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AN EFFICIENT METHOD FOR THE K-CONTINUATION OF A POLYNOMIAL

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This note deals with the problem of continuing a polynomial by adding higher order terms such that all its zeros will lie on the unit circle. Two new methods are introduced which result in possibly the lowest degree of the continuation known. It is also shown that the number of distinct points of the unit circle needed to construct the continuation is bounded by a number depending only on the degree of the initially given polynomial.

1. Introduction

The problem of the S-continuation of a polynomial consists of continuing a given polynomial

$$P(z) = 1 + a_1 z + a_2 z^2 + \cdots + a_n z^n \tag{1}$$

by adding monomials of degree exceeding n such that the resulting polynomial vanishes only at points of an a *priori* give set S in the complex plane.

The problem was first treated by Tschebotareff who proved the existence of an S-continuation where S is a smooth starlike curve containing the origin in its interior [7]. Later[3] Gavrilov extended Tschebotareff's result to include closed piecewise smooth Jordan curves containing the origin as an interior point. Gavrilov also studied in greater detail the particular case when S is the unit circle K[1,2,4,5].

Even in the latter case the question of the degree of the continued polynomial was not treated by the above authors.

More recently Shapiro [6] suggested an equivalent problem to the K-continuation in the form of constructing a mapping from K^m to C^n. This problem was solved by Wilker and Odoni who gave an upper bound on the degree of the continued polynomial.

In the present note two simple constructions of a K-continuation of an n^{th} degree polynomial are given yielding possibly the best estimate known on the degree of the continued polynomial.

2. The Main Theorems

For a polynomial $Q(z)$ of degree n we denote

$$Q^*(z) = z^n\overline{Q}(\frac{1}{z}) = z^n\overline{Q(\frac{1}{\bar{z}})}.$$

The K-continuation problem of the polynomial $P(z)$ in (1) is equivalent to solving the system of the equation of the form

$$\sum_{i=1}^{m} w_i^k = T_k, \qquad k = 1, 2, \ldots, n \tag{2}$$

for given T_k, where the unknowns $m, w_1, \ldots, w_m$ are subject only to $w_i \in K, m \in N, m \geq n$.

Indeed if

$$Q(z) = P(z) + a_{n+1}z^{n+1} + \cdots + a_m z^m, \quad a_m \neq 0$$

is a K-continuation of $P(z)$, then

$$Q^*(z) = z^m + a_1 z^{m-1} + \cdots + a_n z^{m-n} + \cdots + a_m.$$

Therefore the power sums

$$S_k = \sum_{i=1}^{m} \zeta_i^k, \quad k = 1, 2, \ldots, n$$

of the zeros of $Q^*(z)$ satisfy the Newton equations

$$\begin{aligned} S_1 + a_1 &= 0 \\ S_2 + a_1 S_1 + 2a_2 &= 0 \\ &\vdots \\ S_n + a_1 S_{n-1} + \cdots + a_{n-1} S_1 + n a_n &= 0. \end{aligned} \tag{3}$$

The solution of the system (2) with $T_k = S_k$ is thus equivalent to constructing the polynomial $Q^*(z)$ or $Q(z)$.

We shall need

Lemma 1. *Given an arbitrary complex number S and an integer p, $1 \leq p \leq n$ the system (2) with $T_k = \delta_{kp} S$ (δ the Kronecker delta) $k = 1, 2, \ldots, n$ can be solved with*

$$m \leq n + pr_p + 1 \tag{4}$$

with

$$r_p = Max(\left[\frac{n}{p}\right], t_p) \tag{5}$$

where $t_p = t_p(S)$ is the smallest positive integer for which the zeros of

$$S_k(\varsigma) = 1 + \varsigma + \frac{\varsigma^2}{2!} + \cdots + \frac{\varsigma^k}{k!} \tag{6}$$

all lie in the region $|\varsigma| \geq \frac{|S|}{p}$ for $k \geq t_p$.

Proof: The system of equations (3) corresponding to the values of $T_k, k = 1, 2, \ldots, n$ of Lemma 1 is

$$\begin{aligned} S + pa_p &= 0 \\ a_p + 2pa_{2p} &= 0 \\ a_{2p} + 3pa_{3p} &= 0 \\ &\vdots \end{aligned}$$

so that

$$a_{jp} = (-1)^j \frac{S^j}{j!p^j}, \quad j = 1, 2, \ldots, \left[\frac{n}{p}\right]$$

and $a_i = 0$ for all other values of i, $0 \leq i \leq n$. Therefore the polynomial $P_p(z)$ which corresponds to (1) is

$$P_p(z) = 1 - \frac{S}{p} z^p + \frac{S^2}{2!p^2} z^{2p} + \cdots + (-1)^l \frac{S^l}{l!p^l} z^{pl}$$

,where $l = \left[\frac{n}{p}\right]$. Set $\varsigma = z^p$ and $x_p = (-1)^p \frac{S}{p}$. Then

$$P_p(z) = S_l(x_p \varsigma) \tag{7}.$$

By hypothesis the polynomial $\tilde{P}_p(z)$ defined by $\tilde{P}_p(z) = S_{r_p}(x_p \varsigma)$ has all its zeros in the region $|z| \geq 1$ and by (5) and (7) is a continuation of $P_p(z)$. One verifies easily that the polynomial $R_p(z)$ defined by

$$R_p(z) = \tilde{P}_p(z) + z^{n+1} \tilde{P}_p^*(z)$$

has its zeros all lie on the unit circumference K and its degree is $(n+pr_p+1)$. Since $\deg P_p(z) = p[n/p] \leq n$ and $\tilde{P}_p(z)$ is a continuation of $P_p(Z)$, so is $R_p(z)$. The reciprocals of the zeros of $R_p(z)$ consist a solution of the system (3) and their number satisfies the inequality (4). One also notices that the number t_p in the statement of lemma 1 is well defined since $S_k(\varsigma)$ is the Taylor section of the nonvanishing exponential function. This completes the proof of the lemma.

We can now state

Theorem 1. *The K–continuation problem of the polynomial* (1) *has a solution of degree N not exceeding*

$$n(n+1)(1+\frac{t}{2})+n^2 \tag{8}$$

where $t = t(S_1, S_2, \ldots, S_n)$ is the smallest positive integer for which the zeros of $S_k(\varsigma)$ in (6) *all lie in the region $|\varsigma| \geq \max_{1\leq j\leq n} \frac{|S_j|}{j}$.*

Proof: The system (2) for $T_k = S_k$, $k = 1, 2, \ldots, n$ can be solved by adjoining the solutions of (2) with $T_k = \delta_{kp}S_p$, $k = 1, 2, \ldots, n$, for $p = 1, 2, \ldots, n$ each time applying lemma 1. This procedure yields the estimate

$$N \leq \sum_{p=1}^{n}(n+pr_p+1) \leq n(n+1)+\sum_{p=1}^{n}p([\frac{n}{p}]+t_p) \leq n(n+1)(1+\frac{t}{2})+n^2 \quad (9).$$

Since t is independent of n the inequality (9) implies that $N = O(n^2)$. J. B. Wilker and R. W. K. Odoni in [6] suggest that they have achieved the estimate $O(n^\lambda)$ with $2 < \lambda < 3$ proving only $N = O(n^3)$. We conjecture that $O(n^2)$ is the best possible estimate for the K–continuation problem.

It is interesting to ask whether the above mentioned bounds can be improved if we only count the number of distinct points of K in the solution of the system (2). We shall establish below that the number of distinct points is bounded by a constant depending solely on n.

To start we prove

Lemma 2. *For any given numbers $S_1, S_2, \ldots S_n$ and any $p \geq n+1$ there exists a positive number ρ_0 such that for every ρ, $0 \leq \rho \leq \rho_0$ the system of equations*

$$\sum_{i=1}^{n+p} z_i^k = S_k\rho^k,\ k = 1, 2, \ldots, n \tag{10}$$

has a so;ution $\{z_i\}$ with $z_i \in K$.

Proof: Given $S_k, k = 1, 2, \ldots, n$ we solve for a_k in the system (3) and construct the polynomial $P(z)$ of the form (1). Let ρ_0 be the minimim modulus of the zeros of $P(z)$. For a fixed ρ, $0 < \rho \leq \rho_0$ (for $\rho = 0$ claim of the lemma is trivial using as z_i suitable roots of unity) consider the polynomial $Q(z) = P(z\rho)$ and form

$$F(z) = Q(z) + z^p Q^*(z).$$

It is straight to conclude that all the zeros of $F(z)$ lie on K for any $p = 0, 1, 2, \ldots$. Now for $p \geq n+1$

$$G(z) = F(\frac{z}{p}) = P(z) + b_{n+1}z^{n+1} + \cdots + b_{n+p}z^{n+p}$$

so that the polynomial

$$z^{n+p}G(\frac{1}{z}) = z^{n+p} + a_1 z^{n+p+1} + \cdots + a_n z^p + b_{n+1}z^{p-1} + \cdots + b_{n+p}$$

has all its zeros, say $\varsigma_1, \ldots, \varsigma_{n+p}$, on the circle $|z| = \frac{1}{\rho}$. Because of the nature of the system (3) we have

$$\sum_{i=1}^{n+p} \varsigma_i^k = S_k,\ k = 1, 2, \ldots, n.$$

It remains to set $z_i = \rho\varsigma_i$ to obtain relation (10). One notices that if $\rho_0 \geq 1$ the system (10) can be solved with $\rho = 1$. This solution also solves the K−continuation problem. We also notice that we can set $p = n+1$ independently of the value of ρ.

We can now state

Theorem 2. *For arbitrary given numbers $T_1, T_2, \ldots, T_n$ there exist at most $m = 2n(n+1)$ distinct points $w_j \in K$, $j = 1, 2, \ldots, m$ which solve the system (2).*

Proof: We fix $k, 1 \leq k \leq n$ and take $p = n+1$ in lemma 2. By this lemma we solve thwe two systems

$$\sum_{i=1}^{2n+1} z_{ik}^j = \rho_{0k}^j \delta_{jk} T_j$$

and

$$\sum_{i=1}^{2n+1} w_{ik}^j = \rho_{1k}^j \delta_{jk} T_j,\ j = 1, 2, \ldots, n \tag{11}$$

with $z_{ik}, w_{ik} \in K$ and where ρ_{0k} and ρ_{1k} are nonnegative numbers which satisfy the relations $0 \leq \rho_{1k} < \rho_{0k}$ and

$$\left[\frac{1}{\rho_{0k}^k}\right] + \rho_{1k}^k = 1 \qquad (12).$$

It is clear from equations (11) and (12) that combining multiple copies of the sets $\{z_{ik}\}$ and a single copy of the set $\{w_{ik}\}, i = 1, 2, \ldots, 2n+1$ lying on K we can solve the system

$$\sum_{i=1}^{m} w_i^k = \delta_{jk} T_j, \; j = 1, 2, \ldots, n$$

where the set $\{w_i\}$, $i = 1, 2, \ldots, m$ consists of at most $2(2n+1)$ distinct points of K. Applying the above argument for $k = 1, 2, \ldots, n$ we arrive at the theorem.

We notice readily that the method of the $K-$continuation of polynomials of magnitude $O(n^2)$. Theorem 2 also provides an upper bound on the number of distinct zeros of the continued polynomial. As we noticed earlier lower bounds for the degree of the set of $K-$continuations of a given polynomial are also of interest. However this problem has not yet been treated in the literature.

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