

EXTENDED CLASSES OF ANALYTIC FUNCTIONS OF SEVERAL VARIABLES

ADDENDUM*

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1. Introduction.

An oversight appears to have crept in the discussion on the structure of the space Γ_1 introduced in Section 7, p. 47 of the above paper. As such, we provide a more comprehensive account of this space, now replaced by Y_β in this note.

Our notation concerning $\{p_m\}$, $\{q_n\}$ and unexplained terms hereafter remain the same as in the original paper.

2. The Space Y_β : Let us define

$$Y_\beta \equiv Y_\beta(p, q) = \{a \in Q: \overline{\lim}(m+n) |a_{mn}| \frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}} \leq \beta\}, \beta > 0. \quad (1)$$

For each $\epsilon > 0$, let us introduce the function $\|\cdot, \beta + \epsilon\|$ on Y_β as follows:

$$\|a, \beta + \epsilon\| = |\cap_{a_{00}}| + \sum_{m+n \geq 1} \sum |a_{mn}| \left(\frac{m+n}{\beta + \epsilon}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}}, a \in Y_\beta. \quad (2)$$

At the outset we have

Proposition 2.2. The collection $D_\beta = \{\|\cdot, \beta + \epsilon\|, \epsilon > 0\}$ defines a decreasing family of norms on the space Y_β and (Y_β, T_β) , where T_β is the topology generated by D_β , is a complete metrizable locally convex space.

Proof: Following Lemma 3.7 or 7.7 of our paper we can easily see that $\|\cdot, \beta + \epsilon\|$ is a norm on Y_β .

If $\{r_n\}$ is a decreasing sequence of real numbers such that $r_n \rightarrow 0$, then the countable family $\{\|\cdot, \beta + r_n\|: n \geq 1\}$ also generates the topology T_β , thereby making (Y_β, T_β) a metrizable space.

For showing the completeness of the space Y_β , consider a Cauchy sequence $\{a^t\}$ in Y_β . So for each $\eta > 0$, there exists $t_0 \equiv t_0(\eta)$ such that

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$$|a_{00}^t - a_{00}^s| + \sum_{m+n \geq 1} \sum_{m+n \geq 1} |a_{mn}^t - a_{mn}^s| \left(\frac{m+n}{\beta+\epsilon}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} < \eta, \quad \forall t, s \geq t_0. \quad (3)$$

Consequently, for each fixed pair (m, n) of positive integers, $\{a_{mn}^t\}$ is a Cauchy sequence in \mathbb{K} . So we can find a matrix $a = [a_{mn}]$ in Ω such that $a_{mn}^t \rightarrow a_{mn}$ as $t \rightarrow \infty$. Therefore using (3) we have

$$|a_{00}^t - a_{00}| < \eta; \quad |a_{mn}^t - a_{mn}| \left(\frac{m+n}{\beta+\epsilon}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} < \eta, \quad \forall m+n \geq 1. \quad (4)$$

As $a^t \in Y_\beta$, for $\epsilon_0 < \epsilon$ and $m+n \geq N_0$ (say) we get

$$\left(\frac{m+n}{\beta+\epsilon}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} |a_{mn}| < \eta + \left(\frac{\beta+\epsilon_0}{\beta+\epsilon}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}}. \quad (5)$$

Since the sequence of scalars appearing in the right handside of (5) tends to zero, we find a positive constant K such that

$$(m+n) |a_{mn}| \leq (\beta+\epsilon) K \left(\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}\right)^{\frac{M^{-1}+N^{-1}}{M^{-1}+N^{-1}}}. \quad (6)$$

Hence $a \in Y_\beta$. Making use of (3) once again, we conclude that $a^t \rightarrow a$ in T_β .

To characterize the dual of Y_β we will require

Lemma 2.3. For each $a \in Y_\beta$, the series

$$\sum_{m+n \leq 1} \sum_{m+n \leq 1} (m+n) \left(\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} a_{mn} c_{mn} \quad (7)$$

converges if and only if for some $\epsilon > 0$,

$$|c_{mn}| \left(\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}\right)^{\frac{M^{-1}+N^{-1}}{M^{-1}+N^{-1}}} \leq \frac{1}{\beta+\epsilon}, \quad \forall m+n \geq 1 \text{ and } |c_{00}| \leq 1. \quad (8)$$

Proof. The sufficiency part is similar to Lemma 7.7.

Conversely, assume that the series (7) converges but (8) does not hold. So for each $k \geq 1$, there exists $\{m_k\}$ and $\{n_k\}$ such that

$$|c_{m_k n_k}| > \left(\frac{1}{\beta+\frac{1}{k}}\right)^{\frac{p_{m_k}^{-1}+q_{n_k}^{-1}}{M^{-1}+N^{-1}}}. \quad (9)$$

Now if $a \in \Omega$ is given by

$$\begin{aligned}
 \frac{p_m^{-1} + q_n^{-1}}{M^{-1} + N^{-1}} a_{mn} &= \begin{cases} \left(\beta + \frac{1}{k}\right) \frac{p_m^{-1} + q_n^{-1}}{M^{-1} + N^{-1}}, & m=m_k; n=n_k; \\ 0 & , \text{ elsewhere,} \end{cases}
 \end{aligned}$$

then $a \in Y_\beta$. But

$$\left| \frac{p_{m_k}^{-1} + q_{n_k}^{-1}}{M^{-1} + N^{-1}} a_{m_k n_k} c_{m_k n_k} \right| > 1, \quad \forall k \geq 1, \tag{10}$$

which is contradiction to (7). Hence the result follows.

The topological dual of (Y_β, T_β) is characterized as follows

Theorem 2.4. A continuous linear functional F on Y_β is precisely of the form

$$F(a) = a_{00} c_{00} + \sum_{m+n \geq 1} \sum_{(m+n)} \frac{p_m^{-1} + q_n^{-1}}{M^{-1} + N^{-1}} a_{mn} c_{mn}, \tag{11}$$

where for some $\epsilon > 0$

$$|c_{mn}| \frac{M^{-1} + N^{-1}}{p_m^{-1} + q_n^{-1}} \leq \frac{1}{\beta + \epsilon}, \quad \forall m, n \geq 1 \text{ and } |c_{00}| \leq 1. \tag{12}$$

Proof. The sufficiency part is straightforward.

Conversely, let F be an arbitrary continuous linear functional on Y_β . If $a^{(s)}$ is the s -th plane section of a in Y_β , then

$$F(a) = a_{00} F(e^{00}) + \lim_{s \rightarrow \infty} \sum_{1 \leq m+n \leq s} \sum a_{mn} F(e^{mn}). \tag{13}$$

Also, by continuity of F , we have

$$|F(a)| \leq \|a\|, \quad \beta + \epsilon$$

for all $a \in Y_\beta$ and some $\epsilon > 0$. Therefore, if $c \in \Omega$ is given by

$$c_{mn} = \left[\begin{array}{l} \frac{F(e^{mn})}{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}}; m+n \neq 0 \\ F(e^{00}) \quad ; m=0, n=0, \end{array} \right.$$

then, $c=(c_{mn})$ satisfies (12) and from (13) F has the required representation.

In order to establish the relation between the spaces $Y_\beta \equiv Y_\beta(p,q)$ and $\Gamma_2(p,q)$ we need

Definition 2.5. The sequences $\{p_m\}$ and $\{q_n\}$ are said to satisfy density condition, if

$$\lim m^{pm} > M; \lim n^{qn} > N \quad (14)$$

where M and N have the usual meaning.

Theorem 2.7. If p, q also satisfy the density condition (14), then $\Gamma_2(p,q)$ is a subspace of $Y_\beta(p,q)$ for each $\beta > 0$.

Proof. Let $a \in \Gamma_2(p,q)$. Then for each $\epsilon > 0$, there exists an integer $N_0 \equiv N_0(\epsilon)$ such that

$$(m+n)|a_{mn}| \frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}} < \frac{(m+n)\epsilon}{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}}, m+n \geq N_0. \quad (15)$$

In order to show the boundedness of the right hand side sequence, let us observe that

$$\frac{(m+n)}{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} \equiv \frac{(m+n)C}{[e^{-(m+n)}(m+n)^{m+n+\frac{1}{2}}\sqrt{2\pi}] \frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}},$$

where C is a constant. We can find an integer N_1 such that

$$p_m^{-1}+q_n^{-1} > (m+n)\ell - 2N_1\theta, \quad \forall m, n \geq N_1$$

where $\ell = \min\{h, k\}$ and $\theta = \max\{h, k\}$. Hence there exists a constant K such that

$$e \frac{(m+n)^{M^{-1}+N^{-1}}}{p_m^{-1}+q_n^{-1}} \leq K, \quad \forall m+n \geq 0. \quad (16)$$

Also, we can find $N_2 \in \mathbb{N}$ such that

So

$$\lim_{m+n \rightarrow \infty} \frac{(m+n)}{[(m+n)^{m+n+\frac{1}{2}}] \frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} \leq 1. \tag{18}$$

Also

$$(\sqrt{2\pi}) \frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}} \rightarrow 1.$$

Therefore by (15), 16) and (18), we have

$$\overline{\lim}(m+n) |a_{mn}| \frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}} = 0 \tag{19}$$

Thus $a \in Y_{\beta}(p,q)$ for each $\beta > 0$.

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