LIBERTAS MATHEMATICA, VOL. 8, 1988

AN EXISTENCE THEOREM FOR THE MODIFIED BARGAINING SET OF A COOPERATIVE N-PERSON CONVEX GAME*

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The classical bargaining sets have been introduced by R. J. Aumann and M. Maschler (see [3]). In a previous paper [1], a modified bargaining set of a cooperative n-person game with coalition structures has been introduced and a necessary and sufficient condition for the appartenance of an admissible payoff vector to this set has been given. For a convex game, it has been shown in a subsequent paper [2], that an admissible payoff vector x belongs to the modified bargaining set and does not belong to the core, if and only if the highest excess for x is positive and is reached for at least two coalitions. The purpose of the present paper is that of showing that for any coalition structure, except the coalition structure consisting of individual players and the coalition structures consisting of one two-player coalition and individual players, there exists at least one admissible payoff vector which is not in the core and belongs to the modified bargaining set. Moreover, the proofs are constructive, so that such a payoff can be computed. The main tool that has been used was the Stearns transfers, which were introduced by E. Stearns as the basic modifications of the payoff vectors in a dynamic procedure for finding a point in the kernel and in the classical bargaining set (see [4]).

1. Preliminaries. Consider a cooperative n-person game with coalition structures $G = (N, v, F)$, where N is the set of players, $|N| = n$, $v\colon 2^N \to R$ is the characteristic function, $v(\phi) = 0$, and F is the set of admissible payoff vectors. Any partition of N, $\mathcal{S} = (S_1, S_2,\ldots,S_p)$, is a coalition structure of G and the set of admissible payoff vectors for $\mathcal{S}$ is

$$F_{\mathcal{S}} = \{\, x : x\in R^n,\ x(S) = v(S),\ \forall S\in\mathcal{S}\}, \tag{1.1}$$

where $x(S) = \sum_{i\in S} x_i$. F is the union of all $F_{\mathcal{S}}$. The core of G is

$$C(G) = \{\, x : x\in F,\ e(x,S) \le 0,\ \forall\, S \subseteq N\}\,, \tag{1.2}$$

where $e(x,S) = v(S)-x(S)$ is the excess of S for x.

In [1] a modified bargaining set M_0 for G has been defined by imposing stronger conditions than in the definition of the classical bargaining set by Aumann-Maschler (see [3]). More precisely, all objections supported by the same coalition structure are supposed to be countered by all counter objections supported by the same coalition structure. Other slighter modifications have also been done (see [1]). The core of G, if nonempty, is supposed to be included in M_0.

A game with coalition structures is convex, if

$$v(S) + v(T) \leq v(S\cup T) + v(S\cap T), \quad \forall\ S,T \subseteq N. \tag{1.3}$$

In [2] a characterization of the elements in $M_0 - C(G)$ has been proved.

Theorem 1.1: If G is convex, then $x \in F$ belongs to $M_0 - C(G)$, if and only if

$$\begin{aligned} &\text{(a)} \quad \max_S e(x,S) = e(x,S_0) > 0 \\ &\text{(b)} \quad V_0^+(x) = \{ S \subseteq N : e(x,S) = e(x,S_0)\} \text{ has } |V_0^+(x)| > 1. \end{aligned} \tag{1.4}$$

This theorem will extensively be used in this paper, in order to prove the main result: for any nontrivial and nonquasi-trivial coalition structure $\mathcal{I}$, from any $x \in F_{\mathcal{I}}$, $x \notin C(G)$, one can reach $y \in F_{\mathcal{I}}$, $y \in M_0 - C(G)$, by one or two Stearns transfers; an $x \in F_{\mathcal{I}}$, $x \notin C(G)$ can also be found by one Stearns transfer. Note that such transfers of gain between members of the same coalition have been used in [4], in order to lower the value of the highest excess. Here, we use the transfers in order to raise the value of another excess to the value of the highest excess such that Theorem 1.1 can be applied.

In the next section we discuss the trivial and the quasi-trivial coalition structures, in order to determine when an element in $M_0 - C(G)$ does exist. In the last section we prove the main result mentioned above.

2. Trivial and quasi-trivial coalition structures.

Definition 2.1: A coalition structure $\mathcal{I}$ consisting of individual players is called a trivial coalition structure (TCS).

For $\mathcal{I}$ a TCS, the unique $x \in F_{\mathcal{I}}$ is $x_i = v(i)$, $i = 1,\ldots,n$, and we have

$$e(x,S) = v(S) - \sum_{i\in S} v(i), \quad \forall\ S \subseteq N. \tag{2.1}$$

Note that, if G is convex, then $e(x,S) \geq 0$, $\forall\ S \subseteq N$. The following result is obvious.

Theorem 2.2: If G is convex and $\mathcal{S}$ is a TCS, then: (a) if G is additive, then $x \in F_{\mathcal{S}}$ belongs to the core; (b) if G is not additive, then $x \notin C(G)$ and either $|V_0^+(x)| = 1$, $x \notin M_0 - C(G)$, or $|V_0^+(x)| > 1$, $x \in M_0 - C(G)$.

Lemma 2.3: If $\mathcal{S}$ is not a TCS, then there exists $x \in F_{\mathcal{S}}$, $x \notin C(G)$, and x can be obtained by a Stearns transfer.

Proof: For $\bar{S} \in \mathcal{S}$ with $|\bar{S}| \geq 2$ and $i \in \bar{S}$, $j \in \bar{S}$, if for $x \in F_{\mathcal{S}}$ we have $x \in C(G)$, then we use the Stearns transfer

$$y_i = x_i + \delta, \; y_j = x_j - \delta, \; y_k = x_k, \; k \neq i, j, \tag{2.2}$$

and we get for all S:

$$e(y,S) = \begin{cases} e(x,S) + \delta & \text{if } i \notin S, \; j \in S, \\ e(x,S) - \delta & \text{if } i \in S, \; j \notin S, \\ e(x,S) & \text{otherwise.} \end{cases} \tag{2.3}$$

For $\delta > 0$ large enough and some S^* with $i \in S^*$, $j \notin S^*$, we get $e(y,S^*) > 0$ and $y \in F_{\mathcal{S}}$, hence an $y \in F_{\mathcal{S}}$, $y \notin C(G)$ has been found.

As we intend to discuss in the following nontrivial coalition structures, by Lemma 2.3 we can make without loss of generality: Assumption (A): For $\mathcal{S}$ a non TCS an $x \in F_{\mathcal{S}}$, $x \notin C(G)$ is available. Note that by this assumption, condition (1.4) (a) holds and $V_0^+(x) \neq \phi$ for such a payoff. Note also that in Lemma 2.3 the game was not supposed to be convex.

Definition 2.4: A coalition structure $\mathcal{S}$ consisting of one 2-player coalition $\bar{S}$ and individual players is called a quasi-trivial coalition structure (QTCS).

We suppose $\bar{S} = \{1, 2\}$, but the results can be stated for $\bar{S} = \{i, j\}$. For $\mathcal{S}$ a QTCS we can write

$$F_{\mathcal{S}} = \{ x = x(\lambda) : x_1 = h + \lambda, x_2 = h - \lambda, x_k = v(k), \; k \neq 1,2, \; \lambda \in R\}, \tag{2.4}$$

where $h = 1/2v(1,2)$. It follows that for $x \in F_{\mathcal{S}}$ we have

$$e(x,S) = \begin{cases} E(S) & \text{if } 1 \notin S, \; 2 \notin S, \\ E(S) - E(1,2) & \text{if } 1 \in S, \; 2 \in S, \\ E(S) + v(1) + h - \lambda & \text{if } 1 \in S, \; 2 \notin S, \\ E(S) + v(2) - h + \lambda & \text{if } 1 \notin S, \; 2 \in S, \end{cases} \tag{2.5}$$

where

$$E(S) = v(S) - \sum_{i \in S} v(i), \; \forall \, S \subseteq N. \tag{2.6}$$

If G is convex we have

$$E(S) \geq 0, \qquad \forall\ S \subseteq N,$$
$$E(S) \geq E(1,2) \quad \forall\ S \in N \text{ with } 1 \in S,\ 2 \in S. \tag{2.7}$$

In this section we shall use the notations

$$T_{12} = \{\ S \subset N:\ 1 \in S,\ 2 \notin S\},\ \ T_{21} = \{\ S \subset N:\ 1 \notin S,\ 2 \in S\}, \tag{2.8}$$

and for a given $x \in F_{\mathcal{G}}$ we define S_0^1, S_0^2, S_0^0 by

$$e(x,S_0^1) = \max_{S \in T_{12}} e(x,S),\ \ e(x,S_0^2) = \max_{S \in T_{21}} e(x,S),\ \ e(x,S_0^0) = \max_{S \notin T_{12} \cup T_{21}} e(x,S). \tag{2.9}$$

From (2.5) to (2.9), we notice that, in fact

(1) $\max_{S \in T_{12}} e(x,S)$ equals $e(x,S_0^1)$ and $\max_{S \in T_{21}} e(x,S_0^2)$ equals $e(x,S_0^2)$ for all $x \in F_{\mathcal{G}}$, because the excesses are linear functions of λ;

(2) $\max_{S \notin T_{12} \cup T_{21}} e(x,S)$ equals $e(x,S_0^0)$ for all $x \in F_{\mathcal{G}}$, because $e(x,S)$ with $S \notin T_{12} \cup T_{21}$ does not depend on x; moreover, if G is convex we have $e(x, S_0^0) \geq 0$.

These remarks and Theorem 1.1 prove:

Lemma 2.5: If $\max_{S \in T_{12}} e(x,S)$ is also reached for other coalitions besides S_0^1, then

$$\{\ x:\ x \in F_{\mathcal{G}},\ \ \max_{S \in T_{12}} e(x,S) > e(x, S_0^0)\} \subseteq M_0 - C(G)\ ; \tag{2.10}$$

similarly, if $\max_{S \in T_{21}} e(x,S)$ is also reached for other coalitions besides S_0^2, then

$$\{\ x:\ x \in F_{\mathcal{G}},\ \ \max_{S \in T_{21}} e(x,S) > e(x, S_0^0)\} \subseteq M_0 - C(G)\ ; \tag{2.11}$$

these are nonempty sets.

The same remarks are helpful in proving:

Lemma 2.6: If G is convex and $\max_{S \in T_{12}} e(x,S)$ and $\max_{S \in T_{21}} e(x,S)$ are both reached for unique coalitions, namely S_0^1 and S_0^2, respectively, then for a unique $x^* \in F_{\mathcal{G}}$ we have

$$e(x^*, S_0^1) = e(x^*, S_0^2) \tag{2.12}$$

and this number is less than or equal to $e(x^*, S_0^0)$.

Proof: From (2.5) we get

$$x_1^* = h + \lambda^*, \quad x_2^* = h - \lambda^*, \quad x_k^* = v(k), \quad k \neq 1,2 \tag{2.13}$$

where

$$\lambda^* = 1/2\,[\,E(S_0^1) - E(S_0^2) + v(1) - v(2)] + h. \tag{2.14}$$

For $x^* \in F_{\mathcal{Y}}$ defined by (2.13) and (2.14), we have

$$e(x^*, S_0^1) = e(x^*, S_0^2) = 1/2\,[\,e(x^*, S_0^1) + e(x^*, S_0^2)]\,. \tag{2.15}$$

If G is convex, we get

$$e(x^*, S_0^1) + e(x^*, S_0^2) \leq e(x^*, S_0^1 \cup S_0^2) + e(x^*, S_0^1 \cap S_0^2)\,. \tag{2.16}$$

We have $1 \in S_0^1 \cup S_0^2$, $2 \in S_0^1 \cup S_0^2$, and $1 \notin S_0^1 \cap S_0^2$, $2 \notin S_0^1 \cap S_0^2$, hence each term in the right hand side is less than or equal to $e(x^*, S_0^0)$. We get

$$1/2\,[\,e(x^*, S_0^1) + e(x^*, S_0^2)] \leq e(x^*, S_0^0)\,, \tag{2.17}$$

hence from (2.15) and (2.17) the result follows.

Lemma 2.7: If G is convex and $\max_{S \in T_{12}} e(x,S)$ and $\max_{S \in T_{21}} e(x,S)$ are both reached for unique coalitions, namely S_0^1 and S_0^2, respectively, and we have

$$\begin{aligned} E(S) &= 0, \quad \forall\, S \subset N \text{ with } 1 \notin S,\ 2 \notin S, \\ E(S) &= E(1,2) \quad \forall\, S \subseteq N \text{ with } 1 \in S,\ 2 \in S, \end{aligned} \tag{2.18}$$

then there is no $x \in F_{\mathcal{Y}}$ such that $x \in M_0 - C(G)$.

Proof: An inequality like (2.17) holds also for all $x \in F_{\mathcal{Y}}$, therefore, from (2.18) we get

$$e(x, S_0^1) + e(x, S_0^2) \leq 0, \quad \forall\, x \in F_{\mathcal{Y}}, \tag{2.19}$$

because $e(x, S_0^0) = 0$. Then, for all $x \in F_{\mathcal{Y}}$, either $\max_S e(x,S) = e(x, S_0^1) > 0$ only for S_0^1, or $\max_S e(x,S) = e(x, S_0^2) > 0$ only for S_0^2, and these maxima can not be equal and positive, by (2.19) and Lemma 2.6, or $x \in C(G)$. Hence, either $|\, V_0^+(x) \,| = 1$ or $x \in C(G)$ for all $x \in F_{\mathcal{Y}}$.

Lemma 2.8: If G is convex and $e(x, S_0^0) > 0$, then starting from $x \in F_{\mathcal{Y}}$, $x \notin C(G)$ such that

$e(x, S_0^2) > e(x, S_0^0)$, (or $e(x, S_0^1) > e(x, S_0^0)$), one can find $y \in F_{\mathcal{S}}$, $y \in M_0 - C(G)$, by one Stearns transfer.

Proof: An $x \in F_{\mathcal{S}}$ such that $e(x, S_0^2) > e(x, S_0^0)$, (or $e(x, S_0^1) > e(x, S_0^0)$), can be found, as shown by Lemma 2.3. We shall discuss only the first case, the second is similar. Consider the Stearns transfer

$$y_1 = x_1 - \delta, \; y_2 = x_2 + \delta, \; y_k = x_k, \; k \neq 1,2$$
$$\delta = e(x, S_0^2) - e(x, S_0^1) > 0. \tag{2.20}$$

Obviously, $y \in F_{\mathcal{S}}$ and $e(y,S_0^2) = e(x,S_0^0) = e(y,S_0^0)$. We use again an inequality similar to (2.17):

$$e(x, S_0^1) + e(x, S_0^2) \leq 2e(x, S_0^0) \tag{2.21}$$

and from (2.20) and (2.21) we get

$$e(y, S_0^1) = e(x, S_0^1) + \delta \leq e(x, S_0^0) = e(y, S_0^0). \tag{2.22}$$

Therefore, the remarks done above show that

$$\max_S e(y,S) = e(y, S_0^2) = e(y, S_0^0) \geq e(y, S_0^1), \tag{2.23}$$

hence Theorem 1.1 applies and $y \in M_0 - C(G)$.

Now, from Lemmas 2.5–2.8 follows:

Theorem 2.9: Consider G a convex game and $\mathcal{S}$ a QTCS with $\bar{S} = \{1, 2\} \in \mathcal{S}$. If either $\max_{S \in T_{12}} E(S)$ or $\max_{S \in T_{21}} E(S)$ is not unique, or these maxima are unique and $E(S) > 0$ for some S with $1 \notin S$, $2 \notin S$ or/and $E(S) > E(1,2)$ for some S with $1 \in S$, $2 \in S$, then there exists $y \in F_{\mathcal{S}}$, $y \in M_0 - C(G)$. Otherwise, there is no $y \in F_{\mathcal{S}}$ belonging to $M_0 - C(G)$.

Proof: Notice that for any $x \in F_{\mathcal{S}}$ we have

$$\max_{S \in T_{12}} e(x,S) = \max_{S \in T_{12}} E(S) + v(1) + h - \lambda,$$

$$\max_{S \in T_{21}} e(x,S) = \max_{S \in T_{21}} E(S) + v(2) - h + \lambda, \tag{2.24}$$

$$\max_{S \notin T_{12} \cup T_{21}} e(x,S) = \max \Big(\max_{S:\, 1 \notin S,\, 2 \notin S} E(S), \max_{S:\, 1 \in S,\, 2 \in S} [E(S) - E(1,2)] \Big),$$

so that the existence statement follows from Lemmas 2.5 and 2.8.

The nonexistence statement follows from Lemma 2.7.

3. The general case. In this section we consider the coalition structures subject to: Assumption (B): $\mathcal{S}$ is neither a trivial, nor a quasi-trivial coalition structure.

Consider $\mathcal{S}$ and $x \in F_{\mathcal{S}}$ subject to Assumptions (B) and (A), respectively. If $\max_S e(x,S)$ is not unique, then by Theorem 1.1 we have $x \in M_0 - C(G)$. Therefore, the situation to be discussed must be subject to: Assumption (C): $\mathcal{S}$ is subject to (B), $x \in F_{\mathcal{S}}$ is subject to (A) and $\max_S e(x,S)$ is unique. Let $S_0 \subseteq N$ be defined by

$$e(x, S_0) = \max_S e(x,S). \tag{3.1}$$

Theorem 3.1: For G a convex game, consider $\mathcal{S}$ and $x \in F_{\mathcal{S}}$ subject to (C), where S_0 is defined by (3.1). If in some $\bar{S} \in \mathcal{S}$ there are two players i and j, either both in S_0, or both not in S_0, then there exists $y \in F_{\mathcal{S}}$ such that $y \in M_0 - C(G)$. This y can be obtained by one Stearns transfer.

Proof: Let S^* be defined by

$$e(x, S^*) = \max_{S \in T_{ji}} e(x,S), \quad T_{ji} = \{ S \subset N: i \notin S, j \in S \}. \tag{3.2}$$

Consider the Stearns transfer

$$y_i = x_i + \delta, \; y_j = x_j - \delta, \; y_k = x_k, \; k \neq i,j, \tag{3.3}$$
$$\delta = e(x, S_0) - e(x, S^*) > 0.$$

For S with $i \in S, j \in S$, or $i \notin S, j \notin S$, we get $e(y, S) = e(x, S)$. So, for all $S \in \mathcal{S}$ we have $e(y, S) = 0$, hence $y \in F_{\mathcal{S}}$. Then, we have $e(y, S_0) = e(x, S_0)$ and for any other $S \neq S_0$, we get $e(y, S) < e(y, S_0)$, by (C). For $S \in T_{ji}$, we get

$$e(y,S) = e(x,S) + \delta = e(y, S_0) - [e(x,S^*) - e(x,S)] \leq e(y, S_0), \tag{3.4}$$

by the choice of δ. In particular, $e(y, S^*) = e(y, S_0)$.

For $S \in T_{ij}$ we get

$$e(y,S) = e(x,S) - \delta < e(x, S_0) = e(y, S_0). \tag{3.5}$$

To summarize: $y \in F_{\mathcal{S}}$, we have $e(y, S_0) = e(y, S^*)$ and $e(y,S) \leq e(y, S_0)$ for all $S \subseteq N$. Then, by Theorem 1.1 we get $y \in M_0 - C(G)$. Note that this y has been obtained by one Stearns transfer.

Corollary 3.2: For G a convex game, consider $\mathcal{S}$ and $x \in F_{\mathcal{S}}$ subject to (C) and S_0 defined by (3.1). If either there is $S \in \mathcal{S}$ with $|S| > 2$, or for all $S \in \mathcal{S}$ we have $|S| \leq 2$, but there is $\bar{S} \in \mathcal{S}$ with $\bar{S} \subseteq S_0$ or $\bar{S} \cap S_0 = \phi$, then there exists $y \in F_{\mathcal{S}}$ such that $y \in M_0 - C(G)$.

Proof: In any coalition S with $|S| > 2$, there are two players i and j, either both in S_0, or both not in S_0, hence Theorem 3.1 applies. The same theorem applies in the second case.

By Corollary 3.2, the only remaining case is that of coalition structures subject to: Assumption (D): $\mathcal{S}$ and $x \in F_{\mathcal{S}}$ are subject to (C) and for all $S \in \mathcal{S}$ we have $|S| \leq 2$, and each S with $|S| = 2$ has one player in S_0 and one player not in S_0. Note that by (B), a coalition structure $\mathcal{S}$ subject to (D) has at least two 2-player coalitions $S' = \{i', j'\}$, $S'' = \{i'', j''\}$, such that $i' \in S_0$, $j' \notin S_0$, $i'' \in S_0$, $j'' \notin S_0$.

Theorem 3.3: If G is a convex game, for $\mathcal{S}$ and $x \in F_{\mathcal{S}}$ subject to (D) there exists $y \in F_{\mathcal{S}}$ such that $y \in M_0 - C(G)$. Such a y can be obtained by two Stearns transfers.

Proof: Let S' and S'' be described as above and denote:

$$T_1 = \{ S \subset N:\ i' \notin S,\ j' \in S,\ i'' \in S,\ j'' \notin S\},$$

$$T_2 = \{ S \subset N:\ i'' \in S,\ i',\ j', j'' \notin S, \text{ or } j' \in S, i', i'', j'' \notin S, \text{ or } i', i'', j' \in S,\ j'' \notin S, \text{ or } i'',\ j',\ j'' \in S,\ i' \notin S\}, \tag{3.6}$$

$$S_1^* \text{ defined by: } e(x, S_1^*) = \max_{S \in T_1} e(x,S),$$

$$S_2^* \text{ defined by: } e(x, S_2^*) = \max_{S \in T_2} e(x,S).$$

Consider the double Stearns transfer:

$$\begin{array}{ll} y_{i'} = x_{i'} + \delta, & y_{j'} = x_{j'} - \delta, \\ y_{i''} = x_{i''} - \delta, & y_{j''} = x_{j''} + \delta, \end{array} \quad y_k = x_k,\ k \neq i',\ i'', j', j'', \tag{3.7}$$

where $\delta > 0$ is defined by

$$\delta = \begin{cases} 1/2\,[e(x,S_0) - e(x,S_1^*)] & \text{if } e(x,S_2^*) \leq 1/2\,[e(x,S_0) + e(x,S_1^*)], \\ e(x,S_0) - e(x,S_2^*) & \text{if } e(x,S_2^*) \geq 1/2\,[e(x,S_0) + e(x,S_1^*)]. \end{cases} \tag{3.8}$$

We intend to show that in both cases depending on the choice of δ, we have $y \in F_{\mathcal{S}}$, $y \in M_0 - C(G)$. Anyway, from (3.7) we get

$$e(y,S) = \begin{cases} e(x,S) + 2\delta & \text{if } S \in T_1, \\ e(x,S) + \delta & \text{if } S \in T_2, \end{cases} \quad \forall S \in T_1 \cup T_2, \tag{3.9}$$

and

$$e(y,S) \leq e(x,S), \quad \forall S \notin T_1 \cup T_2. \tag{3.10}$$

Obviously, for all $S \in \mathcal{S}$ we have $e(y,S) = e(x,S) = 0$, hence $y \in F_{\mathcal{S}}$, and $e(y,S_0) = e(x,S_0)$. The last equality shows that the value of the excess for S_0 is unchanged and we intend to prove that this is the maximum excess of y, but besides S_0 there is another coalition, either S_1^* or S_2^* with the same highest excess of y.

Case (1): $e(x,S_2^*) \leq 1/2\,[e(x,S_0) + e(x,S_1^*)]$.

From (3.7), (3.8), (3.9) we get:

$$e(y,S) = e(x,S_0) - [e(x,S_1^*) - e(x,S)] \leq e(x,S_0) = e(y,S_0), \quad \forall\ S \in T_1\ , \tag{3.11}$$

in particular, $e(y,S_1^*) = e(y,S_0)$; further, we get:

$$\begin{aligned} e(y,S) = e(x,S_0) &- \{1/2\,[e(x,S_0) + e(x,S_1^*)] - e(x,S_2^*)\} \\ &- [e(x,S_2^*) - e(x,S)] \leq e(x,S_0) = e(y,S_0)\ , \quad \forall S \in T_2\ . \end{aligned} \tag{3.12}$$

From (3.7), (3.8), (3.10) and (C), we get:

$$e(y,S) \leq e(x,S) < e(x,S_0) = e(y,S_0)\ , \qquad \forall S \notin T_1 \cup T_2,\ S \neq S_0\ . \tag{3.13}$$

To summarize: in case (1), we have $e(y,S_1^*) = e(y,S_0)$ and $e(y,S) \leq e(y,S_0)$ for all $S \subseteq N$, so that by Theorem 1.1 we get $y \in M_0 - C(G)$.

Case (2): $e(x,S_2^*) \geq 1/2\,[e(x,S_0) + e(x,S_1^*)]$.

From (3.7), (3.8), (3.9) we get:

$$\begin{aligned} e(y,S) &= e(x,S) + 2[\,e(x,S_0) - e(x,S_2^*)] \leq \qquad \forall S \in T_1, \\ &\leq e(x,S_0) - 2\{e(x,S_2^*) - 1/2\,[e(x,S_0) + e(x,S_1^*)]\} \leq e(x,S_0) = e(y,S_0)\ , \end{aligned} \tag{3.14}$$

and

$$e(y,S) = e(x,S_0) - [e(x,S_2^*) - e(x,S)] \leq e(x,S_0) = e(y,S_0), \quad \forall S \in T_2\ , \tag{3.15}$$

in particular, $e(y,S_2^*) = e(y,S_0)$.

From (3.7), (3.8), (3.10) and (C) we get:

$$e(y,S) \leq e(x,S) < e(x,S_0) = e(y,S_0), \quad \forall S \notin T_1 \cup T_2,\ S \neq S_0. \tag{3.16}$$

To summarize: in case (2), we have $e(y,S_2^*) = e(y,S_0)$ and $e(y,S) \leq e(y,S_0)$, $\forall S \subseteq N$, so that by Theorem 1.1 we get $y \in M_0 - C(G)$. Note that $y \in F_{\mathcal{S}}$, $y \in M_0 - C(G)$, has been obtained by two Stearns transfers.

Corollary 3.4: For G a convex game, consider $\mathcal{S}$ and $x \in F_{\mathcal{S}}$ subject to (C), where S_0 is defined by (3.1). Then, there exists $y \in F_{\mathcal{S}}$, $y \in M_0 - C(G)$, and such a y can be obtained by one or two Stearns transfers.

Proof: Follows from Corollary 3.2 and Theorem 3.3.

Now, if we remove the assumption (C) and keep only the assumption (B), we get from Corollary 3.4 the main result of the paper:

Theorem 3.5: If G is a convex game and $\mathcal{S}$ is neither a trivial, nor a quasi-trivial coalition structure, then there exists $y \in F_{\mathcal{S}}$ such that $y \in M_0 - C(G)$. The payoff y can be obtained from $x \in F_{\mathcal{S}}$, $x \notin C(G)$, $x \notin M_0 - C(G)$, by one or two Stearns transfers.

Note that this result is based on Theorem 1.1, so that for non convex games a different technique should be used in an existence proof.

Acknowledgement: The author is indebted to Professor Michael Maschler and Stef Tijs for suggestions and helpful discussions around the topic.

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*Paper presented at AMS Meeting, No. 838, U.C.L.A., November 1987.

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