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To the memory of
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IMPROPER IMMERSIONS IN A LORENTZIAN MANIFOLD ADMITTING A SASAKIAN STRUCTURE

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INTRODUCTION

Odd dimensional Lorentzian manifolds which admit a Sasakian structure have been defined in [1].

Let $\tilde{M}(\phi,\tilde{\eta},\xi,\tilde{g})$ be such a manifold, whose dimension is $2m + 1$. The structure tensors $(\phi,\tilde{\eta},\xi,\tilde{g})$ are: the Sasakian tensor field ϕ $(\phi^2 = -1)$, the canonical contact 1 - form $\tilde{\eta}$, the canonical vector field ξ, and the metric tensor $\tilde{g}$ of the manifold $\tilde{M}$.

Let then $x:M_c \to \tilde{M}$ be the improper immersion of a co-isotropic hypersurface [2], [3] in M, and U_c the *characteristic* vector field [2] of M_c. If $\tilde{\nabla}$ is the covariant differentiation operator (with respect to the Levi-Civita connection in M), we say that U_c is *null ϕ-concircular* if $\tilde{\nabla}U_c$ is coplaner with U_c and ϕU_c, tensored with two suitable 1-forms.

In this case, the following properties are proved:

(i) M_c is *almost minimal* (in the sense defined by the author [4]), and is a *generic* submanifold [5] of $\tilde{M}$;

(ii) M_c is foliated by hypersurfaces that are orthogonal to ϕU_c;

(iii) ϕU_c is a geodesic direction and defines an *infintesimal homothety* on M_c;

(iv) the Ricci curvature of U_c and ϕU_c are expressed by $2(m-1)$ and $-2(m-1)$, respectively.

Next, one considers the improper immersion of the *mixed-isotropic* 2-codimensional submanifold M_u of $\tilde{M}$.

The following properties can be stated:

(i) any normal section on M_u is quasi-umbilical;

(ii) the restrictions $U = U_c/M_u$, and $\eta = \tilde{\eta}/M_u$ define on M_u a *null-contact structure*, and U is an infinitesimal automorphism for all globally defined (2p+1)-forms $\Theta_p = 4L^p\eta$ $(L^p\eta = \eta \wedge (\frac{d\eta}{2})^p)$;

(iii) M_u is a proper [6] CR submanifold of M, which is *mixed totally geodesic*, and if D denotes the horizontal distribution, then U is D-*transformed by parallelism*.

1. PRELIMINARIES

Let $(\tilde{M},\tilde{g})$ be a $(2m+1)$-dimensional Lorentzian manifold, whose metric tensor $\tilde{g}$ is of normal hyperbolic type (i.e., $\tilde{g}$ is of index 1, [2]).

Suppose $\tilde{M}$ is endowed with a constant structure defined by a canonical 1-form $\tilde{\eta}$, and a canonical vector field ξ. Furthermore, if $\tilde{M}$ is equipped with a tensor field ϕ $(\phi^2 = -1)$ of type (1,1), and $\tilde{\nabla}$ is the covariant differentiation operator, then if for any vector fields $\tilde{Z}$, $\tilde{Z}'$ of $\tilde{M}$ one has

$$(1.1)\qquad \begin{cases} \phi^2\tilde{Z} = -\tilde{Z} + \tilde{\eta}(\tilde{Z})\xi, \quad \phi\xi = 0, \quad \tilde{\eta}(\phi\tilde{Z}) = 0, \\ d\tilde{\eta}(\tilde{Z},\tilde{Z}') = 2\tilde{g}(\phi\tilde{Z},\tilde{Z}'), \quad \tilde{\nabla}_{\tilde{Z}}\xi = -\phi\tilde{Z}, \end{cases}$$

we have called [1] $\tilde{M}(\phi,\tilde{\eta},\xi,\tilde{g})$ a Lorentzian manifold admitting a Sasakian structure.

Let $\mathcal{O}(\tilde{M})$ be the bundle of orthonormal frames on $\tilde{M}$, and $\mathcal{O} = \{e_A;\ A = 1,2,\ldots,2m+1\}$ an element of $\mathcal{O}(\tilde{M})$. The vector basis of $\mathcal{O}$ is defined by a space $\{e_\alpha;\ \alpha = 1,2,\ldots,2m\}$ and a time-like vector field $e_{2m+1} = \xi$, which satisfy

$$(1.2)\qquad \tilde{g}(e_A,e_B) = \varepsilon_A\delta_{AB};\quad \varepsilon_\alpha = -1,\ \varepsilon_{2m+1} = 1.$$

Let $\{\tilde{\omega}^A\}$ be the dual basis of $\mathcal{O}$, and $d\tilde{p}$ and $\tilde{\omega}^A_B = \gamma^A_{BC}\tilde{\omega}^C$ the *line element* (i.e., the canonical vector 1-form on $\tilde{M}$) and the *connection forms* corresponding to $\{\tilde{\omega}^A\}$ respectively.

Setting $a = 1,2,\ldots,m$, $a^* = a + m$, and $\alpha,\beta,\gamma = 1,2,\ldots,2m$, the line element $d\tilde{p}$ of $\tilde{M}(\phi,\tilde{\eta},\xi,\tilde{g})$ is

$$(1.3)\qquad d\tilde{p} = -\tilde{\omega}^\alpha \otimes e_\alpha + \tilde{\eta} \otimes \xi,$$

and by (1.1) the connection equations are

$$(1.4)\quad \begin{cases} \tilde{\nabla} e = \tilde{\omega}_a^{\beta} \otimes e_{\beta} - \tilde{\omega}^{a*} \otimes \xi, \\ \tilde{\nabla} e_{a*} = \tilde{\omega}_{a*}^{\beta} \otimes e_{\beta} + \tilde{\omega}^{a} \otimes \xi, \\ \tilde{\nabla}\xi = \tilde{\omega}^{a} \otimes e_{a*} - \tilde{\omega}^{a*} \otimes e_a = -\phi d\tilde{p}. \end{cases}$$

Since $e_{a*} = \phi e_a$, one obtains from (1.4)

$$(1.5)\quad \tilde{\omega}_b^{a} = \tilde{\omega}_{b*}^{a*}, \quad \tilde{\omega}_b^{a*} = \tilde{\omega}_a^{b*}.$$

Next, the structure equations (E. Cartan) are

$$(1.6)\quad \begin{cases} d\tilde{\omega}^{a} = \tilde{\omega}^{\beta} \wedge \tilde{\omega}_{\beta}^{a} + \tilde{\eta} \wedge \tilde{\omega}^{a*}, \\ d\tilde{\omega}^{a*} = \tilde{\omega}^{\beta} \wedge \tilde{\omega}_{\beta}^{a*} - \tilde{\eta} \wedge \tilde{\omega}^{a}, \\ d\tilde{\eta} = 2 \sum \tilde{\omega}^{a} \wedge \tilde{\omega}^{a*}, \end{cases}$$

and

$$(1.7)\quad \begin{cases} d\tilde{\omega}_{\alpha}^{\beta} = \tilde{\Omega}_{\alpha}^{\beta} + \tilde{\omega}_{\alpha}^{\gamma} \wedge \tilde{\omega}_{\gamma}^{\beta} - \tilde{\omega}_{\alpha}^{2m+1} \wedge \tilde{\omega}_{2m+1}^{\beta}, \\ d\tilde{\omega}_{\alpha}^{2m+1} = \tilde{\Omega}_{\alpha}^{2m+1} + \tilde{\omega}_{\alpha}^{\beta} \wedge \tilde{\omega}_{\beta}^{2m+1}, \end{cases}$$

where $\tilde{\Omega}_B^A$ are the curvature forms, and

$$(1.8)\quad \tilde{\omega}_a^{2m+1} = -\tilde{\omega}^{a*}, \quad \tilde{\omega}_{a*}^{2m+1} = \tilde{\omega}^{a}.$$

2. CO-ISOTROPIC HYPERSUFRACE OF $\tilde{M}(\phi,\tilde{\eta},\xi,\tilde{g})$

Let $x:M_c \to \tilde{M}(\phi,\tilde{\eta},\xi,\tilde{g})$ be the improper immersion of a co-isotropic [2] hypersurface M_c in $\tilde{M}$, and let U_c be the *characteristic* vector field (2) of M_c (U_c is simultaneously normal and tangent to M_c). Referring to [6], we say that U_c is *null ϕ-concircular,* if it satisfies

$$(2.1)\quad \tilde{\nabla} U_c = w \otimes U_c + w' \otimes \phi U_c; \quad w, w' \in \Lambda^1(\tilde{M}).$$

Without loss of generality, one may suppose that M_c is defined by

$$\omega^{2m} = \eta, \tag{2.3}$$

(The elements induced by x are denoted by letters after suppressing ~). Then by (1.3) and (1.2) one obtains

$$U_c = \xi - e_{2m} \Rightarrow g(U_c, U_c) = 0,$$

while (1.1) leads to

$$\phi U_c = e_m. \tag{2.4}$$

If j is the canonical isomorphism defined by the metric g of M_c, we set

$$j(\phi U_c) = u \in \Lambda^1(M_c). \tag{2.5}$$

So, by (1.3), (2.2), (2.4) and (2.5), the line element dp of M_c is expressed by

$$dp = -\omega^i \otimes e_i - \omega^{i*} \otimes e_{i*} - u \otimes \phi U_c + \eta \otimes U_c, \tag{2.6}$$

where we have set $i = 1,2,\ldots m-1$; $i^* = i + m$. Taking account of (1.2) one derives from (2.6)

$$g = \langle dp, dp \rangle = -\sum(\omega^i)^2 - \sum(\omega^{i*})^2 - (u)^2,$$

which shows as known [2] that the metric g is space-like and is of *defect* 1 [2].

If $T_p(M_c)$ and $T_1(M_c)^\perp$ are the tangent space and the normal space of at $p \in M_c$, one easily checks by (2.6) that $U_c \in T_p(M_c) \cap T_p(M_c)^\perp$, that is, U_c is the *characteristic vector field* of M_c.

Since $\{U_c\} = T_p(M_c)^\perp$, and $\phi U_c \in T_p(M_c)^+$, reference to the definition given in [8] proves that any co-isotropic hypersurface M_c of $\tilde{M}(\phi,\tilde{\eta},\xi,\tilde{g})$ is a *generic submanifold* (it is known that this property is valid for all proper hypersurfaces of a Sasakian manifold). Since, in the case under discussion, U_c satisfies (2.1), we derive by taking exterior differential of (2.2) and referring to (1.5):

(2.8) $$\omega_i^{2m} = \omega^{i*}, \quad \omega_{i^*}^{2m} = -\omega^i, \quad \omega_m^{2m} + \eta = 0.$$

Therefore, by (2.2), (2.4), (1.4), and (2.7) we get:

(2.9) $$\nabla\xi = \omega^i \otimes e_{i*} - \omega^{i*} \otimes e_i + u \otimes \phi^2 U_c - \eta \otimes \phi U_c,$$

and

(2.10) $$\tilde{\nabla} U_c = -u \otimes U_c - 2\eta \otimes \phi U_c,$$

(that is, $\omega = u$, $\omega' = 2\eta$).

Next, with the help of (1.4) and (1.5) we obtain from (2.4)

(2.11) $$\nabla \phi U_c = -dp - u \otimes U_c - 2\eta \otimes \phi^2 U_c,$$

and we quickly find

(2.12) $$\nabla_{\phi U_c} \phi U_c = 0$$

Hence, ϕU_c is a *geodesic direction* and one easily checks by means of (2.7)

(2.13) $$du = 0.$$

So, the above equation and (2.5) prove that M_e is foliated by hypersurfaces M_u, orthogonal to ϕU_c. Further, denote by σ the canonical volume element of M_c. Then one has

(2.14) $$\sigma = \eta \wedge \sigma_u.$$

where

(2.15) $$\sigma_u = \omega^1 \wedge \dots \wedge \omega^{m-1} \wedge \omega^{1*} \wedge \dots \wedge \omega^{m*-1} \wedge u.$$

If $\mathcal{L} = d \circ i + i \circ d$ means the $\mathcal{L}$ie derivative, one has as known $\mathcal{L}_{U_c}\sigma = (\mathrm{div}\, U_c)\sigma$. Referring to (2.14), (2.15), (2.3) and (2.13), one finds after a direct computation

(2.16) $$\mathcal{L}_{U_c}\sigma \equiv d\sigma_u = 0 \Rightarrow \mathrm{div}\, U_c = 0.$$

But U_c being the characteristic vector of M_c, it follows from (2.16), and from the definition given in [4], that M_c is an *almost minimal hypersurface*. This property is in accordance with concept of *$\tilde{M}$-vector field* [7] of a proper hypersurface M of $\tilde{M}$.

Next, by (2.14), (2.1) and (1.5) one quickly gets

$$\mathcal{L}_{\phi U_c}\sigma = 2(m-1)\sigma \Rightarrow \operatorname{div}\phi U_c = 2(m-1), \tag{2.17}$$

that is ϕU_c defines an *infinitesimal homothety* on M_c. Now being given s Riemannian or Pseudo-Reimannian manifold (M,g) and an orthonormal frame $\{e_A\}$ of M, one has the integral formula (K. Yano [8]).

$$\begin{aligned} \operatorname{div}(\nabla_Z Z) - \operatorname{div}(\operatorname{div} Z)Z + (\operatorname{div} Z)^2 \\ = \operatorname{Ric}(Z) + \textstyle\sum_{A,B} g(\nabla_{e_A} Z, e_B)g(e_A, \nabla_{e_B} Z). \end{aligned} \tag{2.18}$$

In (2.18), Ric means the Ricci tensor field and Z is any vector field of M_c. But in the case under discussion one gets from (2.10)

$$\nabla_{U_c} U_c = -2\phi U_c \tag{2.19}$$

and by (2.12), (2.16), (2.17) and in view of (2.18) we obtain:

$$\begin{cases} \operatorname{Ric}(U_c) = 2(m-1), \\ \operatorname{Ric}(\phi U_c) = -2(m-1). \end{cases} \tag{2.20}$$

Finally, since U_c is also normal to M_c, the second quadratic fundamental form of M_c is [2] $II = -\langle dp, \tilde{\nabla} U_c\rangle$ One readily finds by (2.1) and (2.5) $II = 2\eta \otimes u$.

Therefore, one has $II(U_c, U_c) = 0$ and $II(\phi U_c, \phi U_c) = 0$, and this proves that U_c and ϕU_c are both *asymptotic* directions of M_c.

Theorem. Let $x: M_c \to \tilde{M}(\phi,\tilde{\eta},\xi,\tilde{g})$ be the improper immersion of a co-isotropic hypersurface M_c in a $(2m+1)$-dimensional manifold $\tilde{M}$, admitting a Sasakian structure $(\phi,\tilde{\eta},\xi,\tilde{g})$. Then M_c is always a geodesic submanifold of $\tilde{M}$. If the characteristic vector U_c of M_c is null ϕ-concircular, then one has the following properties:

(i) M_c is almost minimal, and is foliated by hypersurfaces M_u orthogonal to ϕU_c;

(ii) ϕU_c is a geodesic direction and defines an infinitesimal homothety on M_c;

(iii) the Ricci curvatures of U_c and ϕU_c are expressed by $2(m-1)$ and $-2(m-1)$ respectively;

(iv) both U_c and ϕU_c are asymptotic directions on M_c.

3. MIXED ISOTROPIC CR SUBMANIFOLDS OF M

It has been shown in section 2 that M_c is foliated by hypersurfaces M_u orthogonal to ϕU_c. Consider now the improper immersion $x_u: M_u \to \tilde{M}(\phi,\tilde{\eta},\xi,\tilde{g})$, where M_u is a codimension 2-submanifold of $\tilde{M}$, defined by

$$u = 0, \quad \omega^{2m} = \eta. \tag{3.1}$$

Referring to (2.5), the line element dp of M_u is expressed by

$$dp = -\omega^i \otimes e_i - \omega^{i*} \otimes e_{i*} + \eta \otimes U \tag{3.2}$$

(We denote the elements induced by x_u, by the same letters as the elements induced by x.) Since $< dp,U > = 0$ $(U = U_c/M_u)$, it folows by (3.2) that the tangent space $T_p(M_u)$ and the normal space $T_p(M_u)^\perp$, at each point $p \in M_u$, are defined by $\{e_i, e_{i*}, U\}$ and $\{\phi U, U\}$ respectively.

Obviously, one has $T_p(M_u) \cap T_p(M_u)^\perp \neq 0$. But $T_p(M_u)$ (resp. $T_p(M_u)^\perp$) is not isotropic (resp. co-isotropic), nor *self-orthogonal* [9], that is, $T_p(M_u) \neq T_p(M)^\perp$.

Therefore, we agree to call M_u a *mixed isotropic* submanifold. We shall also call U and ϕU the *characteristic normal* section and the *proper normal* section, respectively. Next, denote $\ell_U = - < dp,\tilde{\nabla}U >$, and $\ell_{\phi U} = < dp,\tilde{\nabla}\phi U >$

the second fundamental quadratic forms associated with the improper immersion x_u. By (3.2), (2.9) and (2.10) one readily finds

$$\begin{cases} \ell_U = 0 \\ \ell_{\phi U} = \langle dp, dp \rangle = g. \end{cases} \tag{3.3}$$

From the above equations it follows that U and ϕU are a *geodesic section,* and an *umbilical* section of M_u, respectively.

Since $\langle U,U \rangle = 0$, any normal section N is expressed by $N = \lambda U + \varepsilon\phi U$, where $\lambda \in C^\infty(M)$, and $\varepsilon = \pm 1$. Without loss of generality, one may take $\varepsilon = +1$. Then, by (2.9), (2.10) and (3.1), one gets:

$$\tilde{\nabla} N = dp - 2\eta \otimes \phi^2 U + (d\lambda + 2\lambda^2\eta) \otimes U - 2\lambda\eta \otimes N, \tag{3.4}$$

and since $\langle U, \phi^2 U \rangle = 1$, one obtains

$$\ell_N = - \langle dp, \tilde{\nabla} N \rangle = g + 2\eta \otimes \eta. \tag{3.5}$$

According to the known definition, it follows that any normal section N of M_u is *quasi-umbilical*. By (2.4), we agree to call the curvature 2-forms Ω_i^m, Ω_{i*}^m $(i = 1,2,\ldots,m-1;\ i^* = i + m)$ the proper *mixed curvature* 2-forms (abr. p.m.c.) of M_u. Making use of (1.7), one finds by (2.1), (2.6) and (3.1):

$$\begin{cases} \Omega_i^m = 2\eta \wedge \omega^{i*}, \\ \Omega_{i*}^m = 2\eta \wedge \omega^i. \end{cases} \tag{3.6}$$

Hence, all (p. m. c.) - 2-forms are conformed to the restriction $\eta = \tilde{\eta}/M_u$ of the contact form $\tilde{\eta}$, If we set

$$\Omega = \frac{1}{2}\, d\eta = \sum \omega^i \wedge \omega^{i*}, \tag{3.7}$$

then one quickly gets from (3.6)

$$\Theta_1 = \sum(\Omega_{i*}^m \wedge \omega^{i*} - \Omega_i^m \wedge \omega^i) = 4\eta \wedge \Omega.$$

If L is the operator of type (1.1), defined by $L^1_\alpha = \alpha \wedge \Omega$ for any p-form α, and $L^p_\alpha = \alpha \wedge \Omega^p$, then one may write $\Theta_1 = 4L^1\eta$. Since $i_U\Theta_1 = 4\Omega$, we easily deduce by (3.7)

$$\mathcal{L}_U \Theta_1 = 0, \tag{3.9}$$

and in the same manner, $\mathcal{L}_U \Theta_p = 0$, where $\Theta_p = 4L^p\eta$. Hence, the characteristic vector field U of M_u is an *infinitesimal automorphism* of all forms θ_p. We shall make now the following considerations.

Equations (2.3) and (3.7) allow to write

$$(d\eta)^{m-1} \wedge \eta \neq 0, \quad \eta(U) = 1, \tag{3.10}$$

and since M_u is of odd dimension, the above equations define a *contact structure* $s_U = (\eta,U)$ whose structure vector field U is null. We agree to call s_U a *null contact structure*. It is worthwhile to remark that we have already defined in [3] a contact structure of the same type. Consider now the following two complementary distributions $D_p = \{e_i, e_{i^*}\} \subset T_p(M_u)$ and $D^\perp_p = \{U\} \subset T_p(M_u)$. Clearly, by (1.1) and (2.3), one has $\phi D_p \subset T_p(M_u)$ and $\phi D^\perp_p \subset T_p(M_u)^\perp$. Hence, according to the definitions given in [8], [10], we see that M_u is a CR-submanifold of $\tilde{M}$, and D and $D^\perp$ are the *horizontal* and *vertical* distributions of M_u, respectively. For any vector field X, tangent to M_u, one sets [8]

$$\phi x = Px + Fx, \tag{3.11}$$

where Px and Fx are the tangential and normal components of ϕx, respectively (P is an endomorphism of TM_u and F is a normal-bundle-valued 1-force in TM_u [8]).

If τ is the component of U in the expression of X, one easily finds $PX \in D_p \subset T_p(M_u)$, and $FX = \tau\phi U \in T_p(M_u)^\perp$.

Since $Px \neq 0$, $Fx \neq 0$, then according to [11] one may say that M_u is a *proper* CR submanifold. If $\nabla^\perp$ is the connection in the normal bundle $T^\perp M_U$ then for any vector field N of $T^\perp M_u$ one has the Weingarten formula

(3.12) $$\tilde{\nabla}_X N = -A_N X + \overset{\perp}{\nabla}_X N,$$

where X is any tangent vector field of M_u.

In the case under discussion, one may set $N = aU + b\phi U$ $(a,b \in C^\infty(M_u))$. Then, if X is any tangent vector of D, one finds by (2.9), (2.10) and (3.1) $A_N X = bX \in D$, and this condition proves that M_u is *mixed totally geodesic* [11].

Finally, one quickly gets from (2.9) and (3.1), $\nabla_X U = 0$, $\forall X \in D$. Hence, one may say that U is D-transformed by *parallelism*.

Theorem. The co-isotropic hypersurface M_c of $\tilde{M}(\phi,\tilde{\eta},\xi,\tilde{g})$, discussed in section 2, is foliated by 2-codimensional mixed isotropic submanifolds M_u of $\tilde{M}$. Let $T_p(M_u)$ and $T_p^\perp(M_u)$ be the tangent and the normal space to M_u, at $p \in M_u$. Then any normal section of $T^\perp(M_u)$ is quasi-umbilical, and all proper mixed curvature 2-forms of M_u are conformal to the restriction $\eta = \tilde{\eta}/M_u$. The characteristic vector fields U and η endows M_u with a null contact structure, and U is an infinitesimal automorphism for all globally defined $(2p+1)$-forms $\Theta_p = 4L^p\eta$.

Furthermore, M_u is a proper CR submanifold of $\tilde{M}$ whose vertical distribution $D^\perp$ is defined by $\{U\}$.

In addition, M_u is mixed totally geodesic, and if D denotes the horizontal distribution, then U is D-transformed by parallelism.

REFERENCES

[1] Roşca, R., "Variétés Lorentziennes à structure Sasakienne at admettant un champ vectoriel ϕ-quasi-concirculaire," C. R. Acad. Sci. Paris, t. 291, Série A, 45-47 (1980).

[2] Roşca, R., "Sur les hypersurfaces isotopes de défaut 1 incluses dans une espace hyperbolique de type normal," Ann. Inst. H. Poincaré, Vol. XX, no. 3, 237-243 (1974).

[3] Roşca, R., CR hypersurfaces à champ normal covariant décomposable incluses dans une variété pseudo-riemannienne neutre," C. R. Acad. Sci. Paris, t. 292, Série I, 287-289 (1981).

[4] Rosca, R., "On null hypersurfaces of a Lorentzian manifold," Tensor, N. S., t. 23 (1972).

[5] Yano, K., Kon, M., "Generic submanifolds of Sasakian manifolds," Kodai Math. J., Vol. 3, 169-196 (1980).

[6] Rosca, R., Verstraken, L., "On submanifolds admitting a normal section which is quasi-concircular with respect to a corresponding principal tangent section," Bull. Math. Soc. Math. Roumaine, t. 20 (68), 399-409 (1976).

[7] Hasanis, T., "Minimal hypersurfaces of a manifold," Tensor, N. S., t. 35, 262-266 (1981).

[8] Yano, K., Kon, M., "Differential geometry of CR submanifolds," Geometric Dedicata, Vol. 60, 369-391 (1981).

[9] Souriau, J. M., *Structure des Systèmes Dynamiques*, Dunod, Paris, 1970.

[10] Bejancu, D., "CR-submanifolds of a Kaehler manifold, I" Proc. Amer. Math. Soc., v. 69, 135-142 (1978).

[11] Kobayashi, M., "CR-submanifolds of a Sasakian manifold," Tensor, N. S., t. 35, 297-307 (1981).