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To the Memory of
Miron Nicolescu

GREEDY AND OPTIMAL PATHS IN A WEIGHTED GRAPH WITHOUT CIRCUITS

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In several recent papers B. Korte and L. Lovasz considered a mathematical structure called a simple language on which a greedy algorithm can operate, (see [3], [4], [5], [6]). The concept of greedoid has been defined by relaxing an axiom and strengthening another axiom from the definition of the matroid. Under some constraints imposed to the objective function of a combinatorial optimization problem defined on a language, they showed that if the language is a greedoid, then a greedy algorithm solves always the problem. An algorithmic characterization of the greedoids, similar to that of the matroids, was further searched; the effort was motivated by several examples of combinatorial optimization problems defined on greedoids.

In this paper we discuss a class of languages associated to a directed graph without circuits, namely the elements of the languages are the paths from all vertices of 0 indegree to all other vertices, a path being considered as a sequence of vertices. Two combinatorial optimization problems of finding a maximum weight path are defined on such languages by using two kinds of weightings, namely weightings defined by means of a non negative weighting of vertices and weightings subjected to the Korte-Lovasz axioms. The purpose of our research has been that of finding necessary and sufficient conditions under which a greedy algorithm solves each of the two classes of problems.

In the first section, we prove that a greedy algorithm works for all non negative and in some sense additive weightings if and only if the directed graph without circuits is a so-called greedy graph (Theorem 7). In the second section, we show that a greedy algorithm works also for all non negative non decreasing Korte-Lovasz weightings if the language is associated to a greedy graph (Theorem 8). As any non negative additive weighting on a language associated to a directed graph without circuits is a Korte-Lovasz weighting, we conclude that a greedy algorithm works for all non negative decreasing Korte-Lovasz weightings if and only if the directed graph without circuits is a greedy graph, (Theorem 9). In the last section the language associated to a directed graph without circuits is shown to be a greedoid if and only if the graph is a greedy graph (Theorem 10). So, we derived immediately an algorithmic characterization of the greedoids on the languages associated to a directed graph without circuits (Theorem 11). This last result enabled us to compare our results to the main algorithmic results given in [3], [4].

Similar problems connected to more general classes of languages and applications to multiobjective discrete programming problems will be discussed in further papers.

1. MAXIMUM WEIGHT PATHS IN A DIRECTED GRAPH WITHOUT CIRCUITS AND THE GREEDY ALGORITHM

Let $G = (V,A)$ be a directed graph without circuits and define recurrently the subsets of vertices $V_0, V_1, \ldots, V_r$, $0 \leq r \leq |V|$, as follows: V_0 is the subset of vertices of 0 indegree in $G_0 = G$; if $V_0, \ldots, V_{q-1}$ have been defined, $q \geq 1$, and G_q is the subgraph of G generated by $V - (V_0 \cup V_1 \cup \ldots \cup V_{q-1}) \neq \phi$, then V_q is the subset of vertices of 0 indegree in G_q. The fact that there is an integer r, $0 \leq r \leq |V|$, such that $V_0, V_1, \ldots, V_r$ do exist and define a partitioning of V, follows from the absence of circuits in $G_0, G_1, \ldots, G_{r-1}$. In this section we assume $V_0 = \{x_0\}$ and x_0 will be called the root of G due to the property (C_2) stated below. Denote

$$A(x) = \{y : y \in V,\ (x,y) \in A\}; \tag{1.1}$$

then, one can easily prove the following properties of G to be further used:

(C_1) for every $q = 0,1,\ldots,r-1$, if $x \in V_q$, then $A(x) \subseteq V_{q+1} \cup \ldots \cup V_r$;

(C_2) for every $q = 0,1,\ldots,r$, if $x \in V_q$, then there is at least one path with $q+1$ vertices from x_0 to x;

(C_3) if $x \in V_q$, $0 \leq q \leq r$, and $D = [x_0, x_{i_1}, \ldots, x_{i_q}]$ is a path in G with $q+1$ vertices from x_0 to x then we have

$$x_{i_h} \in V_h,\quad h = \overline{1,q}\ ; \tag{1.2}$$

(C_4) if $x \in V_q$, $0 \leq q \leq r$, then there is no path in G with $q+2$ vertices from x_0 to x.

Let $w(x)$ be a non negative weight function defined on the set of vertices of a rooted directed graph without circuits G and P be the set of all paths from the root x_0 to all vertices. We define a non negative and in some sense additive weight function on P as follows: if $D \in P$, then

(1.3) $$W(D) = \sum_{x \in D} w(x).$$

In the following we shall consider the combinatorial optimization problem:

(P) find $\bar{D} \in P$ such that

(1.4) $$W(\bar{D}) = \max_{D \in P} W(D).$$

This is a generalized longest path problem with weights associated to the vertices; obviously, the algorithms for solving similar problems with weights associated to the arcs can be applied to problem (P). In this section, we characterize the class of those rooted directed weighted graphs without circuits for which a greedy algorithm solves (P).

A greedy algorithm consists of the following steps: the initial path is $D_0 = [x_0]$; if a path D_{q-1} has already been found, $D_{q-1} = [x_0, x_{i_1}, \ldots x_{i_{q-1}}]$, $q \geq 1$, $i_0 = 0$, then choose $x_{i_q} \in V$ such that $\{x_0, x_{i_1}, \ldots, x_{i_{q-1}}, x_{i_q}\}$ defines a path $D_q = [x_0, x_{i_1}, \ldots, x_{i_{q-1}}, x_{i_q}]$, i.e., $(x_{i_{q-1}}, x_{i_q}) \in A$, and

(1.5) $$W(D_q) = \max\{W([x_0, x_{i_1}, \ldots, x_{i_{q-1}}, x]) : [x_0, x_{i_1}, \ldots, x_{i_{q-1}}, x] \in P\}.$$

If x_{i_q} is found, then pass to a new step; otherwise, stop. A path in P has been found, but it is not necessarily a maximum weight path; so, our problem is to characterize the class of those rooted directed weighted graphs without circuits for which this algorithm gives such a path.

Note that (1.5) can be written as

(1.6) $$w(x_{i_q}) = \max\{w(x) : x \in A(x_{i_{q-1}})\},$$

but (1.5) has been preferred above, because the given formulation of a greedy algorithm is also applicable for exploring maximum weight path problems with more general objectives than $W(D)$ defined by (1.3). This will be done in the next section; however, (1.6) will also be used in some proofs of this section.

A path $D \in P$ is called a *non extendible* path if there is no path in P having D as a proper subpath.

A path $D \in P$ is called a *longest* path if it has $r+1$ vertices.

A path $\bar{D} \in P$ defined by (1.4) is called an *optimal* path with respect to the weighting W.

A path $D^* = [x_0, x_{i_1}, \ldots, x_{i_q}]$ is called a *greedy* path with respect to the weighting W if

(g_1) for each vertex $x_{i_{j+1}} \in D^*$, $j = 0,1,\ldots,q-1$, we have

$$(1.7) \qquad W(x_0, x_{i_1}, \ldots, x_{i_j}, x_{i_{j+1}}) = \max\{W([x_0, x_{i_1}, \ldots, x_{i_j}, x]) : x \in A(x_{i_j})\};$$

(g_2) D^* is a non extendible path.

Note that any path found by a greedy algorithm is a greedy path and conversely, any greedy path can be found by a greedy algorithm.

Note also that in a rooted graph without circuits with positive weights assigned to the vertices the set of longest paths, the set of optimal paths and the set of greedy paths are in general distinct subsets of the set of non extendible paths; if there are null weights, then optimal extendible paths might be present in the graph.

We consider first the problem of characterizing the non extendible paths of a rooted graph without circuits which have the property of becoming optimal paths for all non negative weightings W making them greedy paths. The main concept to be used for this purpose will be that of partial transversal of a family of subsets of a finite set V. A set $T \subseteq V$ is a transversal of the family $U = \{U_j : U_j \subseteq V,\ j \in K\}$, where K is a finite index set, if there is a bijection $B: T \to K$ such that

$$(1.8) \qquad x \in U_{B(x)}, \quad \forall x \in T.$$

A set $T \subseteq V$ is a partial transversal of U if T is a transversal of a subfamily $U^* = \{U_j : U_j \subseteq V,\ j \in J \subseteq K\}$ of U.

Consider any path $D = [x_0, x_{i_1}, \ldots, x_{i_k}] \in P$, $0 \le k \le r$, and denote $K = \{1,\ldots,k\}$. Define a family $U(D)$ of subsets of vertices associated to the path D by

$$(1.9) \qquad U(D) = \{U_0(D)\} \cup \{U_j(D) : U_j(D) = A(x_{i_{j-1}}),\ j \in K\}, \quad U_0(d) = \{x_0\}.$$

In other words, $U_j(D)$, $j \in K$, is the set of vertices that can be reached from $x_{i_{j-1}}$ by an arc of G.

Lemma 1. If $D^* \in P$ is a non extendible path and all paths in P are partial transversals of $U(D^*)$, then D^* is an optimal path for all non negative weightings W making D^* a greedy path.

Proof. D^* is a longest path, because any longest path must be a transversal of $U(D^*)$; denote $D^* = [x_0, x_{i_1}, \ldots, x_{i_r}]$ and let $D = [x_0, x_{j_1}, \ldots, x_{j_k}]$ be any path in P, $0 \le k \le r$. As D is a partial transversal of $U(D^*)$, there are k distinct indices s_h, $h = 1,\ldots,k$ such that

$$(1.10) \qquad x_{j_h} \in U_{s_h}(D^*), \quad h = \overline{1,k}.$$

If D^* is a greedy path w.r.t. W defined by (1.3), then from (1.6), (1.10) we have

$$(1.11) \qquad w(x_{j_h}) \le w(x_{i_{s_h}}), \quad h = \overline{1,k}.$$

As $w(x) \ge 0$, $\forall x \in V$, from (1.3), (1.11) we get $W(D^*) \ge W(D)$ and the result follows, because D was an arbitrary path in P.

Lemma 2. If $D^* \in P$ is a non extendible path, but there is a path $D \in P$ which is not a partial transversal of $U(D^*)$, then there is a non negative weighting W such that D^* is a greedy path but it is not an optimal path w.r.t. this weighting.

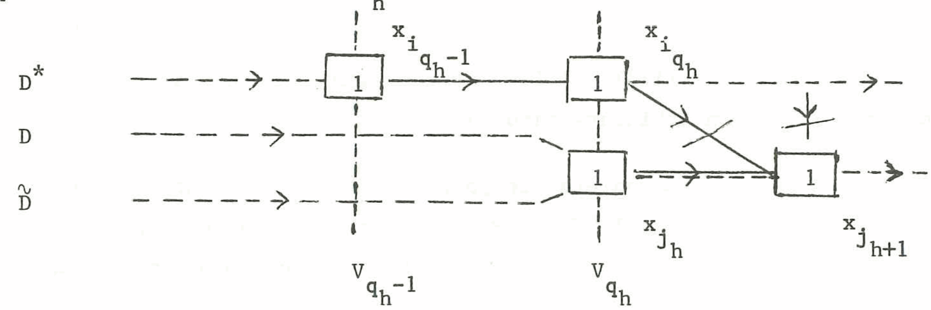
Proof. If D^* is not a longest path we may define W by (1.3) with $w(x_0) = 0$, $w(x) = 1$, $\forall x \in V - \{x_0\}$ and W makes D^* a greedy path with $W(D^*) = |D^*|$, but any longest path in D has $W(D) = r > W(D^*)$, hence D^* is not an optimal path. If D^* is a longest path, $D^* = [x_0 . x_{i_1}, \ldots, x_{i_r}]$, then according to (C_3) we must have

$$(1.12) \qquad x_{i_q} \in V_q, \quad q = \overline{1,r}.$$

Let $D = [x_0, x_{j_1}, \ldots, x_{j_k}] \in P$ be a path which is not a partial transversal of $U(D^*)$. From (C_1) it follows that there is an increasing sequence of distinct indices $1 \leq q_1 \leq \ldots q_k \leq r$ such that

$$(1.13) \qquad x_{j_1} \in V_{q_1}, \ q_1 \geq 1; \ x_{j_2} \in V_{q_2}, \ q_2 \geq 2; \ \ldots \ ; \ x_{j_k} \in V_{q_k}, \ q_k \geq k.$$

Now, denote by $h \in \{1, \ldots, k-1\}$ the first index such that the subpath $D_h = [x_0, x_{j_1}, \ldots, x_{j_h}] \subset D$ is a partial transversal of $U(D^*)$, but the subpath $D_{h+1} = [x_0, x_{j_1}, \ldots, x_{j_h}, x_{j_{h+1}}] \subseteq D$ is not. As $x_{j_h} \in V_{q_h}$, $q_h \geq h$, according to (C_2) there is a path $\bar{D} \in P$ with q_h+1 vertices from x_0 to x_{j_h}. Denote by $\tilde{D} \in P$ the path with q_h+2 vertices consisting of $\bar{D}$ and the arc $(x_{j_h}, x_{j_{h+1}})$. In G we have no arc from $x_{i_{q_h}}$ or any further vertex of D^* to $x_{j_{h+1}}$, otherwise D_{h+1} would be a partial transversal of $U(D^*)$. Of course, this does not mean that we can not add further vertices of D to obtain larger partial transversals than D_h.



Define a non negative weighting W by (1.3), where

$$w(x) = \begin{cases} 1 & \text{if } x \in V_0 \cup \ldots \cup V_{q_h} \quad \{x_{j_{h+1}}\} , \\ 0 & \text{otherwise} \end{cases} \tag{1.13}$$

As from (C_1) we have $A(x_{i_{q_h}}) \subseteq V_{q_h+1} \cup \ldots \cup V_r$ and no arc $(x_{i_e}, x_{j_{h+1}})$, $q_h \leq e \leq r$, does exist in G, the path D^* is a greedy path and $W(D^*) = q_h+1$. On the other hand we have $W(D) = q_h+2 > W(D^*)$, hence D^* is not an optimal path.

Theorem 3. In a rooted graph without circuits, a non extendible path D^* belonging to the set of all paths from the root to all vertices is an optimal path for all non negative weightings W making it a greedy path, if and only if all paths in P are partial transversals of $U(D^*)$; in this case D^* must be a longest path.

Proof. The result follows from Lemma 1 and Lemma 2.

Let us connect Theorem 3 to matroid theory. Consider the following well known results (see [7], p. 103): if U is a family of subsets of a finite set V, then the collection of partial transversals of U is the set of independent sets of a matroid on V. Such a matroid is called a transversal matroid. If U has a transversal, then the bases of this matroid have the same cardinality. If these results are applied to $U = U(D^*)$, then taking into account on the fact that $U(D^*)$ has the transversal D^*, we shall get from Theorem 3:

Corollary 4. In a rooted graph without circuits, if a non extendible path $D^* \in P$ is an optimal path for all non negative weightings making it a greedy path, then the set P of all paths from the root to all vertices is a subset of the set of independent sets of a transversal matroid on V and D^* is a basis.

Note that in general P is a proper subset of the set of independent sets of a transversal matroid, because there are partial transversals of $U(D^*)$ which are not paths in P, as soon as $r > 1$. Note also that a converse result is not true.

Now, we shall consider a subclass of rooted graphs without circuits in which any non extendible path has the property expressed in the statement of theorem 3 and we shall give an algorithmic characterization of this subclass.

A rooted directed graph without circuits G is called a *greedy graph* if for all $q = 1,\dots,r$ we have

$$A(x) = V_q, \quad \forall x \in V_{q-1}. \tag{1.14}$$

In other words, in a greedy graph each vertex $x \in V_{q-1}$, $q = \overline{1,r}$, is connected by arcs of G to all vertices of V_q and only to these vertices. Of course, a greedy graph is determined by the set of numbers r, $n_1 = |V_1|$, $n_2 = |V_2|$, ... $n_r = |V_r|$, because the subgraph of G generated by any two successsive sets V_{q-1} and V_q is a complete directed bipartite graph.

Note that in a greedy graph the set of all non extendible paths and the set of all longest paths coincide.

Lemma 5. In a greedy graph, if D^* is a non extendible path, then any other path $D \in P$ is a partial transversal of $U(D^*)$.

Proof. D^* is a longest path, $D^* = [x_0, x_{i_1}, \dots, x_{i_r}]$, according to the above remark and (C_3) shows that $x_{i_q} \in V_q$, $q = \overline{1,r}$. If $D = [x_0, x_{j_1}, \dots, x_{j_k}]$, $0 \le k \le r$ is any other path then $x_{j_q} \in V_q$, $q = \overline{1,k}$ because $x_{j_{q-1}} \in V_{q-1}$, $x_{j_q} \in A(x_{j_{q-1}})$ and $A(x_{j_{q-1}}) = V_q$ imply $x_{j_q} \in V_q$. As $A(x_{i_{q-1}}) = V_q$, $q = \overline{1,r}$; we have $x_{j_q} \in A(x_{i_q})$, $q = \overline{1,k}$ so that D is a partial transversal of $U(D^*)$.

Theorem 6. In a greedy graph with a non negative weighting W any non extendible path is an optimal path if and only if it is a greedy path w.r.t. the weighting W.

Proof. Let $D = [x_0, x_{j_1}, \dots, x_{j_r}]$ be any non extendible path in a greedy graph G. If D is a greedy path, then according to Theorem 3 it will be an optimal path, because Lemma 5 shows that all paths of G are partial transversals of $U(D)$. So, we must prove only the converse result. Let us suppose that D is an optimal path but it is not a greedy path. Denote by $D^* = [x_0, x_{i_1}, \dots, x_{i_r}]$ a non extendible path which is a greedy path. According to the part already proved of our theorem, D* is also an optimal path, hence we have W(D*) = W(D) or

$$(1.15)\qquad \sum_{q=1}^{r} [w(x_{i_q})-w(x_{j_q})] = 0.$$

From (C_3) and (1.14) we get

$$(1.16)\qquad x_{i_q} \in V_q,\ x_{j_q} \in V_q,\ V_q = A(x_{i_{q-1}}) = A(x_{j_{q-1}}),\ q = \overline{1,r}$$

hence the greediness of D* implies

$$(1.17)\qquad w(x_{i_q}) \geq w(x_{j_q}),\ q = \overline{1,r}.$$

So, from (1.15), (1.16), (1.17) we get

$$(1.18)\qquad w(x_{j_q}) = w(x_{i_q}) = \max_{x \in A(x_{i_{q-1}})} w(x) = \max_{x \in A(x_{j_{q-1}})} w(x).$$

This proves that D is also a greedy path, which contradicts the assumption and the theorem follows.

Theorem 6 says that in a greedy graph the set of non extendible optimal paths and the set of greedy graphs w.r.t. a non negative weighting coincide. Of course, an extendible optimal path, if any, can not be a greedy path; such paths might exist only if all vertices belonging to the levels in which these paths can be extended have null weights and in this case all the non extendible extensions of these paths will be optimal and greedy paths.

Now we shall give the algorithmic characterization of the subclass of greedy graphs in the class of rooted graphs without circuits. We shall say that a greedy algorithm applied to a problem (P) in a rooted graph G without circuits "works," if any non extendible path of G will be an optimal path for all non negative weightings W making it a greedy path.

Theorem 7. A greedy algorithm works in a rooted graph G without circuits if and only if G is a greedy graph.

Proof. If G is a greedy graph, then the stated property of any non extendible path in G follows from Theorem 6, so that we must prove only the converse result. Under our hypothesis all non extendible paths of G must be longest paths, according to Theorem 3. Moreover, for all $q = \overline{1,r}$ if $x \in V_{q-1}$ then from (C_1) we have $A(x) \subseteq V_q \cup \ldots \cup V_r$. However, if a vertex $x \in V_{q-1}$ would have an image in a set V_e with $e > q$, then a non extendible path with less than $r + 1$ vertices would exist. Therefore, for all $q = \overline{1,r}$ if $x \in V_{q-1}$, then $A(x) \subseteq V_q$. So, in order to prove that G is a greedy graph we must prove that $A(x) \subset V_q$ for some $q \in \{1,\ldots,r\}$ and some $x \in V_{q-1}$ is impossible.

Let us suppose that G is not a greedy graph, hence there is $y \in V_q$ such that $y \notin A(x)$ for some $q \in \{1,\ldots,r\}$ and some $x \in V_{q-1}$. According to (C_2) there is a path with $q + 1$ vertices from x_0 to $y \in V_q$; denote it by $D = [x_0, x_{i_1}, \ldots x_{j_q}]$, $x_{j_q} = y$. Consider any non extendible, hence longest path in G passing through x; according to (C_1), (C_2), (C_3) such a path $D^* = [x_0, x_{i_1}, \ldots, x_{i_r}]$, $x_{i_q} = x$ does exist. Notice that $x_{i_{q-1}} \neq x_{j_{q-1}}$, $x_{i_q} \neq x_{j_q}$ otherwise $y \in A(x)$.

Now, according to Theorem 3 if D^* is an optimal path for all weightings making it a greedy path, then D must be a partial transversal of $U(D^*)$. So, the q vertices $x_{j_1}, \ldots, x_{j_q}$ in D must be representatives of q subsets among the r subsets $A(x_0)$, $A(x_{i_1})$, ..., $A(x_{i_{r-1}})$. According to (C_3) we have $x_{j_k} \in V_k$, $k = \overline{1,q}$ and our partial result proved above shows that

$$A(x_0) \subseteq V_1,\ A(x_{i_1}) \subseteq V_2,\ \ldots,\ A(x_{i_{r-1}}) \subseteq V_r. \tag{1.19}$$

Therefore we get

$$x_{j_k} \in A(x_{i_{k-1}}),\ k = \overline{1,q} \tag{1.20}$$

i.e. the representatives must belong exactly to $A(x_0)$, $A(x_{i_1})$, ..., $A(x_{i_{q-1}})$. For $k = q$ we have $x_{j_q} \in A(x_{i_{q-1}})$, or $y \in A(x)$, which is a contradiction. G must be a greedy graph.

Theorem 7 says that a greedy algorithm will always discover an optimal solution of problem (P) in a greedy graph. Moreover, we are able to discover all non extendible optimal solutions if we discover all greedy paths.

At the beginning of this section we assumed that the set of vertices of G of 0 indegree has the cardinality one, i.e. G has a root. This assumption can easily be removed as follows. If this set of vertices has a cardinality greater than one, then we shall denote it by V_1 and we shall continue to define recurrently the partition of V as above. Further, we shall introduce an artificial vertex x_0 with $w(x_0) = 0$ and artificial arcs (x_0,x), $\forall x \in V_1$, so that we shall get a directed rooted graph G^* without circuits. The problem (P*) of finding a maximum weight path in G w.r.t. a non negative weighting defined by (1.3) is equivalent to the problem (P) in G^*. All concepts introduced above in G^* can be redefined with minor changes in G. So, if we delete the word "rooted" from the difinition of a greedy graph and we initialize a greedy algorithm in G by choosing as x_{i_1} the vertex of V_1 with a ma.'mum weight, then by deleting the same word from Theorem 7 we shall get a valid result in G.

2. MAXIMUM WEIGHT PATHS WITH RESPECT TO KORTE-LOVASZ WEIGHTINGS IN A DIRECTED GRAPH WITHOUT CIRCUITS

Let $G = (V,A)$ be a directed graph without circuits and define $V = V_0 \cup V_1 \cup \ldots \cup V_r$ as in the preceeding section. We shall assume again $V_0 = \{x_0\}$ and P will denote the set of all paths from the root x_0 to all vertices.

A non negative non decreasing function W defined on P will be called a *Korte-Lovasz weighting* if

(KL_1) for all $D \in P$ and all $x \in V$, $y \ddot{\in} V$ such that

$$[D,x] \in P, \quad [D,y] \in P, \quad W([D,x]) \geq W([D,y]), \tag{2.1}$$

we have

$$W([D,D',x,D'']) \geq W([D,D',y,D'']) \tag{2.2}$$

for all paths D', D" such that $[D,D',x,D''] \in P$, $[D,D',y,D''] \in P$. Such weightings have been considered in [3], [4], [5] and the sense of the axiom (KL_1) will become clear in the proof of the main result of this section.

Consider the combinatorial optimization problem:

(R) find $\bar{D} \in P$ such that

$$W(\bar{D}) = \max \{W(D) : D \in P\}. \tag{2.3}$$

If W is a non negative weighting defined by (1.3) by means of a non negative function w, then W is non decreasing and the axiom (KL_1) is satisfied, because

$$W([D,D',x,D'']) - W([D,x]) = W([D,D',y,D'']) - W([D,y]) \tag{2.4}$$

for all $x \in V$, $y \in V$ such that $[D,x] \in P$, $[D,y] \in P$, $[D,D',x,D''] \in P$, $[D,D',y,D''] \in P$. So, the problem (R) is more general than the problem (P) considered in the preceeding section. A greedy algorithm for solving (R) has been stated at the beginning of that section, where the concept of greedy path has also been defined for any function W defined on the set of paths P. Now an optimal path will be a path $\vec{D} \in P$ defined by (2.3). Note that the definition of a non extendible path and that of a longest path do not depend on W and that of the expression "a greedy algorithm works" depends on W only by means of the concept of optimal path.

Now, we shall characterize the subclass of the class of rooted graphs without circuits for which a greedy algorithm works in solving (R). As the problem (P) is a particular case of the problem (R), this subclass is also a subclass of the class of greedy graphs. The main result will be that a greedy algorithm works for greedy graphs with Korte-Lovasz weightings, too.

Theorem 8. If G is a greedy graph, then a greedy algorithm works for all Korte-Lovasz weightings.

Proof. Consider any greedy path $D^* = [x_0, x_{i_1}, \ldots, x_{i_r}]$ w.r.t. a K-L weighting W and any other path $D = [x_0, x_{j_1}, \ldots x_{j_q}] \in P$, $q \le r$; we shall prove that $W(D^*) \ge W(D)$. As G is a greedy graph we have

$$x_{i_k} \in V_k, \quad x_{j_k} \in V_k, \quad k = \overline{1,q}. \tag{2.5}$$

If $q = 1$ the result follows from the greediness of D^*. If $q > 1$, then the greediness of D^* can be expressed by

(2.6) $$W([D_k,x_{i_k}]) \geq W([D_k,y]), \ \forall y \in V_k, \quad k = \overline{1,q}$$

where $D_k = [x_0,x_{i_1},\ldots,x_{i_{k-1}}]$, $i_0 = 0$. Remark that (2.6) is exactly (2.1) for $D = D_k$, $x = x_{i_k}$, $k = \overline{1,q}$; therefore, if we denote $D' = \phi$, $D''_k = [x_{j_{k+1}},\ldots,x_{j_q}]$, $k < q$, and we consider $y = x_{j_k} \in V_k$ we get

(2.7) $$W([x_0,x_{i_1},\ldots,x_{i_{k-1}},x_{i_k},x_{j_{k+1}},\ldots,x_{j_q}])$$
$$\geq W([x_0,x_{i_1},\ldots,x_{i_{k-1}},x_{j_k},x_{j_{k+1}},\ldots,x_{j_q}]), \quad k = \overline{1,q-1} .$$

The inequalities (2.7) give a sense to the axiom (KL_1), namely it says that the weight of the path in heavy lines in the Figure 2 is at least equal to the weight of the path using the same arcs except the arcs of the mesh where it takes the lower arcs.

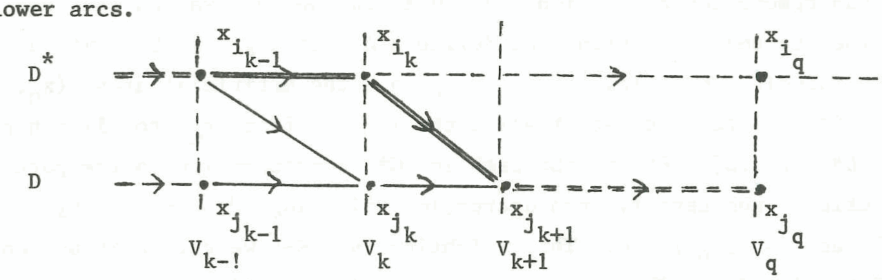


Figure 2

From (2.6) for $k = q$ and $y = x_{j_q} \in V_q$ we also get

(2.8) $$W([x_0,\ldots,x_{i_{q-1}},x_{i_q}]) \geq W([x_0,\ldots,x_{i_{q-1}},x_{j_q}]).$$

Now, from (2.7) for $k = \overline{1,q-1}$ and (2.8) we have

(2.9) $$W([x_0,x_{i_1},\ldots,x_{i_{q-1}},x_{i_q}]) \geq W([x_0,x_{j_1},\ldots,x_{j_{q-1}},x_{j_q}]).$$

where the left hand side has been the left hand side of the inequality (2.8), the right hand side has been the right hand side of the inequality (2.7) for $k = 1$ and all intermediate terms have been cancelled. As W is non decreasing, from (2.9) we get $W(D^*) \geq W(D)$ and this proves the optimality of D^*, because D has been an arbitrary path.

Theorem 9. A greedy algorithm works in a rooted directed graph G without circuits with respect to all Korte-Lovasz weightings if and only if G is a greedy graph.

Proof. The result follows form Theorem 7 and Theorem 8.

Theorem 9 says that a greedy algorithm will always discover an optimal solution of problem (R) in a greedy graph. On the other hand, it represents an algorithmic characterization of the subclass of the class of rooted graphs without circuits for which a greedy algorithm works for all Korte-Lovasz weightings. Note that in the proof we have not used intermediate results like Theorem 3 and Theorem 6.

Now, we can remove the assumption $|V_0| = 1$ in the same way as this has been done in the preceeding section. We define a directed graph G^* without circuits by introducing an artificial root x_0 and the artificial arcs (x_0,x), $\forall\, x \in V_1$. If P^* denotes the set of all paths in G^* from x_0 to all other vertices and $D^* = [x_0,D] \in P^*$ is the path in G^* corresponding to the path $D \in P$, we define a non negative non decreasing weighting W^* on P^* by $W^*(D^*) = W(D)$ and $W^*([x_0]) = 0$. This weighting is a K-L weighting if and only if W is a K-L weighting. Therefore, the problem (R*) of finding a maximum weight path in G w.r.t. W is equivalent to the problem (R) in G^*. So, by deleting from Theorem 9 the word "rooted" we shall get a valid result in G.

Theorem 8 rises an interesting question, namely that of finding more general classes of weightings for which a maximum weight path in a greedy graph can always be found by a greedy algorithm with respect to such weightings.

3. GREEDOIDS AND THE MAXIMUM WEIGHT PATH PROBLEMS IN A DIRECTED GRAPH WITHOUT CIRCUITS

The maximum weight path problem has a practical interest that motivates the efforts to discover particular graphs in which an efficient technique like a greedy algorithm can solve the problem with respect to a particular class of weightings. However, the results of the preceeding sections are also connected to the abstract problem of characterizing algorithmically the languages for which a maximum weight word can be computed by a greedy algorithm with respect to a given class of weightings. This problem has recently been considered by B. Korte and L. Lovasz in [3], [4], [5], [6]. In this section we shall give algorithmic characterizations of the languages associated to a directed graph without circuits for which a maximum weight word can be computed by a greedy algorithm with respect to the two classes of weightings discussed in the preceeding two sections. These results will enable us to compare our results to the results given in [3], [4]. In fact all these problems are connected to matroid theory. Therefore, a short introduction describing this connection will also be given.

A matroid on a finite set V is a pair $M = (V,F)$ in which F is a collection of subsets of V such that:

(A) $$D \in F,\quad D' \subset D \Rightarrow D' \in F;$$

(B) if $X \in F$, $Y \in F$ and $|X| = |Y| + 1$, then there is $x \in X - Y$ such that $Y \cup \{x\} \in F$.

Let $w: V \to R^+$ be a weight function extended on 2^V by

$$W(\phi) = 0 \quad \text{and} \quad W(D) = \sum_{x \in D} w(x), \quad \forall D \in 2^V. \tag{3.1}$$

Consider the combinatorial optimization problem $(\bar{P})$: find $\bar{D} \in F$ such that

$$W(\bar{D}) = \max \{W(D) : D \in F\}. \tag{3.2}$$

One of the basic results of the matroid theory is the following: $\bar{D} \in F$ can be found by a greedy algorithm if and only if $M = (V,F)$ is a matroid on V, (see [7], chap. 19). This is an algorithmic characterization of the matroids on F subjected to (A), by means of a combinatorial optimization problem. In [3], [4] B. Korte and L. Lovasz remarked that a greedy algorithm can operate on a more general structure than that defined by the axiom (A). They considered a collection of sequences of distinct elements of V, let us denote it by P, subjected to the axiom:

(A*) if D is a sequence of P, then all the initial subsequences of D belong also to P.

Such a pair $N = (V,P)$ is called a *simple hereditary language*([4]) and P is called a *simple language* ([5]). Obviously, (A) implies (A*), but in general (A*) does not imply (A). Any $D \in P$ is called a *word* of P.

A *greedoid* on a finite set V is a pair $N = (V,P)$ in which P is subjected to the axiom (A*) and to a second axiom

(B*) if $X \in P$, $Y \in P$ and $|X| = |Y| + 1$, then there is $x \in X - Y$ such that the sequence $[Y,x]$ belongs to P.

The structural properties of the greedoids and the relationships with the concept of selector introduced by H. Crapo have further been discussed in [5], [6]. An algorithmic characterization of the languages P for which a combinatorial optimization problem $(\bar{P})$ with $F = P$ can be solved by a greedy algorithm with respect to a class of weightings more general than that defined by (3.1) has also been searched. The main algorithmic results will be discussed at the end of this section, where their connections with our results will be shown, too.

Let $G = (V,A)$ be a directed graph without circuits and $V = V_0 \cup V_1 \cup \ldots \cup V_r$ be the partitioning used in the first two sections. We shall assume $V_0 = \{x_0\}$ and P will denote the set of all paths from the root x_0 to all vertices.

If a path $D \in P$ is considered as a sequence of its vertices, then obviously $N = (V,P)$ is a language associated to the rooted directed graph G without circuits.

Theorem 10. The language $N = (V,P)$ associated to a rooted graph $G = (V,A)$ without circuits is a greedoid on V, if and only if G is a greedy graph.

Proof. Consider $X = [x_0, x_{i_1}, \ldots, x_{i_q}, x_{i_{q+1}}]$, $Y = [x_0, x_{j_1}, \ldots, x_{j_q}]$, $0 \le q < r$, i.e. $X \in P$, $Y \in P$ with $|X| = |Y| + 1$. If G is a greedy graph, then we have

$$x_{i_q} \in V_q, \quad x_{j_q} \in V_q, \quad x_{i_{q+1}} \in V_{q+1} = A(x_{j_q}). \tag{3.4}$$

So, $(x_{j_q}, x_{i_{q+1}}) \in A$ shows that $[Y,x] \in P$, where $x_{i_{q+1}} \in X - Y$, hence (B*) is satisfied and $N = (V,P)$ is a greedoid on V.

Suppose that $N = (V,P)$ is a greedoid on V, but G is not a greedy graph, i.e. there are $q \in \{1,\dots,r\}$ and $\bar{x} \in V_{q-1}$ such that $A(\bar{x}) \neq V_q$. According to (C_1) we have $A(\bar{x}) \subseteq V_q \cup \dots \cup V_r$. First, we shall show that $A(\bar{x}) \supseteq V_q$, secondly we shall prove that $A(\bar{x}) \supset V_q$ is impossible and in this way we shall get a contradiction showing that G must be a greedy graph.

If $A(\bar{x}) \not\supseteq V_q$, then there is $y \in V_q$, $y \notin A(\bar{x})$. Consider $X \in P$, $Y \in P$ where X is a path with $q+1$ vertices from x_0 to y and Y is a path with q vertices from x_0 to $\bar{x}$; according to (C_2) such paths do exist and we have $|X| = |Y| + 1$. As N is a greedoid, there is $x \in X - Y$ such that $[Y,x] \in P$. We can not have $x \neq y$, because this would contradict (C_4); on the other hand, $x = y$ would contradict $y \notin A(\bar{x})$. So, the hypothesis has been false, hence $A(\bar{x}) \supseteq V_q$.

If $A(\bar{x}) \supset V_q$, there is $y \in V_k$, $q+1 \leq k \leq r$, $y \in A(\bar{x})$. According to (C_2), (C_3), there is a path $D = [x_0, x_{j_1}, \dots, x_{j_{k-1}}, x_{j_k}]$ with $x_{j_k} = y$. Consider $x_{j_h} \in V_h$, $h < k$, the last vertex of D except x_{j_k} such that $x_{j_h} \in A(\bar{x})$ and remark that $h \geq q$, because $x_{j_q} \in V_q$ and $V_q \subseteq A(\bar{x})$ according to the first part of the proof. Consider $X = [x_0, x_{i_1}, \dots, x_{i_{q-1}}, x_{j_h}, x_{j_{h+1}}]$, where $x_{i_{q-1}} = \bar{x}$ and $[x_0, x_{i_1}, \dots, x_{i_{q-1}}]$ is a path with q vertices from x_0 to $\bar{x} \in V_{q-1}$. Obviously, if $h = k-1$ we have $x_{h+1} = y$, otherwise $x_{h+1} \neq y$. Consider $Y = [x_0, x_{i_1}, \dots, x_{i_{q-1}}, y]$ with $x_{i_{q-1}} = \bar{x}$ and remark that $|X| = q+2$, $|Y| = q+1$, hence $|X| = |Y| + 1$. As N is a greedoid, there is $x \in X - Y$ such that $[Y,x] \in P$. However, this is impossible because the piece of X from x to $x_{j_{h+1}}$, the piece of D from $x_{j_{h+1}}$ to $x_{j_k} = y$, if $h < k-1$, and (y,x) would form a circuit of G. So, the contradiction shows that the hypothesis has been false, hence $A(\bar{x}) = V_q$ and G must be a greedy graph.

Theorem 10 holds if the word "rooted" is deleted from the statement; this can be shown by the same procedure which has already been used in the preceeding sections.

Note that according to Theorem 10 the languages associated to the greedy graphs are examples of greedoids.

The algorithmic characterization for both classes of weightings follows from Theorem 9 and Theorem 10 and can be given in the common statement:

Theorem 11. Let $N = (V,P)$ be a language associated to the directed graph $G = (V,A)$ without circuits. A greedy algorithm solves the combinatorial problem

$$\text{(S)} \qquad \max\{W(D) : D \in P\}$$

for all non negative non decreasing Korte-Lovasz (or additive) weightings if and only if N is a greedoid on V.

Theorem 11 enables us to compare our results with the main algorithmic results given in [3], [4]:

(a) Let $N = (V,P)$ be any language on V. If N is a greedoid, then a greedy algorithm solves (S) for all non negative non decreasing Korte-Lovasz weightings (see Theorem 4.1, [4]);

(b) Let $N = (V,P)$ be a language on V, whose maximal elements of P have the same cardinality. If a greedy algorithm solves (S) for all generalized bottleneck functions W, then N is a greedoid on V (see Theorem 4.5, [4]).

Note that a Korte-Lovasz weighting W has been defined in [4] by two axioms, namely (KL_1) stated in the preceeding section and a second axiom which is automatically satisfied if W is defined on the language associated to a directed graph without circuits.

Note also that any generalized bottleneck function W is a Korte-Lovasz weighting, as it has been shown in [4].

The last two remarks show that:

1. The "if" part of our Theorem 11 is a particular case of Theorem 4.1, namely we suppose that N is the particular language associated to a directed graph without circuits.

2. In order to prove a converse result to Theorem 4.1 we bounded ourselves to a class of particular languages. In fact, this has also been done in Theorem 4.5 by considering only the languages whose maximal elements have the same cardinality. Note that the maximal elements of P in a language associated to a directed graph without circuits can have different cardinalities. On the other hand, the assumption concerning the objective function of (S) in Theorem 4.5 is weaker than our assumption, because the generalized bottleneck functions are Korte-Lovasz weightings. So, the "only if" part of our Theorem 11 is an alternative result to Theorem 4.5; in order to prove a converse result to Theorem 4.1,

we imposed some hypotheses only to the considered languages. Of course, an open question is whether such a converse result can be proved for a more general class of languages or not.

Finally, let us remark that the languages associated to a directed graph without circuits, despite of their particular properties, represent an important class of languages. This matter and several applications of the above results will be discussed in a further paper; various details are given in [2]. The same results enable us to state a greedy subroutine of an algorithm for finding the generalized nucleolus of a finite set and to use it for solving some multiobjective discrete programming problems: the main algorithm is given in [1] and the greedy subroutine will also be discussed in a further paper.

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