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GENERALIZATION OF LAGRANGE'S EQUATIONS

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Summary

In this paper Lagrange's equations are generalized, in the case of holonomic relations, with the view of establishing a system of equations, in which the higher order accelerations might be obtained as solutions of these equations. For $n=3$, the classical equations result from the general equations n being the order of the acceleration.

1. Introduction.

The higher order accelerations appear directly in the study of all phenomena whose evolution in time is very fast. Their determination by the method of successive derivations lead, in most cases, to great difficulties in the mathematical calculus. A direct method for determining the $n>2$ order accelerations is possible by passing from the differential equations of the classical mechanics to the equations of the mathematical physics.

In this paper Lagrange's equations are generalized in view of establishing a system of equations in which the higher order accelerations might be obtained as solutions of these equations.

To make the order of acceleration coincide with the order of the derivative, we call x the zero order acceleration, $\dot{x}$ the first order acceleration, $\ddot{x}$ the second order acceleration, and the $n>2$ order ones were called higher order accelerations [3], [4], [5].

2. Classical considerations

From the Analytical Mechanics we know that in the case of holonomic relations, when the system has s degrees of freedom, Lagrange's equations are [1], [6]:

$$\frac{d}{\partial t}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q_k}=Q_k,\ (k=1,2,\cdots,\ s), \tag{1}$$

where T is the kinetic energy of the system. The equations (1) constitute a system of s second order differential equations, with the unknown functions $q_k(t)$, $(k = 1, 2, \cdots, s)$. To these equations the initial conditions

$$q_k(t_o)=q_{o,k},\ \dot{q}_k(t_o)=\dot{q}_{o,k}\ ,$$

are added, fixing the position of the system at the moment t_o and the state of velocities at the same moment.

If the system does not contain but the material points M_i, $(i=1,2,\cdots,n)$, we have in particular

$$T=\frac{1}{2}\sum_{i=1}^{n} m_i\cdot\bar{v}_i^2,\ Q_k=\sum_{i=1}^{n}\bar{F}_i\frac{\partial\bar{r}_i}{\partial q_k},\ \bar{r}_i=\bar{r}_i(q_k). \tag{2}$$

In case that all the forces are conservative, equations (1) are written in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)-\frac{\partial L}{\partial q_k}=0,\ (k=1,2,\cdots,s), \tag{3}$$

where the kinetic potential is the scalar function $L=T+U$, U being "the function of the common force $U(x,y,z)$".

When all the forces soliciting the material system are conservative, one can determine a " function of generalized forces" $U(t,q_1,\ q_2,\cdots,q_s)$, so that

$$\frac{\partial U}{\partial q_k}=Q_k,\ (k=1,\ 2,\ \cdots,\ s). \tag{4}$$

The q_k variables, $(k=1,\ 2,\ \cdots,\ s)$ are independent geometrical parameters (distances, angles) which determine the system's position completely.

In the second member of the classical form (1), the expressions Q_k appear as result of the presence of the given forces. When the given forces admit a potential, the function $L(q,\dot{q},t)$ is introduced, depending on the parameters q_1, $\cdots$, q_s, $\dot{q}_1,\cdots,\ \dot{q}_s$, t by the (definition) relation

$$L(q,\dot{q},t) = T(q,\dot{q},t) + U(q,t), \tag{5}$$

where, in a concise form, the arguments on which the considered functions depend are indicated. Function L is "the kinetic potential of Lagrange". Lagrange's equations (1), as result of relations (4) and (5), take the remarkable form (3).

Lagrange's method gives us a simple pattern for making up the second order differential equations of the holonomous systems dynamics. All is reduced to the effective expression of function T, the system's kinetic energy, with the aid of q, $\dot{q}$, t variables considered as independent, and to the integration of the classical system (1), with the given initial conditions [1], [6].

3. The New Mathematical Form of Lagrange's Equations.

The classical equations (1) and (3) are written in the new form

$$\frac{\partial \dot{T}}{\partial \dot{q}_k} - 2\frac{\partial T}{\partial q_k} = Q_k, \quad (k = 1,2,\cdots,s), \tag{6}$$

$$\frac{\partial \dot{L}}{\partial \dot{q}_k} - 2\frac{\partial L}{\partial q_k} = 0, \quad (k = 1,2,\cdots,s). \tag{7}$$

The identity between the equations (1) and (6), respectively (3) and (7) may be verified for the different expressions of function T and L. From (1) and (6), and from (3) and (7), it follows

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \frac{\partial \dot{T}}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k}, \tag{8}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial \dot{L}}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k}. \tag{9}$$

If $\frac{\partial T}{\partial q_k} = 0$ and $\frac{\partial L}{\partial q_k} = 0$, the expressions (8) and (9) become

$$\left(\frac{\partial T}{\partial \dot{q}_k}\right)^{\cdot} = \frac{\partial \dot{T}}{\partial \dot{q}_k}, \quad \left(\frac{\partial L}{\partial \dot{q}_k}\right)^{\cdot} = \frac{\partial \dot{L}}{\partial \dot{q}_k}.$$

To pass from the classical forms (1) and (3), to the new forms (6) and

(7), is necessary for Lagrange's equations generalization.

4. Generalization of the Equations.

By taking the derivatives $\dot{T}$, $\dot{L}$, $\frac{\partial \dot{T}}{\partial \dot{q}_k}$ and $\frac{\partial \dot{L}}{\partial \dot{q}_k}$, it follows

$$\frac{\partial \dot{T}}{\partial \dot{q}_k} = \frac{\partial^2 T}{\partial \dot{q}_k \partial q_k}\dot{q}_k + \frac{\partial T}{\partial q_k} + \frac{\partial T}{\partial q_k} + \frac{\partial^2 T}{\partial \dot{q}_k^2}\ddot{q}_k + \frac{\partial^2 T}{\partial \dot{q}_k \partial t}, \tag{10}$$

$$\frac{\partial \dot{L}}{\partial \dot{q}_k} = \frac{\partial^2 L}{\partial \dot{q}_k \partial q_k}\dot{q}_k + \frac{\partial L}{\partial q_k} + \frac{\partial^2 T}{\partial \dot{q}_k^2}\ddot{q}_k + \frac{\partial^2 L}{\partial \dot{q}_k \partial t}. \tag{11}$$

By substituting (10) in (6), and (11) in (7), one obtains

$$\frac{\partial^2 T}{\partial \dot{q}_k \partial q_k}\dot{q}_k + \frac{\partial^2 T}{\partial \dot{q}_k^2}\ddot{q}_k + \frac{\partial^2 T}{\partial \dot{q}_k \partial t} - \frac{\partial T}{\partial q_k} = Q_k, \quad (k = 1, 2, \cdots, s), \tag{12}$$

$$\frac{\partial^2 L}{\partial \dot{q}_k \partial q_k}\dot{q}_k + \frac{\partial^2 T}{\partial \dot{q}_k^2}\ddot{q}_k + \frac{\partial^2 L}{\partial \dot{q}_k \partial t} - \frac{\partial L}{\partial q_k} = 0, \quad (k = 1, 2, \cdots, s). \tag{13}$$

By using the transformations

$$\dot{q}_k(t) = \sum_{\sigma=0}^{n-2} q_k^{(1+\sigma)}(0)\,\frac{t^\sigma}{\sigma!} + \int_0^t \frac{(t-s)^{n-2}}{(n-2)!}\,q_{k,n}(s)ds, \tag{14}$$

$$\ddot{q}_k(t) = \sum_{\sigma=0}^{n-3} q_k^{(2+\sigma)}(0)\,\frac{t^\sigma}{\sigma!} + \int_0^t \frac{(t-s)^{n-3}}{(n-3)!}\,q_{k,n}(s)ds, \tag{15}$$

$$q_{k,n}(t) = q_k(t), \quad (n = 3, 4, 5, \cdots), \quad (k = 1, 2, \cdots, s),$$

and by substituting (14) and (15) in (12) and (13), it follows

$$\frac{\partial^2 T}{\partial \dot{q}_k \partial t} - \frac{\partial T}{\partial q_k} + \left[\frac{\partial^2 T}{\partial \dot{q}_k \partial q_k} \sum_{\sigma=0}^{n-2} q_k^{(1+\sigma)}(0) + \frac{\partial^2 T}{\partial \dot{q}_k^2} \sum_{\sigma=0}^{n-3} q_k^{(2+\sigma)}(0)\right]\frac{t^\sigma}{\sigma!} +$$

$$+\left[\frac{\partial^2 T}{\partial \dot{q}_k \partial q_k}\int_0^t \frac{(t-s)^{n-2}}{(n-2)!} + \frac{\partial^2 T}{\partial \dot{q}_k^2}\int_0^t \frac{(t-s)^{n-3}}{(n-3)!}\right] q_{k,n}(s)ds = Q_k,$$

$$(k = 1, 2, \cdots, s), \quad (n = 3, 4, 5, \cdots), \tag{16}$$

$$\frac{\partial^2 L}{\partial \dot{q}_k \partial t}-\frac{\partial L}{\partial q_k}+\left[\frac{\partial^2 L}{\partial \dot{q}_k \partial q_k}\sum_{\sigma=0}^{n-2} q_k^{(1+\sigma)}(0)+\frac{\partial^2 T}{\partial \dot{q}_k^2}\sum_{\sigma=0}^{n-3} q_k^{(2+\sigma)}(0)\right]\frac{t^\sigma}{\sigma!}+$$

$$+\left[\frac{\partial^2 T}{\partial \dot{q}_k \partial q_k}\int_0^t \frac{(t-s)^{n-2}}{(n-2)!}+\frac{\partial^2 T}{\partial \dot{q}_k^2}\int_0^t \frac{(t-s)^{n-3}}{(n-3)!}\right] q_{k,n}(s)ds=0, \qquad (17)$$

$(k=1,\ 2,\ \cdots,\ s)$, $(n=3,4,5,\cdots)$.

The integro-differential equations (16) and (17) are "Lagrange's generalized equations", permitting to determine "the higher order generalized accelerations". For $n=3$, from (16) and (17) the equations (6) and (7) result.

5. Application

Let us determine the fourth order generalized acceleration of a load $\bar{P}$, suspended by a cable of length ℓ and total weight $\bar{F}$, which unfolds on a tambour of radius R and weight $\bar{G}$. By neglecting the frictions, one considers that at the initial moment the unfolded length of the cable is x_0 (Fig.1).

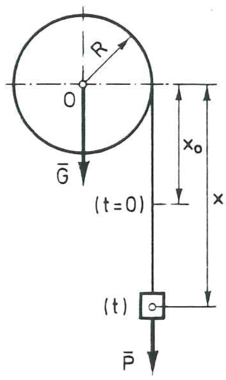

Fig.1

The system having only one degree of freedom, the generalized coordinate is x, the generalized velocity (the first order generalized acceleration) is $\dot{x}$, and thus $\dot{x} = R\omega$. All the cable, unfolded or folded, has the velocity $\dot{x}$.

By putting T_1 as kinetic energy of the tambour, T_2 as kinetic energy of the cable and T_3 the kinetic energy of the load, for the ensemble it follows

$$T = T_1 + T_2 + T_3 = \frac{\dot{x}^2}{2g}\left(\frac{G}{2} + F + P\right). \tag{18}$$

For a displacement of dx, the elementary mechanical work is

$$dW = Pdx + \frac{F}{\ell}\cdot xdx,$$

and thus the generalized force Q has the expression

$$Q = \frac{dW}{dx} = P + \frac{F}{\ell}x . \tag{19}$$

By applying Lagrange's classical equation

$$\left(\frac{\partial T}{\partial \dot{x}}\right)^{\cdot} - \frac{\partial T}{\partial x} = Q,$$

and having

$$\frac{\partial T}{\partial \dot{x}} = \frac{\dot{x}}{g}\left(\frac{G}{2} + F + P\right), \quad \frac{\partial T}{\partial x} = 0,$$

it follows

$$\frac{\ddot{x}}{g}\left(\frac{G}{2} + F + P\right) = P + \frac{F}{\ell}x. \tag{20}$$

For the problem, equation (16) becomes

$$\frac{\partial^2 T}{\partial \dot{x}\partial t} - \frac{\partial T}{\partial x} + \frac{\partial^2 T}{\partial \dot{x}\partial x} = \sum_{\sigma=0}^{2} x_0^{(1+\sigma)} + \frac{\partial^2 T}{\partial \dot{x}^2}\left[\sum_{\sigma=0}^{1} x_0^{(2+\sigma)}\right]\frac{t^\sigma}{\sigma!} +$$
$$+\left[\frac{\partial^2 T}{\partial \dot{x}\partial x}\int_0^t \frac{(t-s)^2}{2} + \frac{\partial^2 T}{\partial \dot{x}^2}\int_0^t (t-s)\right]\varphi_4(s)ds = Q, \tag{21}$$

where $\varphi_4(t) = x^{(4)}(t)$.

Having

$$\frac{\partial^2 T}{\partial \dot{x}\partial t} = 0, \quad \frac{\partial^2 T}{\partial \dot{x}\partial x} = 0, \quad \frac{\partial^2 T}{\partial \dot{x}^2} = 2A, \quad \left[A = \frac{G + 2(F+P)}{4g}\right],$$

equation (21) becomes

$$\beta x - \int_0^t (t-s)\varphi_4(s)ds = \ddot{x}_0 + \dddot{x}_0 t - \alpha, \tag{22}$$

where

$$\alpha = \frac{P}{2A}, \quad \beta = \frac{F}{2A\ell} .$$

By performing the transformation

$$x = \sum_{\sigma=0}^{3} x_0^{(\sigma)} \frac{t^\sigma}{\sigma!} + \int_0^t \frac{(t-s)^3}{6} \varphi_4(s)ds,$$

the equation (22) becomes

$$\int_0^t N_4(t,s)\cdot\varphi_4(s)ds = f_4(t), \tag{23}$$

where

$$N_4(t,s) = \frac{\beta}{6}(t-s)^3 - (t-s),$$

$$f_4(t) = \ddot{x}_0 - \beta x_0 - \alpha + (\dddot{x}_0 - \beta\dot{x}_0)t -$$
$$- \beta(\ddot{x}_0 \frac{t^2}{2} + \dddot{x}_0 \frac{t^3}{6}).$$

The first kind Volterra type linear integral equation (23) may be transformed into one of the second kind, if conditions [2]

$$N_4(t,t) \neq 0, \quad f_4(0) = 0. \tag{24}$$

are satisfied.

The first condition not being satisfied, we have

$$\left[\frac{\partial N_4(t,s)}{\partial t}\right]_{s=t} = -1 \neq 0, \tag{25}$$

$$\dot{f}_4(t) = \dddot{x}_0 - \beta\dot{x}_0 t + \dddot{x}_0 \frac{t^2}{2}).$$

From $\dot{f}_4(0) = 0$, it follows

$$\dddot{x}_0 = \beta\dot{x}_0. \tag{26}$$

Taking the relation (26) also results the derivative in the differential equation (20), for $t = 0$. The given initial conditions are $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$. From (20), for $t = 0$, one obtains

$$\ddot{x}_0 = \alpha + \beta x_0. \tag{27}$$

With the conditions (25) and (26), the second kind integral equation is

$$\varphi_4(t) + \int_0^t N_4^1(t,s)\varphi_4(s)ds = f_4^1(t), \tag{28}$$

where

$$N_4^1(t,s) = \frac{\partial^2 N_4(t,s)}{\partial t^2} = \beta(t-s),$$
$$f_4^1(t) = \ddot{f}_4(t) = -\beta(\ddot{x}_0 + \dddot{x}_0 t).$$

By applying the method of successive approximations, the solution of equation (28) has the form

$$\varphi_4(t) = \varphi_{4,0}(t) + \varphi_{4,1}(t) + \varphi_{4,2}(t) + \cdots + \varphi_{4,m}(t) + \cdots,$$

where

$$\varphi_{4,0}(t) = f_4^1(t),$$
$$\varphi_{4,1}(t) = -\int_0^t N_4^1(t,s)\cdot\varphi_{4,0}(s)\cdot ds,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots ,$$
$$\varphi_{4,m}(t) = -\int_0^t N_4^1(t,s)\cdot\varphi_{4,m-1}(s)\cdot ds.$$

In the second approximation, the solution of equation (28) is

$$\varphi_4(t) \approx \varphi_{4,0}(t) + \varphi_{4,1}(t),$$

where

$$\varphi_{4,0}(t) = = \beta(\ddot{x}_0 + \dddot{x}_0 t),$$
$$\varphi_{4,1}(t) = \beta^2 \int_0^t (t-s)(\ddot{x}_0 + \dddot{x}_0 s)\cdot ds \ .$$

Performing the integration, it follows

$$\varphi_4(t) \approx -\beta(\ddot{x}_0 + \dddot{x}_0 t) + \frac{\beta^2}{2} t^2 (\ddot{x}_0 + \dddot{x}_0 \ \frac{t}{3}) \ . \tag{29}$$

By substituting (26) and (27) in (29), one obtains the generalized acceleration of the fourth order of the weight P:

$$\varphi_4(t) \approx -\beta\left[\alpha + \beta(x_0 + \dot{x}_0 t)\right] + \frac{\beta^2}{2} t^2 \left[\alpha + \beta(x_0 + \dot{x}_0 \ \frac{t}{3})\right] ,$$

in $m.s^{-4}$.

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