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FIXED POINTS AND COINCIDENCES IN SEQUENTIALLY COMPACT SPACES

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Abstract. An extension of Edelstein's contraction theorem to sequentially compact spaces is given and then a version of Goebel's coincidence theorem to this non-metric setting is presented.

In what follows, X is a Hausdorff topological space, $F: X \times X \to [0,\infty)$ is continuous, and $F(x,y) = 0$ iff $x = y$.

Theorem 1. Let X be sequentially compact. If $H: X \to X$ is such that

(1) $\qquad F(Hx,Hy) < F(x,y)$ whenever $x \neq y$,

then H has a unique fixed point.

Proof. Let $x \in X$, $x_n = H^n x$ (iteration), and $d_n = H(x_n, x_{n+1})$, $n \in \mathbb{N}$. Assume $x_n \neq x_{n+1}$ (for otherwise x_n is a fixed point). Then $x_n \neq x_m$ if $n \neq m$, since otherwise $d_n = d_m < d_{m-1} < \ldots < d_n$ $(m > n)$, a contradiction. The sequence (x_n) has a convergent subsequence $(x_{n(k)})$ with a unique limit $u = \lim x_{n(k)}$. Since (x_n) is a sequence of distinct points, so is $(x_{n(k)})$, and we may assume that u and Hu are not elements of $(x_{n(k)})$. Further, $(x_{n(k)})$ has a subsequence $(x_{n(k(i))})$ for which $(Hx_{n(k(i))})$ is convergent. By (1), $F(\lim Hx_{n(k(i))}, Hu) = \lim F(Hx_{n(k(i))}, Hu) \leq \lim F(x_{n(k(i))}, u) = 0$, hence

(2) $$\lim Hx_{n(k(i))} = Hu \,.$$

In a way identical to the already described one, we can choose a subsequence of $(x_{n(k(i))})$ (which will be denoted by the same symbol) such that

(3) $$\lim H^2x_{n(k(i))} = H^2u \,.$$

Let $a_i = d_{n(k(i))}$ and $b_i = d_{n(k(i))+1}$ for all $i \in \mathbb{N}$. Since $a_1 > b_1 > \ldots > a_i > b_i > \ldots > 0$, both (a_i) and (b_i) are convergent and have a common limit. By (2) and (3) we get

$$\begin{aligned} F(u,Hu) &= F(\lim x_{n(k(i))}, H \lim x_{n(k(i))}) \\ &= \lim a_i \\ &= \lim b_i \\ &= \lim F(Hx_{n(k(i))}, H^2x_{n(k(i))}) \\ &= F(Hu, H^2u). \end{aligned}$$

If $u \neq Hu$, then $F(u,Hu) > F(Hu,H^2u)$, a contradiction. Hence $u = Hu$. Uniquenes follows from (1).

Remark 1. If H is continuous, then Theorem 1 is valid with F satisfying $F(x,x) = 0$ for all $x \in X$. If X is metrizable with F a metric, Theorem 1 reduces to [1, Th. 1]. Under a condition independent of (1), a similar type result (i.e. with a continuous function playing the role of a metric) for a pair of continuous mappings appears in [4]. Note however that a special case of the condition of [4] is of the form $F(Hx,Hy) < a F(x,y)$, $x \neq y$ $(0 < a < 1)$.

That H (in Theorem 1) need not be continuous is useful in proving the following theorem.

Theorem 2. Let Y be a set, $S, T: Y \to X$, $S(Y) \subset T(Y)$, and let $T(Y)$ be a sequentially compact subspace of X. If

(4) $F(Sx,Sy) < F(Tx,Ty)$ whenever $Tx \neq Ty$, and

(5) $Sx = Sy$ whenever $Tx = Ty$

for all $x, y \in Y$, then there exists $z \in Y$ such that $Sz = Tz$.

Proof. The proof is analogous to that of [2, Th. 1 + Remark] except that the appeal to [1, Th. 1] is replaced by an application of Theorem 1. Indeed, for every $a \in T(Y)$, $Ha = S(T^{-1}\{a\}) \subset T(Y)$ is a singleton by (5) (cf. [3]). Now, if $a, b \in T(Y)$, $a \neq b$, then there exist $x \in T^{-1}\{a\}$, $y \in T^{-1}\{b\}$ and $x \neq y$. Then $Sx = Ha$, $Sy = Hb$ and $F(Ha,Hb) = F(Sx,Sy) < F(Tx,Ty) = F(a,b)$, since $Tx = a \neq b = Ty$. So, Theorem 1 applies to H, a selfmapping of $T(Y)$, a sequentially compact Hausdorff space. Thus, $Hc = c$ for some $c \in T(Y)$, and $Sz = Hc = c = Tz$ for every $z \in T^{-1}\{c\}$.

Remark 2. Conditions (4) and (5) can be replaced by any of the following (cf. [3] and [2], respectively):

(6) $F(Sx,Sy) < F(Tx,Ty)$ whenever $Sx \neq Sy$,

(7) $F(Sx,Sy) < F(Tx,Ty)$ whenever $x \neq y$.

This means that H is then a well-defined selfmappings of $T(Y)$ and satisfies condition (1). For T injective (this is the case if (7) holds), there exists a unique coincidence point of S and T. Finally note that with $Y = X$ and T an identity mapping, Theorem 2 reduces to Theorem 1.

References

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