$ *is a* $(\Delta, \Lambda)$-*modulus. Hence if* $\mathcal{A}$ *is a* $\Lambda$-*modulus then* $\mathcal{A}'$ *is a* $\Lambda$-*modulus too.*

By the Proposition $\mathcal{A}'$ is a left $\Delta$-modulus and a right $\Lambda$-modulus. Let $E, F, G, H \in \Omega$ and

$$E \underset{\Lambda}{\overset{a}{\longrightarrow}} F \underset{\mathcal{A}'}{\overset{x'}{\longrightarrow}} G \underset{\Delta}{\overset{b}{\longrightarrow}} H\,.$$

We have

$$\langle x, (bx')a\rangle = \langle ax, bx'\rangle = \langle (ax)b, x'\rangle = \langle a(xb), x'\rangle = \langle xb, x'a\rangle = \langle x, b(x'a)\rangle$$

for every $x \in \mathcal{A}(H,E)$, so

$$(bx')a = b(x'a)). \quad \square$$

Let $(\Omega, \mathcal{A})$, $(\Omega, \mathcal{B})$ be Banach categories. A *functor of Banach categories of $\mathcal{A}$ into $\mathcal{B}$* is a functor $f$ of $\mathcal{A}$ into $\mathcal{B}$ so that

$$f(xy) = f(x)f(y)$$

for all $E, F, G \in \Omega$ and

$$E \underset{\mathcal{A}}{\overset{y}{\rightarrow}} F \underset{\mathcal{A}}{\overset{x}{\rightarrow}} G.$$

Let $(\Omega, \Lambda)$ be a Banach category and $\mathcal{A}, \mathcal{B}$ be left (right) $\Lambda$–moduli. A *functor of left (right) $\Lambda$-moduli of $\mathcal{A}$ into $\mathcal{B}$* is a functor $f$ of $\mathcal{A}$ into $\mathcal{B}$ so that

$$f(ax) = af(x) \qquad\qquad (f(xa) = f(x)a)$$

for all $E, F, G \in \Omega$ and

$$E \underset{\mathcal{A}}{\overset{x}{\rightarrow}} F \underset{\Lambda}{\overset{a}{\rightarrow}} G \qquad\qquad \left(E \underset{\Lambda}{\overset{a}{\rightarrow}} F \underset{\mathcal{A}}{\overset{x}{\rightarrow}} G\right).$$

A *functor of $\Lambda$-moduli* is a functor of left and right $\Lambda$–moduli.

Let $\Lambda$ be a Banach category and $\mathcal{A}, \mathcal{B}$ be two $\Lambda$–categories. A *functor of $\Lambda$-categories of $\mathcal{A}$ into $\mathcal{B}$* is a functor of $\Lambda$–moduli of $\mathcal{A}$ into $\mathcal{B}$ which is at the same time a functor of Banach categories.

**Proposition 6** *Let $(\Omega, \Lambda)$ be a Banach category, $\mathcal{A}, \mathcal{B}$ be left (right) $\Lambda$-moduli, and $f$ be a functor of left (right) $\Lambda$-moduli of $\mathcal{A}$ into $\mathcal{B}$. Then $f'$ is a functor of right (left) $\Lambda$-moduli of $\mathcal{B}'$ into $\mathcal{A}'$.*

We have

$$f'(E,F) = f(F,E)' \in \mathcal{L}(\mathcal{B}(F,E)', \mathcal{A}(F,E)') = \mathcal{L}(\mathcal{B}'(E,F), \mathcal{A}'(E,F))$$

for all $E, F \in \Omega$, so $f'$ is a functor of $\mathcal{B}'$ into $\mathcal{A}'$.

Let $E, F, G \in \Omega$ and

$$E \underset{\Lambda}{\xrightarrow{a}} F \underset{\mathcal{B}'}{\xrightarrow{y'}} G .$$

Then

$$\begin{aligned}\langle x, f'(y'a)\rangle &= \langle fx, y'a\rangle = \langle afx, y'\rangle = \langle f(ax), y'\rangle = \\ &= \langle ax, f'y'\rangle = \langle x, (f'y')a\rangle\end{aligned}$$

for all $x \in \mathcal{A}(G, E)$ so

$$f'(y'a) = (f'y')a . \quad \square$$

**Proposition 7** *Let $(\Omega, \Lambda)$ be a Banach category. $\mathcal{A}$ be a left (right) $\Lambda$–modulus and $j$ be the evaluation of $\mathcal{A}$. Then $j$ is a functor of left (right) $\Lambda$–moduli of $\mathcal{A}$ into its bidual.*

We set

$$\tilde{x} := jx$$

for all $E, F \in \Omega$ and $x \in \mathcal{A}(E, F)$.

Let $E, F, G \in \Omega$ and

$$E \underset{\mathcal{A}}{\xrightarrow{x}} F \underset{\Lambda}{\xrightarrow{a}} G .$$

We have

$$\langle \widetilde{ax}, x'\rangle = \langle ax, x'\rangle = \langle x, x'a\rangle = \langle \tilde{x}, x'a\rangle = \langle a\tilde{x}, x'\rangle$$

for all $x' \in \mathcal{A}'(G, E)$ so

$$\widetilde{ax} = a\tilde{x} . \quad \square$$

## 3. Biduals of Banach categories

Let $(\Omega, \Lambda)$ be a Banach category and $\mathcal{A}$ be a left (right) $\Lambda$–modulus. We set

$$xx' : \Lambda(G, E) \to \mathsf{K}, a \mapsto \langle ax, x'\rangle$$

for all $E, F, G \in \Omega$ and

$$E \underset{\mathcal{A}'}{\xrightarrow{x'}} F \underset{\mathcal{A}}{\xrightarrow{x}} G$$

$$(x'x : \Lambda(G, E) \to \mathsf{K}, a \mapsto \langle xa, x'\rangle)$$

for all $E, F, G \in \Omega$ and

$$E \underset{\mathcal{A}}{\xrightarrow{x}} F \underset{\mathcal{A}'}{\xrightarrow{x'}} G) .$$

If $\mathcal{A} = \Lambda$ then it is easy to see that the above composition law coincides with the multiplication introduced in Proposition 4.

**Proposition 8** *Let $\Lambda$ be a Banach category and $\mathcal{A}$ be a left (right) $\Lambda$-modulus. The composition law introduced above is an $(\mathcal{A}', \mathcal{A}, \Lambda')$-multiplication $((\mathcal{A}, \mathcal{A}', \Lambda')$-multiplication) so that*

$$\begin{aligned} E \xrightarrow{x'} F \xrightarrow{x} G \xrightarrow{a} H &\Rightarrow (ax)x' = a(xx') \\ (E \xrightarrow{a} F \xrightarrow{x} G \xrightarrow{x'} H &\Rightarrow (x'x)a = x'(xa)) , \\ E \xrightarrow{a} F \xrightarrow{x'} G \xrightarrow{x} H &\Rightarrow (xx')a = x(x'a) \\ (E \xrightarrow{x} F \xrightarrow{x'} G \xrightarrow{a} H &\Rightarrow (ax')x = a(x'x)) , \end{aligned}$$

*where $E, F, G, H$ are objects of $\Lambda$, $a$ is a morphism of $\Lambda$, $x$ is a morphism of $\mathcal{A}$, and $x'$ is a morphism of $\mathcal{A}'$. If $\mathcal{A}$ is a $\Lambda$-modulus then*

$$\begin{aligned} E \xrightarrow{x} F \xrightarrow{a} G \xrightarrow{x'} H &\Rightarrow (x'a)x = x'(ax) , \\ E \xrightarrow{x'} F \xrightarrow{a} G \xrightarrow{x} H &\Rightarrow (xa)x' = x(ax') , \end{aligned}$$

*with the same conventions as above.*

The first assertion is easy to verify. We have

$$\langle b, (ax)x' \rangle = \langle b(ax), x' \rangle = \langle (ba)x, x' \rangle = \langle ba, xx' \rangle = \langle b, a(xx') \rangle$$

$$(\langle b, (x'x)a \rangle = \langle ab, xx' \rangle = \langle x(ab), x' \rangle = \langle (xa)b, x' \rangle = \langle b, x'(xa) \rangle) ,$$

$$\langle b, (xx')a \rangle = \langle ab, xx' \rangle = \langle (ab)x, x' \rangle = \langle a(bx), x' \rangle = \langle bx, x'a \rangle = \langle b, x(x'a) \rangle$$

$$(\langle b, (ax')x \rangle = \langle xb, ax' \rangle = \langle (xb)a, x' \rangle = \langle x(ba), x' \rangle = \langle ba, x'x \rangle = \langle b, a(x'x) \rangle)$$

for every $H \underset{\Lambda}{\xrightarrow{b}} E$, which proves the relations. If $\mathcal{A}$ is a $\Lambda$-modulus then

$$\langle b, (x'a)x \rangle = \langle xb, x'a \rangle = \langle a(xb), x' \rangle = \langle (ax)b, x' \rangle = \langle b, x'(ax) \rangle ,$$

$$(\langle b, (xa)x' \rangle = \langle b(xa), x' \rangle = \langle (bx)a, x' \rangle = \langle bx, ax' \rangle = \langle b, x(ax') \rangle ,$$

for every $H \underset{\Lambda}{\xrightarrow{b}} E$, which proves the last assertion. □

**Proposition 9** *Let* $\Lambda$ *be a Banach category. Then* $\Lambda''$ *is a* $\Lambda$*-modulus, the evaluation functor* $j$ *of* $\Lambda$ *is a functor of* $\Lambda$*- moduli, and*

$$(j_{GH}x)x' = xx', \qquad x'(j_{EF}y) = x'y$$

*for all objects* $E, F, G, H$ *of* $\Lambda$ *and*

$$E \underset{\Lambda}{\xrightarrow{y}} F \underset{\Lambda'}{\xrightarrow{x'}} G \underset{\Lambda}{\xrightarrow{x}} H .$$

By Corollary 5 $\Lambda''$ is a $\Lambda$–modulus and by Proposition 7 the evaluation functor of $\Lambda$ is a functor of $\Lambda$–moduli. We have

$$\langle z, (j_{GH}x)x' \rangle = \langle j_{GH}x, x'z \rangle = \langle x, x'z \rangle = \langle zx, x' \rangle = \langle z, xx' \rangle$$

for all $H \underset{\Lambda}{\xrightarrow{z}} F$ and

$$\langle z, x'(j_{EF}y) \rangle = \langle j_{EF}y, zx' \rangle = \langle y, zx' \rangle = \langle yz, x' \rangle = \langle z, x'y \rangle$$

for all $G \underset{\Lambda}{\xrightarrow{z}} E$. □

Let $(\Omega, \Lambda)$ be a Banach category and $\mathcal{A}$ be a left (right) $\Lambda$–modulus. We set

$$x'a'' : \mathcal{A}(G, E) \to \mathbb{K}, \ x \mapsto \langle a'', xx' \rangle$$

for all $E, F, G \in \Omega$ and

$$E \underset{\Lambda''}{\xrightarrow{a''}} F \underset{\mathcal{A}'}{\xrightarrow{x'}} G$$

$$(a''x' : \mathcal{A}(G, E) \to \mathbb{K}, \ x \mapsto \langle a'', x'x \rangle$$

for all $E, F, G \in \Omega$ and

$$E \underset{\mathcal{A}'}{\xrightarrow{x'}} F \underset{\Lambda''}{\xrightarrow{a''}} G).$$

If $\mathcal{A} = \Lambda$ then it is easy to see that the above composition law coincides with the one introduced at the beginning of this section.

**Proposition 10** *Let* $\Lambda$ *be a Banach category,* $\mathcal{A}$ *be a left (right)* $\Lambda$*-modulus, and* $E, F, G, H$ *be objects of* $\Lambda$*. We have:*

*a) The composition law introduced above is a right (left) multiplication on $\mathcal{A}'$ over $\Lambda''$ so that*

$$E \xrightarrow[\Lambda'']{a''} F \xrightarrow[\mathcal{A}']{x'} G \xrightarrow[\mathcal{A}]{x} H \Rightarrow x(x'a'') = (xx')a''$$

$$E \xrightarrow[\mathcal{A}]{x} F \xrightarrow[\mathcal{A}']{x'} G \xrightarrow[\Lambda'']{a''} H \Rightarrow a''(x'x) = (a''x')x\,.$$

*b) If $F \xrightarrow[\mathcal{A}']{x'} G$ and*

$$u : \mathcal{A}(G,H) \to \Lambda'(F,H)\,, x \mapsto xx'$$

$$(u : \mathcal{A}(E,F) \to \Lambda'(E,G)\,, x \mapsto x'x)$$

*then*

$$H \xrightarrow[\Lambda'']{a''} F \Rightarrow u'a'' = x'a''$$

$$(G \xrightarrow[\Lambda'']{a''} E \Rightarrow u'a'' = a''x')\,.$$

a) The first assertion is easy to verify. By Proposition 8

$$\langle a, x(x'a'')\rangle = \langle ax, x'a''\rangle = \langle a'', (ax)x'\rangle = \langle a'', a(xx')\rangle = \langle a, (xx')a''\rangle$$

$$(\langle a, a''(x'x)\rangle = \langle a'', (x'x)a\rangle = \langle a'', x'(xa)\rangle = \langle xa, a''x'\rangle = \langle a, (a''x')x\rangle)$$

for all $H \xrightarrow[\Lambda]{a} E$.

b) We have

$$\langle x, u'a''\rangle = \langle ux, a''\rangle = \langle xx', a''\rangle = \langle x, x'a''\rangle$$

for all $G \xrightarrow[\mathcal{A}]{x} H$

$$(\langle x, u'a''\rangle = \langle ux, a''\rangle = \langle x'x, a''\rangle = \langle x, a''x'\rangle$$

for all $E \xrightarrow[\mathcal{A}]{x} F)$. ◻

Let $(\Omega, \Lambda)$ be a Banach category. We set

$$x'' \dashv y'' : \Lambda'(G,E) \to \mathsf{K},\ x' \mapsto \langle y'', x'x''\rangle$$

$$x'' \vdash y'' : \Lambda'(G,E) \to \mathsf{K},\ x' \mapsto \langle x'', y''x'\rangle$$

for all $E, F, G \in \Omega$ and

$$E \xrightarrow[\Lambda'']{y''} F \xrightarrow[\Lambda'']{x''} G\,.$$

$\vdash$ and $\dashv$ are called the left and right Arens multiplications ([1],[2]); they do not coincide in general.

**Proposition 11** *Let* $\Lambda$ *be a Banach category,* $E, F, G, H$ *be objects of* $\Lambda$*, and* $\mathcal{A}$ *be a left (right)* $\Lambda$*-modulus. Then*

$$E \underset{\Lambda''}{\overset{a''}{\to}} F \underset{\Lambda''}{\overset{b''}{\to}} G \underset{\mathcal{A}'}{\overset{x'}{\to}} H \Rightarrow x'(b'' \dashv a'') = (x'b'')a'' .$$

$$(E \underset{\mathcal{A}'}{\overset{x'}{\to}} F \underset{\Lambda''}{\overset{a''}{\to}} G \underset{\Lambda''}{\overset{b''}{\to}} H \Rightarrow (b'' \vdash a'')x' = b''(a''x')) .$$

By Proposition 10 a)

$$\begin{aligned}\langle x, x'(b'' \dashv a'')\rangle &= \langle b'' \dashv a'', xx'\rangle = \langle a'', (xx')b''\rangle = \\ &= \langle a'', x(x'b'')\rangle = \langle x, (x'b'')a''\rangle\end{aligned}$$

$$\begin{aligned}(\langle x, (b'' \vdash a'')x'\rangle &= \langle b'' \vdash a'', x'x\rangle = \langle b'', a''(x'x)\rangle = \\ &= \langle b'', (a''x')x\rangle = \langle x, b''(a''x')\rangle)\end{aligned}$$

for every $H \underset{\mathcal{A}}{\overset{x}{\to}} E$. □

**Theorem 12** *If* $\Lambda$ *is a Banach category then:*

*a)* $\dashv$ *and* $\vdash$ *are inner multiplications on* $\Lambda''$ .

*b)* $\Lambda''$ *endowed with* $\dashv$ *(with* $\vdash$ *) is a* $\Lambda$*-category; we denote it by* $\Lambda''_{\dashv}$ *(by* $\Lambda''_{\vdash}$*).*

*c) The evaluation functor of* $\Lambda$ *into* $\Lambda''_{\dashv}$ *(into* $\Lambda''_{\vdash}$*) is a functor of* $\Lambda$*-categories.*

*d) If* $\mathcal{A}$ *is a left (right)* $\Lambda$*-modulus then* $\mathcal{A}'$ *is a right* $\Lambda''_{\dashv}$*-modulus (left* $\Lambda''_{\vdash}$*-modulus).*

*e)* $\Lambda'$ *is a left* $\Lambda''_{\vdash}$*-modulus and a right* $\Lambda''_{\dashv}$*-modulus.*

*f) If* $\Lambda'$ *is a* $(\Lambda''_{\vdash}, \Lambda''_{\dashv})$*-modulus then*

$$(z'' \dashv y'') \vdash x'' = z'' \dashv (y'' \vdash x'')$$

*for all objects* $E, F, G, H$ *of* $\Lambda$ *and*

$$E \underset{\Lambda''}{\overset{x''}{\to}} F \underset{\Lambda''}{\overset{y''}{\to}} G \underset{\Lambda''}{\overset{z''}{\to}} H .$$

*g)* *If $j$ denotes the evaluation functor of $\Lambda$ then*

$$x''x = x'' \vdash (j_{EF}x) = x'' \dashv (j_{EF}x),$$

$$yx'' = (j_{GH}y) \vdash x'' = (j_{GH}y) \dashv x''$$

*for all objects $E, F, G, H$ of $\Lambda$ and*

$$E \underset{\Lambda}{\overset{x}{\to}} F \underset{\Lambda''}{\overset{x''}{\to}} G \underset{\Lambda}{\overset{y}{\to}} H.$$

*h)* *The maps*

$$\Lambda''(E,F) \to \Lambda''(E,G), y'' \mapsto x'' \dashv y''$$

$$\Lambda''(G,H) \to \Lambda''(E,G), y'' \mapsto y'' \vdash x''$$

*are continuous with respect to the topologies of pointwise convergence for all objects $E, F, G, H$ of $\Lambda$ and every $F \underset{\Lambda''}{\overset{x''}{\to}} G$.*

It is easy to see that $\dashv$ and $\vdash$ are $(\Lambda'', \Lambda'', \Lambda'')$-multiplications. Let $E, F, G, H$ be objects of $\Lambda$. In order to simplify the writing we do the following conventions: the letters $x, y$ denote morphisms of $\Lambda$, the letter $x'$ denotes a morphism of $\Lambda'$, and the letters $x'', y'', z''$ denote morphisms of $\Lambda''$.

a) By Proposition 9 $\Lambda''$ is a $\Lambda$-modulus. Let

$$E \overset{x''}{\to} F \overset{y''}{\to} G \overset{z''}{\to} H$$

By Proposition 11

$$\begin{aligned}\langle (z'' \dashv y'') \dashv x'', x' \rangle &= \langle x'', x'(z'' \dashv y'') \rangle = \langle x'', (x'z'')y'' \rangle = \\ &= \langle y'' \dashv x'', x'z'' \rangle = \langle z'' \dashv (y'' \dashv x''), x' \rangle,\end{aligned}$$

$$\begin{aligned}\langle (z'' \vdash y'') \vdash x'', x' \rangle &= \langle z'' \vdash y'', x''x' \rangle = \langle z'', y''(x''x') \rangle = \\ &= \langle z'', (y'' \vdash x'')x' \rangle = \langle z'' \vdash (y'' \vdash x''), x' \rangle\end{aligned}$$

for every $H \overset{x'}{\to} E$, so

$$(z'' \dashv y'') \dashv x'' = z'' \dashv (y'' \dashv x''), \qquad (z'' \vdash y'') \vdash x'' = z'' \vdash (y'' \vdash x'').$$

b) $1^{st}$ step $E \xrightarrow{x''} F \xrightarrow{y''} G \xrightarrow{x} H \;\Rightarrow\; \begin{cases} x(y'' \dashv x'') = (xy'') \dashv x'' \\ x(y'' \vdash x'') = (xy'') \vdash x'' \end{cases}$

By Proposition 10

$$\begin{aligned} \langle x(y'' \dashv x''), x' \rangle &= \langle y'' \dashv x'', x'x \rangle = \langle x'', (x'x)y'' \rangle = \langle x'', x'(xy'') \rangle = \\ &= \langle (xy'') \dashv x'', x' \rangle , \end{aligned}$$

$$\begin{aligned} \langle x(y'' \vdash x''), x' \rangle &= \langle y'' \vdash x'', x'x \rangle = \langle y'', x''(x'x) \rangle = \\ &= \langle y'', (x''x')x \rangle = \langle xy'', x''x' \rangle = \langle (xy'') \vdash x'', x' \rangle \end{aligned}$$

for every $H \xrightarrow{x'} E$.

$2^{nd}$ step $E \xrightarrow{x} F \xrightarrow{x''} G \xrightarrow{y''} H \;\Rightarrow\; \begin{cases} (y'' \dashv x'')x = y'' \dashv (x''x) \\ (y'' \vdash x'')x = y'' \vdash (x''x) \end{cases}$

We have

$$\begin{aligned} \langle (y'' \dashv x'')x, x' \rangle &= \langle y'' \dashv x'', xx' \rangle = \langle x'', (xx')y'' \rangle = \\ &= \langle x'', x(x'y'') \rangle = \langle x''x, x'y'' \rangle = \langle y'' \dashv (x''x), x' \rangle , \end{aligned}$$

$$\begin{aligned} \langle (y'' \vdash x'')x, x' \rangle &= \langle y'' \vdash x'', xx' \rangle = \langle y'', x''(xx') \rangle = \\ &= \langle y'', (x''x)x' \rangle = \langle y'' \vdash (x''x), x' \rangle \end{aligned}$$

for every $H \xrightarrow{x'} E$ (Proposition 8)

$3^{rd}$ step $E \xrightarrow{x''} F \xrightarrow{x} G \xrightarrow{y''} H \;\Rightarrow\; \begin{cases} (y''x) \dashv x'' = y'' \dashv (xx'') \\ (y''x) \vdash x'' = y'' \vdash (xx'') \end{cases}$

We have

$$\begin{aligned} \langle (y''x) \dashv x'', x' \rangle &= \langle x'', x'(y''x) \rangle = \langle x'', (x'y'')x \rangle = \\ &= \langle xx'', x'y'' \rangle = \langle y'' \dashv (xx''), x' \rangle , \end{aligned}$$

$$\begin{aligned} \langle (y''x) \vdash x'', x' \rangle &= \langle y''x, x''x' \rangle = \langle y'', x(x''x') \rangle = \\ &= \langle y'', (xx'')x' \rangle = \langle y'' \vdash (xx''), x' \rangle \end{aligned}$$

for every $H \xrightarrow{x'} E$ (Proposition 8)

$4^{th}$ **step b)**

Follows from Proposition 8 and the preceding steps.

c) By Proposition 7 the evaluation functor $j$ of $\Lambda$ is a functor of $\Lambda$–moduli. Let $E \xrightarrow{x} F \xrightarrow{y} G$. We have

$$\langle j(yx), x' \rangle = \langle y, (jx), x' \rangle = \langle y(jx)x' \rangle = \langle jy, (jx)x' \rangle = \langle (jy) \vdash (jx), x' \rangle ,$$

$$\langle j(yx), x' \rangle = \langle (jy)x, x' \rangle = \langle x, x'(jy) \rangle = \langle jx, x'(jy) \rangle = \langle (jy) \dashv (jx), x' \rangle$$

for every $G \xrightarrow{x'} E$ so

$$j(yx) = (jy) \vdash (jx) = (jy) \dashv (jx) .$$

d) follows from b) and Proposition 11.

e) follows form Corollary 5 and d).

f) We have

$$\begin{aligned}\langle (z'' \dashv y'') \vdash x'', x' \rangle &= \langle z'' \dashv y'', x''x' \rangle = \langle y'', (x''x')z'' \rangle = \langle y'', x''(x'z'') \rangle = \\ &= \langle y'' \vdash x'', x'z'' \rangle = \langle z'' \dashv (y'' \vdash x''), x' \rangle\end{aligned}$$

for every $H \xrightarrow{x'} E$.

g) We have

$$\langle x'' \vdash (j_{EF}x), x' \rangle = \langle x'', (j_{EF}x)x' \rangle = \langle x'', xx' \rangle = \langle x''x, x' \rangle ,$$

$$\langle x'' \dashv (j_{EF}x), x' \rangle = \langle j_{EF}x, x'x'' \rangle = \langle x, x'x'' \rangle = \langle x''x, x' \rangle$$

for all $G \xrightarrow{x'} E$ and

$$\langle (j_{GH}y) \vdash x'', x' \rangle = \langle j_{GH}y, x''x' \rangle = \langle y, x''x' \rangle = \langle yx'', x' \rangle ,$$

$$\langle (j_{GH}y) \dashv x'', x' \rangle = \langle x'', x'(j_{GH}y) \rangle = \langle x'', x'y \rangle = \langle yx'', x' \rangle$$

for all $H \xrightarrow{x'} F$ (Proposition 9).

h) We have

$$\lim_{z''\to y''}\langle x'' \dashv z'', x'\rangle = \lim_{z''\to y''}\langle z'', x'x''\rangle = \langle y'', x'x''\rangle = \langle x'' \dashv y'', x'\rangle$$

for all $E \xrightarrow{y''} F, G \xrightarrow{x'} E$, and

$$\lim_{z''\to y''}\langle z'' \vdash x'', x'\rangle = \lim_{z''\to y''}\langle z'', x''x'\rangle = \langle y'', x''x'\rangle = \langle y'' \vdash x'', x'\rangle$$

for all $G \xrightarrow{y''} H, H \xrightarrow{x'} F$. □

**Proposition 13** ([4] Theorem 1) *Let $\Lambda$ be a Banach category, $E, F, G, H$ be objects of $\Lambda$, and $F \underset{\Lambda''}{\xrightarrow{x''}} G$. We denote by $j$ the evaluation functor of $\Lambda'$ and set*

$$u_{x'} : \Lambda(F,G) \to \Lambda'(E,G), x \mapsto xx'$$

$$(\textit{resp. } u_{x'} : \Lambda(F,G) \to \Lambda'(F,H), x \mapsto x'x)$$

*for every $E \underset{\Lambda'}{\xrightarrow{x'}} F$ (resp. $G \underset{\Lambda'}{\xrightarrow{x'}} H$). Then the following assertions are equivalent:*

a) $E \underset{\Lambda''}{\xrightarrow{y''}} F \Rightarrow x'' \vdash y'' = x'' \dashv y''$ (*resp.* $G \underset{\Lambda''}{\xrightarrow{y''}} H \Rightarrow y'' \vdash x'' = y'' \dashv x''$);

b) *the map*

$$\Lambda''(E,F) \to \Lambda''(E,G), y'' \mapsto x'' \vdash y''$$

$$(\text{resp. } \Lambda''(G,H) \to \Lambda''(F,H), y'' \mapsto y'' \dashv x''$$

*is continuous with respect to the topologies of pointwise convergence;*

c) $E \underset{\Lambda'}{\xrightarrow{x'}} F \Rightarrow u''_{x'}x'' = j_{EG}(x''x')$ (*resp.* $G \underset{\Lambda'}{\xrightarrow{x'}} H \Rightarrow u''_{x'}x'' = j_{FH}(x'x'')$);

d) $E \underset{\Lambda'}{\xrightarrow{x'}} F \Rightarrow u''_{x'}x'' \in j_{EG}(\Lambda'(E,G))$ ( *resp.*$G \underset{\Lambda'}{\xrightarrow{x'}} H \Rightarrow u''_{x'}x'' \in j_{FH}(\Lambda'(F,H))$.

a $\Rightarrow$ b follows from Theorem 12 h).

b $\Rightarrow$ a . Let $i$ be the evaluation functor of $\Lambda$. Then $i_{EF}(\Lambda(E,F))$ (resp. $i_{GH}(\Lambda(G,H))$) is dense in $\Lambda''(E,F)$ (resp. in $\Lambda''(G,H)$) with respect to the topology of pointwise convergence. By Theorem 12 g)

$$x'' \vdash y'' = x'' \dashv y'' \qquad (\text{resp. } y'' \vdash x'' = y'' \dashv x'')$$

for all $y'' \in i_{EF}(\Lambda(E,F))$ (resp. $y'' \in i_{GH}(\Lambda(G,\dashv))$. By continuity (b) and Theorem 12 h) we get

$$x'' \vdash y'' = x'' \dashv y'' \qquad (\textit{resp. } y'' \vdash x'' = y'' \dashv x'')$$

for all $E \xrightarrow[\Lambda'']{y''} F$ (resp. $G \xrightarrow[\Lambda'']{y''} H$).

a $\Rightarrow$ c. By Theorem 12 g) and Proposition 10 b)

$$\begin{aligned}\langle y'', u''_{x'}x''\rangle &= \langle x'', u'_{x'}y''\rangle = \langle x'', x'y''\rangle = \langle y'' \dashv x'', x'\rangle = \\ &= \langle y'' \vdash x'', x'\rangle = \langle y'', x''x'\rangle = \langle y'', j_{EG}(x''x')\rangle\end{aligned}$$

for all $G \xrightarrow[\Lambda'']{y''} E$ (resp.

$$\begin{aligned}\langle y'', u''_{x'}x''\rangle &= \langle x'', u'_{x''}y''\rangle = \langle x'', y''x'\rangle == \langle x'' \vdash y'', x'\rangle = \\ &= \langle x'' \dashv y'', x'\rangle = \langle y'', x'x''\rangle = \langle y'', j_{FM}(x'x'')\rangle\end{aligned}$$

for all $H \xrightarrow[\Lambda'']{y''} F$) so

$$u''_{x'}x'' = x''x' \qquad (\text{resp. } u''_{x'}x'' = x'x'').$$

c $\Rightarrow$ d is trivial.

d $\Rightarrow$ b By Proposition 10 b)

$$\langle x'' \vdash y'', x'\rangle = \langle x'', y''x'\rangle = \langle x'', u'_{x'}y''\rangle = \langle u''_{x'}x'', y''\rangle$$

for all $E \xrightarrow[\Lambda'']{y''} F$ and $G \xrightarrow[\Lambda']{x'} E$

$$(\text{resp. } \langle y'' \dashv x'', x'\rangle = \langle x'', x'y''\rangle = \langle x'', u'_{x'}y''\rangle = \langle u''_{x'}x'', y''\rangle$$

for all $G \xrightarrow[\Lambda'']{y''} H$ and $H \xrightarrow[\Lambda']{x'} F$). By d) the map

$$\Lambda''(E,F) \to \Lambda''(E,G), y'' \mapsto x'' \vdash y''$$

$$(\text{resp. } \Lambda''(G,H) \to \Lambda''(F,G), y'' \mapsto y'' \dashv x'')$$

is continuous with respect to the topologies of pointwise convergence. □

Let $(\Omega,\Lambda)$ be a Banach category and $\mathcal{A}$ be a $\Lambda$–modulus ($\Lambda$–category). A $\Lambda$-*submodulus ($\Lambda$–subcategory) of* $\mathcal{A}$ is a Banach subsystem $\mathcal{B}$ of $\mathcal{A}$ which is stable with respect to the multiplications of $\mathcal{A}$. Endowed with the restrictions of these multiplications $\mathcal{B}$ becomes a $\Lambda$–modulus ($\Lambda$–category) and the inclusion functor fo $\mathcal{B}$ into $\mathcal{A}$ is a functor of $\Lambda$–moduli ($\Lambda$–categories). By factorisations we may define the corresponding multiplications on $\mathcal{A}/\mathcal{B}$ which becomes by that a $\Lambda$–modulus ($\Lambda$–category). The quotient functor of $\mathcal{A}$ into $\mathcal{A}/\mathcal{B}$ is then a functor of $\Lambda$–moduli ($\Lambda$–categories).

Let $\Omega$ be a class of Banach spaces. If we denote for every $E,F\in\Omega$ by $\mathcal{K}(E,F)$ the Banach space of compact operators of $E$ into $F$, then $\mathcal{K}$ is a Banach $\mathcal{L}$–subcategory. The Banach $\mathcal{L}$–category $\mathcal{L}/\mathcal{K}$ is called *the Calkin category.*

**Proposition 14** *Let $(\Omega,\Lambda)$ be a Banach category, $\mathcal{A}$ be a $\Lambda$–modulus, and $\mathcal{B}$ be a $\Lambda$–submodulus of $\mathcal{A}$. Then $\mathcal{B}^\circ$ is a $\Lambda$–submodulus of $\mathcal{A}'$. If we denote for every $E,F\in\Omega$ by $f(E,F)$ the canonical isometry*

$$\mathcal{A}'/\mathcal{B}^\circ(E,F)\to\mathcal{B}'(E,F)$$

*then $f$ is an isometric functor of $\Lambda$–moduli of $\mathcal{A}'/\mathcal{B}^\circ$ into $\mathcal{B}'$.*

Let $E,F,G,H\in\Omega$ and

$$E\xrightarrow[\Lambda]{a}F\xrightarrow[\mathcal{B}^\circ]{x'}G\xrightarrow[\Lambda]{b}H\,.$$

We have

$$\langle x,x'a\rangle=\langle ax,x'\rangle=0\,,$$
$$\langle y,bx'\rangle=\langle yb,x'\rangle=0$$

for all

$$G\xrightarrow[\mathcal{B}]{x}E\qquad H\xrightarrow[\mathcal{B}]{y}F\,,$$

so

$$x'a\in\mathcal{B}^\circ(E,G)\qquad bx'\in\mathcal{B}^\circ(F,H)\,.$$

Hence $\mathcal{B}^\circ$ is a $\Lambda$–submodulus of $\mathcal{A}'$.

Let $E,F,G,H\in\Omega$,

$$E\xrightarrow[\Lambda]{a}F\xrightarrow[\mathcal{A}']{x'}G\xrightarrow[\Lambda]{b}H\,,$$

and $q$ be the quotient functor of $\mathcal{A}'$ into $\mathcal{A}'/\mathcal{B}^\circ$. We have

$$\langle x, f((qx')a)\rangle = \langle x, (qx')a\rangle = \langle ax, qx'\rangle = \langle ax, f(qx')\rangle = \langle x, (f(qx'))a\rangle,$$

$$\langle y, f(b(qx'))\rangle = \langle y, b(qx')\rangle = \langle yb, qx'\rangle = \langle yb, f(qx')\rangle = \langle y, bf(qx')\rangle$$

for all

$$G \xrightarrow[\mathcal{B}]{x} E, \qquad H \xrightarrow[\mathcal{B}]{y} F,$$

which proves the last assertion. □

**Proposition 15** *Let $(\Omega, \mathcal{A})$ be a Banach system, $i$ and $j$ be the evaluation functors of $\mathcal{A}$ and $\mathcal{A}'$ respectively, and*

$$p := j \circ i'.$$

*If $\mathcal{A}'$ is a Banach category so that $i(\mathcal{A})$ is an $\mathcal{A}'$-submodulus of $\mathcal{A}''$ then $i(\mathcal{A})^\circ$ is an $\mathcal{A}'''_{\vdash}$-submodulus and an $\mathcal{A}'''_{\dashv}$-submodulus.*

$1^{st}$ step $E \xrightarrow[i(\mathcal{A})]{x''} F \xrightarrow[i(\mathcal{A})^\circ]{x'''} G \xrightarrow[i(\mathcal{A})]{y''} H \Rightarrow x'''x'' = 0, y''x''' = 0.$

We have

$$\langle x', x'''x''\rangle = \langle x''', x''x'\rangle = 0,$$

$$\langle y', y''x'''\rangle = \langle x''', y'y''\rangle = 0$$

for all

$$G \xrightarrow[\mathcal{A}']{x'} E, \quad H \xrightarrow[\mathcal{A}']{y'} F.$$

$2^{nd}$ step $E \xrightarrow[i(\mathcal{A})^\circ]{x'''} F \xrightarrow[i(\mathcal{A})^\circ]{y'''} G \Rightarrow y''' \vdash x''', y''' \dashv x''' \in (i(\mathcal{A})^\circ)(E,G).$

We have

$$\langle y''' \vdash x''', x''\rangle = \langle y''', x'''x''\rangle = 0,$$

$$\langle y''' \dashv x''', x''\rangle = \langle x''', x''y'''\rangle = 0$$

for all

$$G \xrightarrow[i(\mathcal{A})]{x''} E$$

by the first step.

$3^{rd}$ step $E \underset{j(\mathcal{A}')}{\overset{x'''}{\rightarrow}} F \underset{i(\mathcal{A})^\circ}{\overset{y'''}{\rightarrow}} G \underset{j(\mathcal{A}')}{\overset{z'''}{\rightarrow}} H \Rightarrow \left\{ \begin{array}{l} y''' \vdash x''', y''' \dashv x''', z''' \vdash y''', z''' \dashv y''' \\ are\ morphisms\ of\ i(\mathcal{A})^\circ . \end{array} \right.$

Let

$$E \underset{\mathcal{A}'}{\overset{x'}{\rightarrow}} F, \quad G \underset{\mathcal{A}'}{\overset{z'}{\rightarrow}} H$$

with

$$x''' = j_{EF}x', \quad z''' = j_{GH}z'.$$

We have

$$\langle y''' \dashv x''', x'' \rangle = \langle x''', x''y''' \rangle = 0,$$

$$\begin{array}{l} \langle y''' \vdash x''', x'' \rangle = \langle y''', x'''x'' \rangle = \langle y''', (j_{EF}x')x'' \rangle = \langle y''', x'x'' \rangle = 0, \\ \langle z''' \vdash y''', y'' \rangle = \langle z''', y'''y'' \rangle = 0, \\ \langle z''' \dashv y''', y'' \rangle = \langle y''', y''z''' \rangle = \langle y''', y''(j_{GH}z') \rangle = \langle y''', y''z' \rangle = 0 \end{array}$$

for every

$$G \underset{i(\mathcal{A})}{\overset{x''}{\rightarrow}} E, \quad H \underset{i(\mathcal{A})}{\overset{y''}{\rightarrow}} F$$

by the first step and Proposition 9.

$4^{th}$ step $i(\mathcal{A})^\circ$ is an $\mathcal{A}'''_{\vdash}$-submodulus and an $\mathcal{A}'''_{\dashv}$-submodulus.

The assertion follows from the second and the third step and from Proposition 1 d). □

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