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ASYMPTOTICALLY PERIODIC SOLUTIONS
FOR SOME HYPERBOLIC EQUATIONS

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In this paper,we study the asymptotic behavior of certain bounded solutions for some hyperbolic partial differential equations. In (1) the existence problem has been discussed for solutions of a nonlinear hyperbolic equation, after reducing such equation to a Volterra integral equation in two variables.
We sall deal now with the hyperbolic equation of the form

(E) $$u_{xt} + a(x,t)u_x = C(x,t,u),$$

where a is defined for

$$(x,t) \in \Delta' = \{ (x,t): 0 \le x \le \ell , -\infty < t < +\infty \},$$

and $C:\Delta'\times R \longrightarrow R$. If we consider the equation (E) in the strip Δ', say under assumption

(1) $$a(x,t) \ge m > 0 \quad \text{in } \Delta',$$

and the condition on the characteristic

(2) $$u(0,t) = u_0(t), \ -\infty < t < +\infty,$$

then the integral equation generated by these data has the form

(3) $$u(x,t) = u_0(t) + \int_0^x \int_{-\infty}^t e^{-\int_\tau^t a(\xi,\theta)d\theta} C(\xi,\tau,u(\xi,\tau))d\xi d\tau.$$

In order to obtain bounded solutions for equation (E) in the semi-strip

$$\Delta = \{ (x,t): 0 \le x \le \ell , t \ge 0 \},$$

with conditions on the characteristics

(4) $$u(x,0) = \phi(x) , 0 \le x \le \ell ; u(0,t) = u_0(t) , t \ge 0$$

we will assume that

(5) $$a(x,t) \ge m > 0 \quad \text{in } \Delta.$$

It will be assumed that $\phi(0) = u_0(0)$, which leads to the Volterra equation

(6) $$u(x,t) = u_0(t) + \int_0^x \phi'(\xi)e^{-\int_0^t a(\xi,\tau)d\tau} d\xi$$

$$+ \int_0^x \int_0^t e^{-\int_\tau^t a(\xi,\theta)d\theta} C(\xi,\tau,u(\xi,\tau))d\xi d\tau.$$

Based on the following theorem in my dissertation [3], we are going to consider periodic and asymptotically periodic solutions of the equation (E) in Δ'.

Theorem 1. Assume the following hypothesis hold true in regard to equation (3):

(*i*). $u_0(t)$ is continuous and bounded on the real axis $-\infty < t < +\infty$;

(*ii*). $a(x,t)$ is continuous on Δ', and verifies the inequality (1);

(*iii*). $C(x,t,u)$ is continuous on $\Delta' \times R$, with $C(x,t,0)$ bounded on Δ', and satisfies the Lipschitz condition.

$$|C(x,t,u) - C(x,t,\upsilon)| \le L|u - \upsilon|,$$

where L is a positive constant.

Then, there exists a unique continuous and bounded solution(inΔ') of the equation (3). Hence, a unique classical solution of (E), under condition (2).

The proof, by the method of successive approximations, can be carried out without difficulty. (see [1], where a sketch of proof is given).

Corollary. If $u_0(t)$, $a(x,t)$, and $C(x,t,u)$ are periodic in t, with period T, $(T > 0)$, then the unique bounded solution of (3) is also periodic in t, with the same period T.

Proof. Let us note the following property:
If $f(x,t)$ is a continuous function of period T in t for any $x \in [0,\ell]$ then

(7) $$\int_a^{a+T} f(x,u)du = \int_0^T f(x,u)du, \text{ any } a \text{ and } x \in [0,\ell].$$

Consider now the equation (3). One obtains from (3), changing t in t+T:

(8) $$u(x,t+T) = u_0(t+T) + \int_0^x \int_{-\infty}^{t+T} e^{-\int_\tau^{t+T} a(\xi,\theta)d\theta} C(\xi,\tau,u(\xi,\tau))d\xi d\tau$$

Making the substitution

$$\tau = \eta + T,$$

and taking (7) into account, (8) yields

$$u(x,t+T) = u_0(t+T) + \int_0^x \int_{-\infty}^{t} e^{-\int_{\eta+T}^{t+T} a(\xi,\theta)d\theta} C(\xi,\eta+T,u)d\eta d\xi.$$

Taking into account the periodicity of $u_0(t)$, $C(x,t,u)$, and $a(x,t)$, using (7) and

$$\int_{\eta+T}^{t+T} a(\xi,\theta)d\theta = \int_{\eta+T}^{\eta} a(\xi,\theta)d\theta + \int_\eta^\tau a(\xi,\theta)d\theta + \int_t^{t+T} a(\xi,\theta)d\theta = \int_\eta^t a(\xi,\theta)d\theta,$$

one obtains

$$u(x,t+T) = u_0(t) + \int_0^x \int_{-\infty}^t e^{-\int_\eta^t a(\xi,\theta)d\theta} C(\xi,\eta,u(\xi,\eta))d\eta d\xi = u(x,t)$$

which completes the proof of the Corollary.

Before we prove the asymptotic periodicity of solutions u(x,t) for equation (E) in Δ, we need to linearize the equation (E) as follows:

In case $a(x,t) \geq m > 0$ in Δ, as we have seen earlier [3], all solutions to the equation (E) are bounded in Δ.

If u and $\bar{u}$ are any two bounded solutions in Δ, then one obtains from equation (E)

(9) $$(u - \bar{u})_{xt} + a(x,t)(u - \bar{u})_x = C(x,t,u) - C(x,t,\bar{u}).$$

Based on Hadamard's Lemma [2], and assuming $C_u(x,t,u)$ exists and is continuous in Δ×R, then equation (9) can be written as follows

$$(u - \bar{u})_{xt} + a(x,t)(u - \bar{u})_x = \hat{b}(x,t,u,\bar{u})(u - \bar{u})$$

or

$$(u - \bar{u})_{xt} + a(x,t)(u - \bar{u})_x = b(x,t)(u - \bar{u}),$$

where b(x,t) is a continuous function in Δ. Hence, we must deal with the following equation in Δ:

(E') $$v_{xt} + a(x,t)v_x = b(x,t)v,$$

where $v(x,t) = u(x,t) - \bar{u}(x,t)$ is bounded in Δ, and with data on characteristics

(10) $$v(x,0) = \Psi(x), \quad 0 \leq x \leq \ell \ ; \quad v(0,t) = v_0(t), \ t \geq 0.$$

We shall investigate the asymptotic behavior of solutions in the case

(11) $$v_0(t) \longrightarrow 0 \quad \text{as } t \longrightarrow \infty.$$

Equation (E') in the semi-strip Δ, say under assumption (2), and with conditions (10) will reduce to the following integral equation:

(12) $$v_x(x,t) = \Psi'(x)e^{-\int_0^t a(x,\tau)d\tau} + \int_0^t e^{-\int_\tau^t a(x,\theta)d\theta} b(x,\tau)v(x,\tau)d\tau,$$

since $v(0,t) = v_0(t)$, $t \geq 0$, one obtains after integration of both sides of equation (12) from 0 to x:

(13) $$v(x,t) = v_0(t) + \int_0^x \Psi'(\xi)e^{-\int_0^t a(\xi,\tau)d\tau} d\xi$$

$$+ \int_0^x \int_0^t e^{-\int_\tau^t a(\xi,\theta)d\theta} b(\xi,\tau)v(\xi,\tau)d\xi d\tau .$$

By means of successive approximations, with the first approximation $v_0(t)$, the next approximation is

$$v_1(x,t) = v_0(t) + \int_0^x \Psi'(\xi)e^{-\int_0^t a(\xi,\tau)d\tau} d\xi$$

$$+ \int_0^x \int_0^t e^{-\int_\tau^t a(\xi,\theta)d\theta} b(\xi,\tau)v_0(\tau)d\xi d\tau .$$

This leads to the estimate

(14) $$\left| v_1(x,t) \right| \le \left| v_0(t) \right| + \ell \sup_{x \in [0,\ell]} \left| \Psi(x) \right| e^{-mt}$$

$$+ \ell \sup_\Delta \left| b(x,t) \right| \int_0^t e^{-m(t-\tau)} \left| v_0(\tau) \right| d\tau ,$$

and taking (5) into account, we can write $\left| v_0(t) \right| < \frac{m\epsilon}{2}$,

for sufficiently large t, say for $t \ge T$. If we notice that

$$\int_0^t e^{-m(t-\tau)} \left| v_0(\tau) \right| d\tau = \int_0^T e^{-m(t-\tau)} \left| v_0(\tau) \right| d\tau + \int_T^t e^{-m(t-\tau)} \left| v_0(\tau) \right| d\tau ,$$

then

$$\int_0^t e^{-m(t-\tau)} \left| v_0(\tau) \right| d\tau \le \frac{e^{-m(t-\tau)}}{m} \Bigg|_0^T \sup \left| v_0(t) \right| + \frac{e^{-m(t-\tau)}}{m} \Bigg|_T^t \cdot \frac{m\epsilon}{2}$$

$$= \left[\frac{e^{-m(t-T)}}{m} - \frac{e^{-mt}}{m} \right] A + \frac{1 - e^{-m(t-T)}}{m} \cdot \frac{m\epsilon}{2}$$

$$= \frac{1}{m} e^{-mt} \left[1 - e^{mt} \right] A + \frac{1}{m} \left[1 - e^{-m(t-T)} \right] \cdot \frac{m\epsilon}{2}$$

$$\le \frac{A}{m} e^{-mt} + \frac{1}{m} \cdot \frac{m\epsilon}{2}$$

$$\le \frac{\epsilon}{2A} A + \frac{\epsilon}{2} = \epsilon , \text{ for all } t \ge t_1(\epsilon),$$

which means that

(15) $$v_1(x,t) \longrightarrow 0 \text{ as } t \longrightarrow \infty, \text{ uniformly in } x,\ 0 \le x \le \ell .$$

The process of successive approximations leads to the sequence $\{ v_n(x,t) \}$, such that the recurrence formula is

(16) $$v_{n+1}(x,t) = v_0(t) + \int_0^x \Psi'(\xi)e^{-\int_0^t a(\xi,\tau)d\tau} d\xi$$

$$+ \int_0^x \int_0^t e^{-\int_\tau^t a(\xi,\theta)d\theta} b(\xi,\tau)v_n(\tau)d\xi d\tau .$$

If we assume that

(17) $$v_n(x,t) \longrightarrow 0, \quad \text{as } t \longrightarrow \infty,$$

uniformly with respect to x, $0 \le x \le \ell$, then by induction the following estimates are satisfied in Δ:

(18) $$\left| v_{n+1}(x,t) \right| \le \left| v_0(t) \right| + \ell \sup_{[0,\ell]} \left| \psi'(x) \right| e^{-mt}$$

$$+ \ell \sup_{\Delta} \left| b(x,t) \right| \int_0^t e^{-m(t-\tau)} \left| v_n(\tau) \right| d\tau.$$

This inequality is of the same form as the inequality (14), and therefore it leads to the conclusion

(19) $$v_{n+1}(x,t) \longrightarrow 0 \quad \text{as } t \longrightarrow \infty,$$

uniformly with respect to x, $0 \le x \le \ell$.

Therefore, the sequence $\{ v_n(x,t) \}$, which we know [1], [3] to be uniformly convergent, has all its terms tending to zero as $t \longrightarrow \infty$. This easily implies that $v(x,t)$, the solution of (E′) satisfying

$$v(0,t) = v_0(t),\ t \ge 0, \text{ and } v(x,0) = \psi(x),$$

is such that $v(x,t) \longrightarrow 0$ as $t \longrightarrow \infty$, uniformly in x, $0 \le x \le \ell$. Indeed, from

$$\left| v(x,t) \right| \le \left| v(x,t) - v_n(x,t) \right| + \left| v_n(x,t) \right|,$$

one easily obtains

$$\left| v(x,t) \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

provided n is chosen sufficiently large, and $T(\epsilon) > 0$ such that $\left| v_n(x,t) \right| < \frac{\epsilon}{2}$ for $t \ge T(\epsilon)$.

Based on Theorem 1, its corollary, and the discussion conducted above, we can state the following result.

Theorem 2. Consider the equation (E) in (Δ), and assume that a(x,t) and C(x,t,u) are both periodic in t, with period T>0. Moreover, let a(x,t) satisfy (5), and C(x,t,u) be as in Theorem 1. If $u_0(t)$ is a continuous periodic function of period T, then the unique solution of (E), (2) is also periodic of period T.

The bounded solution u(x,t) of (E) in Δ, satisfying

(20) $$u(x,0) = \ell(x), \quad 0 \le x \le \ell,\ u(0,t) = \bar{u}_0(t),\ t \ge 0,$$

where

(21) $$\bar{u}_0(t) - u_0(t) \longrightarrow 0 \quad \text{as } t \longrightarrow \infty,$$

is then asumptotically periodic in t.

In concluding this paper, we notice the fact that the case of almost periodicity can be also dealt with, using more or less the same arguments.

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