

ASYMPTOTIC BEHAVIOUR FOR DIFFERENCE EQUATIONS

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In this article, the objective is to provide criteria for the almost periodicity and convergence of solutions to equations of the form

$$(1) \quad x_{n+1} = f(n, x_n), \quad n \in Z \text{ or } n \in N,$$

or

$$(2) \quad x_{n+2} = f(n, x_n, x_{n+1}), \quad n \in Z \text{ or } n \in N.$$

Existence of almost periodic solutions for systems of the form

$$(3) \quad x_{n+1} = \Lambda x_n + f(n, x_n), \quad n \in Z,$$

will be proved, when f is an almost periodic sequence in respect to the first argument, uniformly with respect to the second in any bounded set.

The following result is known, and we state it here for having a comparison term with the results we shall establish in this chapter.

Theorem 1. Consider the system (1) and assume that f is almost periodic from Z to R^m , uniformly with respect to the second argument in any bounded set. Moreover, let f verify the monotonicity condition

$$(4) \quad \langle f(n, x) - f(n, y), x - y \rangle \geq k|x - y|^2, \quad k > 1,$$

for any $n \in Z$, any $x, y \in R^m$. If (1) has a bounded solution on Z , then this solution is almost periodic.

The proof of this theorem can be found in [1].

Now let us prove the convergence of bounded solutions of the

systems of the form (1). In other words, we are interested in solutions $\{x_n\}$, $n \in \mathbb{N}$, such that $x = \{x_n\} \in c$ (space of convergent sequences).

Theorem 2. Consider the system (1) where $f : \mathbb{N} \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a given map, and assume that,

$$(5) \quad \lim_{n \rightarrow \infty} f(n, x) = f_\infty(x),$$

uniformly with respect to the second argument in any bounded set. Moreover, let f verify the monotonicity condition (4) in Theorem 1, with respect to the second argument, for any $n \in \mathbb{N}$ and any $x, y \in \mathbb{R}^m$. If (1) has a bounded solution $\{x_n\}$ on \mathbb{N} , then $\{x_n\}$ is convergent.

Proof. The condition (5) means, given $M > 0$, for any $\varepsilon > 0$, there exists a positive integer N (depending on ε), such that

$$(6) \quad |f(n, x) - f_\infty(x)| < \varepsilon$$

for $n > N(\varepsilon)$ and $|x| < M$. Now let $\{x_n\}$, $n \in \mathbb{N}$, be the bounded solution of the system (1). Writing the system (1) for $n + p$ instead of n , one obtains

$$(7) \quad x_{n+p+1} = f(n + p, x_{n+p}).$$

Subtracting side by side (7) from (1) one obtains

$$(8) \quad x_{n+p+1} - x_{n+1} = f(n + p, x_{n+p}) - f(n, x_{n+p}) + f(n, x_{n+p}) - f(n, x_n),$$

$$n \in \mathbb{N}, p \geq 1.$$

By scalar multiplication of both sides in (8) by $x_{n+p} - x_n$, one obtains:

$$\begin{aligned} \langle x_{n+p+1} - x_{n+1}, x_{n+p} - x_n \rangle &= \langle f(n+p, x_{n+p}) - f(n, x_{n+p}), x_{n+p} - x_n \rangle \\ &\quad + \langle f(n, x_{n+p}) - f(n, x_n), x_{n+p} - x_n \rangle. \end{aligned}$$

Taking (4) into account we obtain:

$$\begin{aligned} \langle x_{n+p+1} - x_{n+1}, x_{n+p} - x_n \rangle &\geq k |x_{n+p} - x_n|^2 \\ &\quad + \langle f(n+p, x_{n+p}) - f(n, x_{n+p}), x_{n+p} - x_n \rangle, \\ k &> 1. \end{aligned}$$

Using Schwarz inequality we have

$$\begin{aligned} |x_{n+p+1} - x_{n+1}| |x_{n+p} - x_n| &\geq k |x_{n+p} - x_n|^2 \\ &\quad - |f(n+p, x_{n+p}) - f(n, x_{n+p})| |x_{n+p} - x_n|. \end{aligned}$$

Taking supremum with respect to n , for $n \geq N_0$, where N_0 is a large integer, one obtains:

$$\begin{aligned} \sup_{\substack{n \geq N_0 \\ p \in \mathbb{N}}} |x_{n+p+1} - x_{n+1}| \sup_{\substack{n \geq N_0 \\ p \in \mathbb{N}}} |x_{n+p} - x_n| &\geq k \sup_{\substack{n \geq N_0 \\ p \in \mathbb{N}}} |x_{n+p} - x_n|^2 \\ &\quad - \sup_{\substack{n \geq N_0 \\ p \in \mathbb{N}}} |f(n+p, x_{n+p}) - f(n, x_{n+p})| \sup_{\substack{n \geq N_0 \\ p \in \mathbb{N}}} |x_{n+p} - x_n|. \end{aligned}$$

Dividing both sides of inequality by $\sup_{\substack{n \geq N_0 \\ p \in \mathbb{N}}} |x_{n+p} - x_n|$, and taking into

account that

$$\sup_{\substack{n \geq N_0 \\ p \in \mathbb{N}}} |x_{n+p+1} - x_{n+1}| \leq \sup_{\substack{n \geq N_0 \\ p \in \mathbb{N}}} |x_{n+p} - x_n|,$$

we obtain

$$\sup_{\substack{n > N_0 \\ p \in \mathbb{N}}} |x_{n+p} - x_n| \geq k \sup_{\substack{n > N_0 \\ p \in \mathbb{N}}} |x_{n+p} - x_n| - \sup_{\substack{n > N_0 \\ p \in \mathbb{N}}} |f(n+p, x_{n+p}) - f(n, x_{n+p})|,$$

which implies

$$(9) \quad (k-1) \sup_{\substack{n > N_0 \\ p \in \mathbb{N}}} |x_{n+p} - x_n| \leq \sup_{\substack{n > N_0 \\ p \in \mathbb{N}}} |f(n+p, x_{n+p}) - f(n, x_{n+p})|.$$

Dividing both sides of inequality by $(k-1)$, one obtains

$$(10) \quad \sup_{\substack{n > N_0 \\ p \in \mathbb{N}}} |x_{n+p} - x_n| \leq (k-1)^{-1} \sup_{\substack{n > N_0 \\ p \in \mathbb{N}}} |f(n+p, x_{n+p}) - f(n, x_{n+p})|.$$

But according to (5) the right-hand side of (10) can be chosen arbitrarily small, provided N_0 is large enough. This implies that $\{x_n\}$ is a Cauchy sequence, which means that it is convergent.

Therefore, $\lim_{n \rightarrow \infty} x_n$ does exist, which means that $\{x_n\} \in c(\mathbb{N}, \mathbb{R}^m)$.

The next theorem is concerned with discrete processes described by second order difference equations, the almost periodicity property of bounded solutions being emphasized.

Theorem 3. Consider the system (2) and assume that f is almost periodic from Z to \mathbb{R}^m , uniformly with respect to the second and third arguments, in any bounded set. Moreover, let f verify the monotonicity condition:

$$(11) \quad \langle f(n, x, y) - f(n, \bar{x}, \bar{y}), x - \bar{x} + y - \bar{y} \rangle \geq k(|x - \bar{x}|^2 + |y - \bar{y}|^2),$$

$$k > 1$$

for any $n \in Z$, and any $x, y, \bar{x}, \bar{y} \in \mathbb{R}^m$. If (2) has a bounded solution

on Z , say $\{x_n\}$, then this solution is almost periodic.

Proof. Let $\{x_n\}$ be a bounded solution of (2) on Z .

Writing the system (2) for $n+p$ instead of n , and subtracting side by side, one obtains

$$x_{n+p+2} = f(n+p, x_{n+p}, x_{n+p+1}),$$

$$x_{n+p+2} - x_{n+2} = f(n+p, x_{n+p}, x_{n+p+1}) - f(n, x_n, x_{n+1}).$$

By adding and subtracting terms one obtains:

$$(12) \quad x_{n+p+2} - x_{n+2} = f(n+p, x_{n+p}, x_{n+p+1}) - f(n, x_{n+p}, x_{n+p+1}) \\ + f(n, x_{n+p}, x_{n+p+1}) - f(n, x_n, x_{n+1}).$$

By scalar multiplication of both sides in (12) by $x_{n+p} - x_n$ and $x_{n+p+1} - x_{n+1}$, one obtains

$$\langle x_{n+p+2} - x_{n+2}, x_{n+p} - x_n \rangle \\ = \langle f(n+p, x_{n+p}, x_{n+p+1}) - f(n, x_{n+p}, x_{n+p+1}), x_{n+p} - x_n \rangle \\ + \langle f(n, x_{n+p}, x_{n+p+1}) - f(n, x_n, x_{n+1}), x_{n+p} - x_n \rangle,$$

$$\langle x_{n+p+2} - x_{n+2}, x_{n+p+1} - x_{n+1} \rangle \\ = \langle f(n+p, x_{n+p}, x_{n+p+1}) - f(n, x_{n+p}, x_{n+p+1}), x_{n+p+1} - x_{n+1} \rangle \\ + \langle f(n, x_{n+p}, x_{n+p+1}) - f(n, x_n, x_{n+1}), x_{n+p+1} - x_{n+1} \rangle.$$

If we add the above inequalities, we obtain:

$$\begin{aligned}
& \langle x_{n+p+2} - x_{n+2}, x_{n+p} - x_n + x_{n+p+1} - x_{n+1} \rangle \\
&= \langle f(n+p, x_{n+p}, x_{n+p+1}) - f(n, x_{n+p}, x_{n+p+1}), x_{n+p+1} - x_{n+1} \\
&\quad + x_{n+p} - x_n \rangle + \langle f(n, x_{n+p}, x_{n+p+1}) - f(n, x_n, x_{n+1}), x_{n+p} \\
&\quad - x_n + x_{n+p+1} - x_{n+1} \rangle
\end{aligned}$$

one obtains taking (11) into account:

$$\begin{aligned}
& \langle x_{n+p+2} - x_{n+2}, x_{n+p+1} - x_{n+1} \rangle \geq k(|x_{n+p} - x_n|^2 \\
&\quad + |x_{n+p+1} - x_{n+1}|^2) + \langle f(n+p, x_{n+p}, x_{n+p+1}) \\
&\quad - f(n, x_{n+p}, x_{n+p+1}), x_{n+p} - x_n + x_{n+p+1} - x_{n+1} \rangle
\end{aligned}$$

using Schwarz inequality

$$\begin{aligned}
& |x_{n+p+2} - x_{n+2}| |x_{n+p} - x_n + x_{n+p+1} - x_{n+1}| \geq k(|x_{n+p} - x_n|^2 \\
&\quad + |x_{n+p+1} - x_{n+1}|^2) - |f(n+p, x_{n+p}, x_{n+p+1}) \\
&\quad - f(n, x_{n+p}, x_{n+p+1})| |x_{n+p} - x_n + x_{n+p+1} - x_{n+1}|
\end{aligned}$$

By triangle inequality one obtains:

$$\begin{aligned}
& |x_{n+p+2} - x_{n+2}| (|x_{n+p} - x_n| + |x_{n+p+1} - x_{n+1}|) \\
&\quad \geq k(|x_{n+p} - x_n|^2 + |x_{n+p+1} - x_{n+1}|^2) \\
&\quad - |f(n+p, x_{n+p}, x_{n+p+1}) - f(n, x_{n+p}, x_{n+p+1})| (|x_{n+1} - x_n| \\
&\quad\quad + |x_{n+p+1} - x_{n+1}|)
\end{aligned}$$

Taking supremum with respect to $n, n \in Z$, one obtains:

$$(13) \quad 2 \sup_{n \in Z} |x_{n+p} - x_n|^2 \geq 2k \sup_{n \in Z} |x_{n+p} - x_n|^2 \\ - 2 \sup_{n \in Z} |x_{n+p} - x_n| \sup_{n \in Z} |f(n+p, x_{n+p}, x_{n+p+1}) \\ - f(n, x_{n+p}, x_{n+p+1})|, \quad k > 1.$$

Excepting the case in which $\{x_n\}$ is periodic, with period p ,

(13) implies

$$(14) \quad \sup_{n \in Z} |x_{n+p} - x_n| \leq (k-1)^{-1} \sup_{n \in Z} |f(n+p, x_{n+p}, x_{n+p+1}) \\ - f(n, x_{n+p}, x_{n+p+1})|.$$

Taking into account the hypothesis of almost periodicity of f in the first argument, uniformly with respect to the second and third arguments in any bounded set, (14) shows that $\{x_n\}$ is an almost periodic process.

Theorem 4. Consider the system (2), where $f : N \times R^m \rightarrow R^m$ is a given map, and let us assume that

$$(15) \quad \lim_{n \rightarrow \infty} f(n, x, y) = f_{\infty}(x, y),$$

uniformly with respect to the second and third arguments in any bounded set.

Moreover, let f verify the monotonicity condition with respect to the second and third arguments:

$$(16) \quad \langle f(n, x, y) - f(n, \bar{x}, y), x - \bar{x} \rangle \geq k |x - \bar{x}|^2, \quad k > 1, \\ \langle f(n, x, y) - f(n, x, \bar{y}), y - \bar{y} \rangle \geq k |y - \bar{y}|^2, \quad k > 1,$$

for any $n \in N$, and any $x, y, \bar{x}, \bar{y} \in R^m$. If (2) has a bounded solution

$\{x_n\}$ on N , then $\{x_n\}$ is convergent.

Proof. Let $\{x_n\}$ be the bounded solution of (2). Writing the system (2) for $n+p$ instead of n , and subtracting side by side, one obtains:

$$\begin{aligned} x_{n+p+2} &= f(n+p, x_{n+p}, x_{n+p+1}), \\ (17) \quad x_{n+p+2} - x_{n+2} &= f(n+p, x_{n+p}, x_{n+p+1}) - f(n, x_{n+p}, x_{n+p+1}) \\ &\quad + f(n, x_{n+p}, x_{n+p+1}) - f(n, x_n, x_{n+1}). \end{aligned}$$

By scalar multiplication of both sides in (17) by $x_{n+p} - x_n$ and $x_{n+p+1} - x_{n+1}$, we obtain

$$\begin{aligned} \langle x_{n+p+2} - x_{n+2}, x_{n+p} - x_n \rangle &= \langle f(n+p, x_{n+p}, x_{n+p+1}) \\ &\quad - f(n, x_{n+p}, x_{n+p+1}), x_{n+p} - x_n \rangle + \langle f(n, x_{n+p}, x_{n+p+1}) \\ &\quad - f(n, x_n, x_{n+1}), x_{n+p} - x_n \rangle, \end{aligned}$$

and

$$\begin{aligned} \langle x_{n+p+2} - x_{n+2}, x_{n+p+1} - x_{n+1} \rangle &= \langle f(n+p, x_{n+p}, x_{n+p+1}) \\ &\quad - f(n, x_{n+p}, x_{n+p+1}), x_{n+p+1} - x_{n+1} \rangle + \langle f(n, x_{n+p}, x_{n+p+1}) \\ &\quad - f(n, x_n, x_{n+1}), x_{n+p+1} - x_{n+1} \rangle. \end{aligned}$$

If we add the above inequalities and take (16) into account, we obtain:

$$\begin{aligned} \langle x_{n+p+2} - x_{n+2}, x_{n+p} - x_n + x_{n+p+1} - x_{n+1} \rangle &\geq k(|x_{n+p} - x_n|^2 \\ &\quad + |x_{n+p+1} - x_{n+1}|^2) + \langle f(n+p, x_{n+p}, x_{n+p+1}) \\ &\quad - f(n, x_{n+p}, x_{n+p+1}), x_{n+p} - x_n + x_{n+p+1} - x_{n+1} \rangle \end{aligned}$$

Using Schwarz inequality implies:

$$\begin{aligned} & |x_{n+p+2} - x_{n+2}| (|x_{n+p} - x_n| + |x_{n+p+1} - x_{n+1}|) \\ & \geq k(|x_{n+p} - x_n|^2 + |x_{n+p+1} - x_{n+1}|^2) - |f(n+p, x_{n+p}, x_{n+p+1}) \\ & \quad - f(n, x_{n+p}, x_{n+p+1})| (|x_{n+p} - x_n| + |x_{n+p+1} - x_{n+1}|). \end{aligned}$$

Taking supremum with respect to $n \geq N_0$, one obtains:

$$\begin{aligned} \sup_{\substack{n > N_0 \\ p \in \mathbb{N}}} |x_{n+p} - x_n| & \leq (k-1)^{-1} \sup_{\substack{n > N_0 \\ p \in \mathbb{N}}} |f(n+p, x_{n+p}, x_{n+p+1}) \\ & \quad - f(n, x_{n+p}, x_{n+p+1})|. \end{aligned}$$

Taking into account condition (15) the right-hand side can be done arbitrarily small for N_0 large enough, which implies that $\{x_n\}$ is a Cauchy sequence. This completes the proof.

Let us now consider the equation

$$(18) \quad x_{n+1} = Ax_n + b_n, \quad n \in \mathbb{Z},$$

assuming the almost periodicity of the discrete process $\{b_n\}$, which takes its values in the complex vector space of m dimensions - C^m . Since A is a square matrix of order m , with complex entries, it is natural to look for solutions of (18) which are discrete processes with values in C^m . Of course, only bounded solutions of (18) can be discussed in regard to their almost periodicity, since any almost periodic discrete process is bounded on \mathbb{Z} . So this is what we will investigate below.

Theorem 5. Consider the linear system (18) Let $\{b_n\}$ be a bounded discrete process, A is a square matrix of order m , whose moduli of

eigenvalues are different of 1, ($|\lambda| \neq 1$). Assume $\{x_n\}$ is a discrete process satisfying the equation (18). Then there exists a unique bounded solution which satisfies equation (18). Moreover, the following estimate holds true:

$$(19) \quad \sup_{n \in \mathbb{Z}} |x_n| \leq k \sup_{n \in \mathbb{Z}} |b_n| ,$$

where $k > 0$ is a constant depending only on A . If $\{b_n\}$ is almost periodic, then so is $\{x_n\}$.

The proof of this theorem can be found in [1].

The proof of Theorem 5 plays an important role when trying to extend the investigation of almost periodicity of solutions to the nonlinear case of the systems

$$(20) \quad x_{n+1} = Ax_n + f(n, x_n), \quad n \in \mathbb{Z}.$$

Theorem 6. Consider the nonlinear system (20) with $f(n, x)$ an almost periodic map from $\mathbb{Z} \times \mathbb{R}^m$ into \mathbb{R}^m , uniformly with respect to the second argument in any bounded set. Moreover, let f satisfy a Lipschitz condition:

$$(21) \quad |f(n, x) - f(n, y)| \leq L|x - y| .$$

A is a square matrix of order m , and the moduli of its eigenvalues are different from 1, ($|\lambda| \neq 1$). Then the system (20) admits a unique almost periodic solution if the Lipschitz constant L is small enough.

Proof. To prove the existence of a unique almost periodic solution, we will apply the Banach fixed point theorem.

If E is a Banach space, T a mapping $T : E \rightarrow E$, such that

$$|Tx - Ty| \leq \lambda |x - y|, \quad 0 \leq \lambda < 1, \quad \text{for all } x, y \in E,$$

then there is a unique fixed point for $T : x = Tx$.

In order to define the operator T on the space $AP(Z, C^m)$, let us consider the system

$$(22) \quad x_{n+1} = Ax_n + f(n, u_n),$$

where $\{u_n\} = u$ is an almost periodic sequence. In (22), x_n is the unique almost periodic solution, according to Theorem 5. We put $x = Tu$. Let us notice that $\{f(n, u_n)\}$ is an almost periodic sequence (see [1]).

Also, let us write the system (20) in y_n and v_n :

$$(23) \quad y_{n+1} = Ay_n + f(n, v_n), \quad n \in Z.$$

Subtracting side by side of (23) from (22) one obtains:

$$(24) \quad x_{n+1} - y_{n+1} = A(x_n - y_n) + f(n, u_n) - f(n, v_n).$$

Applying formula (19), we obtain from (24)

$$(25) \quad |x - y| = |Tu - Tv| \leq k \sup_{n \in Z} |f(n, u_n) - f(n, v_n)| \leq kL|u - v|.$$

If we now assume

$$(26) \quad kL < 1,$$

then (3.25) shows that T is a contraction and therefore T has a unique fixed point.

This fixed point satisfies

$$x_{n+1} = Ax_n + f(n, x_n),$$

which coincides with (3.20).

The proof is now completed.

Remark. Results similar to Theorem 6 can be obtained in the same way. Indeed, this theorem shows that under the assumption made on A (the moduli of eigenvalues are $\neq 1$), the almost periodicity of $f(n,x)$ in the first argument and the Lipschitz condition imply the existence of a unique almost periodic solution to (3.20).

If we keep the assumption on A and adequately change the assumptions on f then existence and uniqueness of solution for (3.20) will be assured in other sequence spaces.

For instance, if instead of the space $AP(Z, R^m)$ we consider the space $\ell^\infty(Z, R^m)$ (i.e., $f(n,x)$ is uniformly bounded in the first argument), then existence in $\ell^\infty(Z, R^m)$ is obtained.

In case we consider (3.20) only for $n \in N$, the inequality (3.19) of Theorem 5 shows, for instance, that $\{b_n\} \in \ell^p(N, R^m)$ implies $\{x_n\} \in \ell^p(N, R^m)$. This can be easily extended to (3.20).

New results in the dynamics of discrete processes are given in [2].

R E F E R E N C E S

- [1] Corduneanu , C. : Almost periodic discrete processes . Libertas Mathematica , vol. 2 (1982) , 159 - 169 .
- [2] Lakshmikantham , V. , and Trigiante , D. : Theory of Difference Equations with Applications to Numerical Analysis . Academic Press , New York , 1988 .

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