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ON THE CHOOUET-KENDALL THOREM

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§1 We present in this paper an improvement of the Choquet-Kendall theorem. [10]

If E is a real vector space, a convex set $A \subset E$ is linearly compact if $A \cap L$ is either empty or a closed bounded interval in L, for each one dimensional manifold L in E (where L is regarded as a copy of the real line R in E)

A subset $K \subset E$ is a pointed convex cone if,

i) $K + K \subset K$

ii) $(\forall \lambda \in R_+)(\lambda K \subset K)$

iii) $K \cap (-K) = \{0\}$

If $(E, \leqslant)$ is an ordered vector space then $E_+ = \{x \in E \mid x \geqslant 0\}$ is a pointed convex cone.

Conversely, if $K \subset E$ is a pointed convex cone, then E is an ordered vector space with respect to the order,

$$\text{" } x \leqslant y \Longleftrightarrow y - x \in K \text{ "}$$

If $(E, \leqslant)$ is an ordered vector space and for every pair (x,y) of E there exist the supremum $\sup(x,y)$ and the infimum $\inf(x,y)$ then E is a vector lattice.

We say that $B \subset E$ is a base for the convex cone $K \subset E$ if,

b_1) B is convex,

b_2) every $x \in K \setminus \{0\}$ has a unique representation of the form $x = \lambda b$, where $\lambda \in R_+ \setminus \{0\}$ and $b \in B$.

A non-empty convex set $A \subset E$ is a <u>simplex</u> if for every $x, y \in E$ and every $a, b \in R_+$ such that, $(x + aA) \cap (y + bA) \neq \Phi$, there exist $z \in E$ and $c \in R_+$ such that, $(x + aA) \cap (y + bA) = z + cA$.

The following result is the classical Choquet-Kendall theorem [5], [7], [10].

<u>Theorem</u> [Choquet-Kendall]

<u>If E is a vector space, ordered by a generating cone $K(E=K-K)$ with a base B, then E is a vector lattice, if and only if B is a linearly compact simplex.</u>

This theorem is a very important result in the Potential Theory. [2-5] We will prove that in a locally convex space, we have the Choquet-Kendall theorem for a closed generating cone K, with a closed base B if we have that B is only a simplex.

§ 2 We denote by $(E(\tau), \{p_\alpha\}_{\alpha \in \mathcal{A}})$ a locally convex space where the topology τ is defined by <u>a saturated family of seminorms</u> $\{p_\alpha\}_{\alpha \in \mathcal{A}}$, that is,

1°) for every $x \in E$, $x \neq 0$ there exists $\alpha \in \mathcal{A}$ such that, $p_\alpha(x) \neq 0$,

2°) for every $\alpha, \beta \in \mathcal{A}$ there exists $\gamma \in \mathcal{A}$ such that,

$p_\alpha(x), p_\beta(x) \leqslant p_\gamma(x)$, for every $x \in E$.

If $K \subset E$ is a closed convex cone, then a convex subset $B \subset K$ is a closed base for K, if and only if, there exists a continuous linear functional f strictly positive ou K, $(f(x) > 0$, for every $x \in K \setminus \{0\})$ such that, $B = K \cap f^{-1}(1)$, [6], [10]

We say that K is <u>well-based</u> if there exists a bounded convex set A such that $0 \notin \bar{A}$ and $K = \bigcup_{\lambda \geq 0} \lambda A$. If A is compact we obtain that

K is locally compact. [6], [10]

A subset $D \subset E$ is linearly bounded if D has bounded intersection with all lines in E. Obviously, if D is closed and linearly bounded then it is linearly compact.

Perhaps, we find the origin of this concept in Nikodym's paper [9] and it was studied by Klee [8] and used by Kendall [7] and recently by Ray [11].

The following concept is similar but more general as the concept of conically bounded set defined and studied by Bourgin [1].

Let $\mathcal{C}(x)$ be the family of compact convex subsets of E that do not contain the point $x \in E$.

For each $B \in \mathcal{C}(x)$ we denote by $K_x(B)$ the convex cone over B with vertex x, that is, $K_x(B) = \{\lambda b + (1-\lambda)x / \lambda \geq 0, b \in B\}$

If $x = 0$ then, $K_o(B) = \bigcup_{\lambda \geq 0} \lambda B$, that is, $K_o(B)$ is the locally compact cone generated by B.

Definition 1

We say that $A \subset E$ is 0-conically compact(resp. x-conically compact) if for each $B \in \mathcal{C}(0)$(resp. $B \in \mathcal{C}(x)$) the set $A \cap K_o(B)$(resp. $A \cap K_x(B)$) is bounded.

Remark

The concept of conically compact set is strongly dependent of the topology τ.

From definition 1 we deduce immediately the following elementary properties.

Properties

1°) Any subset of a conically compact set is conically compact.

2°) Finite unions of conically compact sets are conically compact.

3°) $A \subset E$ is conically compact if and only if each of its countably infinite unbounded subset is conically compact.

Proposition 1

If $(E(\tau), \{p_\alpha\}_{\alpha \in \mathcal{A}})$ is a locally convex space with respect to the topology τ, defined by the saturated family of seminorms $\{p_\alpha\}_{\alpha \in \mathcal{A}}$, then a set $A \subset E$ is x-conically compact, with $x \neq 0$, if and only if it is 0-conically compact.

Proof

It is sufficient to prove that, if A is 0-conically compact, then it is x-conically compact for $x \neq 0, (x \in E)$.

The proposition will be proved if we prove that whenever $B \in \mathscr{C}(x)$ there exists a set $B_1 \in \mathscr{C}(0)$ for which, $K_x(B) \setminus K_0(B_1)$ is bounded.

Since $x \notin B$ and B is closed, there exists $\alpha \in \mathcal{A}$ such that, $\inf\{p_\alpha((b-x) \mid b \in B\} > 0$ and since $x \neq 0$ there exists $\beta \in \mathcal{A}$ such that, $p_\beta(x) > 0$.

But since $\{p_\alpha\}_{\alpha \in \mathcal{A}}$ is a saturated family of seminorms there exists $\gamma \in \mathcal{A}$ such that, for every $x \in E$ we have, $p_\alpha(x), p_\beta(x) \leqslant p_\gamma(x)$.

Then we have, $d = \inf\{p_\gamma(b-x) \mid b \in B\} > 0$ and $p_\gamma(x) > 0$

Let $\lambda = 3d^{-1} p_\gamma(x)$ and consider the set, $D = \{\lambda b + (1-\lambda)x \mid b \in B\}$.

We observe that D is compact convex and $x \notin D$, that is $D \in \mathscr{C}(x)$

We can prove that $K_x(D) = K_x(B)$ and because,

$p_\gamma(\lambda b + (1-\lambda)x) = p_\gamma(\lambda(b-x)+x) \geqslant p_\gamma(\lambda(b-x)) - p_\gamma(x) \geqslant 3d^{-1} p_\gamma(x) d - p_\gamma(x) \geqslant$
$> 2p_\gamma(x)$; for every $b \in B$

We deduce,

$$\inf\{p_\gamma(b') \mid b' \in D\} \geqslant 2p_\gamma(x) .$$

Let $B_1 = co(D \cup (D-x))$. We have that B_1 is compact. Moreover, $0 \notin B_1$.

Indeed we have for every $b_1, b_2 \in D$ and every $t \in [[0,1]$,

$p_\gamma[t(b_1-x)+(1-t)b_2] \geq p_\gamma[tb_1+(1-t)b_2]-p_\gamma(x) \geq 2p_\gamma(x)-p_\gamma(x)=p_\gamma(x)>0$

Now, let for every $\alpha \in \mathcal{A}$,

$d_\alpha = \sup\{p_\alpha(b'-x) \mid b' \in D\}$.

Since D is compact we get that $d_\alpha < +\infty$

If $y \in K_x(D)$ is an element such that, $p_\alpha(y-x) > d_\alpha$, then there exists a number $t \in]0,1[$ such that, $ty+(1-t)x \in D$

Because,

$ty = t(ty+(1-t)x) + (1-t)(ty+(1-t)x-x)$ and $ty+(1-t)x \in D$, we have that ty is a convex combination of a point of D and one of $D-x$, so that $ty \in B_1$ and $y \in K_0(B_1)$.

It follows that,

$K_x(B) \setminus K_0(B_1) = K_x(D) \setminus K_0(B_1) \subseteq \{y \mid p_\alpha(y-x) \leq d_\alpha\}$ and the proof is complete. #

<u>Proposition 2</u>

<u>Let $(E(\tau), \{p_\alpha\}_{\alpha \in \mathcal{A}})$ be a locally convex space and let $K \subset E$ be a closed pointed convex cone. If $B^* \subset K$ is a closed base for K, then B^* is a 0-conically compact set.</u>

<u>Proof</u>

It must prove that if $B \in \mathcal{C}(0)$ then $B_1 = B^* \cap K_0(B)$ is bounded or empty.

Indeed, if B_1 is nonempty then B_1 is a closed base of $K_1 = K \cap K_0(B)$.

From Klee's theorem [6] $K_0(B)$ is locally compact which implies that K_1 is locally compact, and hence it has a compact base.

But since in a locally compact cone every base is compact we obtain that B_1 is compact and hence bounded. #

Finally we obtain the following form of Choquet-Kendall theorem.

Theorem

Let $E(\tau)$ be a locally convex space where τ is defined by a saturated family of seminorms $\{p_\alpha\}_{\alpha \in \mathcal{A}}$

If E is a vector space ordered by a closed generating cone K with a closed base B, then E is a vector lattice if and only if, B is a simplex.

Proof

If B is a simplex then from Proposition 1 and Proposition 2 we obtain that B is linearly compact and from Choquet-Kendall theorem we obtain that E is a vector lattice.

Conversely, if E is a vector lattice obviously from Choquet-Kendall theorem we deduce that B is a simplex . #

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