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To the Memory of
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DILATION THEORY APPLIED TO LINEAR TRANSPORT

V. Protopopescu

ABSTRACT

The dilation of contractions to isometries is applied to a specific problem in linear transport theory, in order to prove the reality of the rightmost part of the spectrum which dominates the long-time behavior of the solution.

We present in this note a new proof of some known results in linear transport theory, based on the dilation theory of contractions, as constructed by Sz.-Nagy and Foiaş [1].

Let us consider the following initial-boundary value problems arising from neutron transport:

$$\frac{\partial f}{\partial t}(x,\mu,t) = -\mu \frac{\partial f}{\partial t}(x,\mu,t) - f(x,\mu,t) + \tfrac{1}{2}\int_{-1}^{1} f(x,\mu',t)\,d\mu' \tag{1}$$

$$\lim_{t\to 0} f(x,\mu,t) = f_0(x,\mu) \tag{2}$$

$$f(-a,\mu,t) = \alpha(\mu)f(a,\mu,t) \tag{3}$$

$$f(0,-\mu,t) = \alpha(\mu)f(0,\mu,t); \quad \mu > 0 \tag{3'}$$

$$\alpha(\mu) = \begin{cases} \alpha & \mu > 0 \\ \alpha^{-1} & \mu < 0 \end{cases} \qquad 0 < \alpha \leq 1 \tag{4}$$

considered as evolution problems in the Hilbert spaces $H = L_2([-a,a]\times[-1,1];dxd\mu)$ and $H' = L_2((-\infty,0]\times[-1,1];dxd\mu)$ respectively. The monoenergetic linear Boltzmann equations (1) describes the evolution of the one-particle distribution function f depending on position x, velocity μ, and time t, with initial condition $f_0 \in H$ (or H') within the infinite slab $[-a,a]$ (or semiinfinite medium $(-\infty,0]$) with boundary conditions (B.C.)(3)(or(3')). The (B.C.)(3) describes a partial absorbtion-partial retransmission of the distribution at the walls $x = -a$, $x = a$ [2], whereas (3') accounts for a partial absorbtion-partial specular reflection at the boundary $x = 0$ [3]. These problems have been studied in Refs. [2-3] where a complete description of the spectra of the transport operators has been given, providing -inter alia- only real nonvoid spectrum located on $(-1,0]$. The physical importance of this set, situated at the rightmost part of the spectrum, comes from the fact that in principle (and, for our problems, also in fact) it controls the asymptotic behavior in time of the solution of the corresponding initial value problem.

The operator -1 appearing in (1) is completely unessential for the subsequent discussion, as its action is only to translate the spectrum to the left with -1.

Let us then rewrite (1) in the form

$$\frac{\partial f}{\partial t} = Af \equiv (T + J)f, \tag{5}$$

where T is the closed operator generated by $-\mu\frac{\partial}{\partial x}$ under B. C. (3) (or (3')), and J is the one-dimensional projector from H (or H') to the space $L_2([-a,a];dx)$ (or $L_2((-\infty,0];dx))$ of the functions independent on μ.

For being specific, we shall consider now the problem (1)-(3). Then T is a maximal dissipative operator which generates the semigroup of positive contractions $V(t)$ [4]:

$$(V(t)f)(x,\mu) = \chi_{[-a,a]}(x) \sum_{m=-\infty}^{\infty} \alpha(\mu)^m f(x+2ma-\mu t,\mu) \tag{6}$$

where $\chi_{[-a,a]}(x)$ is the characteristic function of the segment $[-a,a]$.

We want to lift these contraction to a continuous semigroup of translations, $\underline{V}(t)$, in a larger Hilbert space $\underline{H}$, with the required properties [1]:

$$H \subset \underline{H}, \quad V(t)f = P_H \underline{V}(t)f \quad \forall f \in H \tag{7}$$

where P_H is the orthogonal projection from $\underline{H}$ to H. Then, $\underline{V}(t)$ will be used for proving - very simply - the reality of the rightmost spectrum.

Following ad literam the procedure from Chapter I, 10 in [1], let us define in H the nonnegative bilinear functional

$$[f,g] = -(f,Tg) - (Tf,g) \qquad f,\ g \in D(T) \tag{8}$$

$$[f,f] = \int_{-1}^{1} d\sigma(\mu)\,|f(-a,\mu)|^2 \geq 0 \qquad f \in D(T) \tag{9}$$

as $d\sigma(\mu) = \mu(\alpha(\mu)^{-2}-1)d\mu$.

The very definition of T entails that for $f \in D(T)$ and $t \geq 0$

$$\frac{d}{dt}\|V(t)f\|^2 = 2\mathrm{Re}(V(t)Tf,\ V(t)f) = -[V(t)f,\ V(t)f] \tag{10}$$

and, consequently

$$\|f\|^2 = \int_0^t [V(s)f,\ V(s)f]ds + \|V(t)f\|^2 \tag{11}$$

wherefrom

$$\|f\|^2 = \int_0^\infty [V(s)f,\ V(s)f]ds + \lim_{t\to\infty}\|V(t)f\|^2. \tag{12}$$

The limit in (12) exists, as $\|V(t)f\|$ is a positive, non-increasing function of t; then $\operatorname{s-lim}_{t\to\infty} V(t)^* V(t)$ exists. Here,

$$(V(t)^* f)(x,\mu) = \chi_{[-a,a]}(x) \sum_{m=-\infty}^{\infty} \alpha(\mu)^m f(x+2ma+\mu t,\mu) \tag{13}$$

(for more details about the adjoint problem for the transport operator see [4]).

Let Q be the square-root of this limit, $Q \geq 0$. One constructs, then, in $\underline{K} \equiv \overline{QH}$, the continuous semigroup of isometries $\{\underline{W}(t)\}_{t \geq 0}$ which verifies

$$QV(t) = \underline{W}(t)Q \qquad t \geq 0. \tag{14}$$

Let $\underline{D}$ be the completion with respect to the metric (8-9) of the pre-Hilbert space $D(T)$, Modulo N_0 ($N_0 = \{f\,|\,[f,f] = 0\}$), and let $\underline{H} \equiv \int_\oplus \underline{D}_s ds$ be the Hilbert space of functions $\underline{f} = \underline{f}(s)$ $(-\infty < s < \infty)$ with values in $\underline{D}$, strongly measurable, and such that

$$(15) \qquad \|\underline{f}\|^2 = \int_{-\infty}^{\infty} ds \|\underline{f}(s)\|^2_{\underline{D}_s} = \int_{-\infty}^{\infty} d\sigma(\mu) |f(-a,\mu)|^2 .$$

We define the injection $j: H \hookrightarrow \underline{H} \oplus \underline{K}$ by identifying $f \in H$ with $f \oplus Qf \in \underline{H} \oplus \underline{K}$ through

$$(16) \qquad f(t,\mu) = (jf)(t,\mu) = \begin{cases} 0 & t < 0 \\ (V(t)f)(-a,\mu) & t \geq 0 \end{cases}$$

and define the new operators $\{\underline{V}(t)\}_{t \geq 0}$ in $\underline{H}$ by

$$(17) \qquad (\underline{V}(t)\underline{f})(s) = \underline{f}(s+t) \qquad -\infty < s < \infty \quad t \geq 0$$
$$\underline{f} \in \underline{H}$$

Then, $\{\underline{V}(t) \oplus \underline{W}(t)\}$ is a continuous semigroup of translations in $\underline{H} \oplus \underline{K}$ and represents an isometric dilation of $V(t)$ in H.

Indeed, for $f \in H$ and $t \geq 0$, we have

$$(18) \qquad (\underline{V}(t) - V(t))f = \underline{f}^{(t)} \oplus 0,$$

where

$$(19) \qquad \underline{f}(t) = \begin{cases} V(t+s)f & -t \leq s \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

is obviously orthogonal on H. This entails $V(t) = P_H\underline{V}(t)$, $t \geq 0$.

We have sketched the construction procedure in general, as it is given in [1], but for the problem (1)-(3) let us observe that the things become even simpler, as $\lim_{t\to\infty} V(t) = 0$ and consequently, Q, $\underline{K}$, and $\underline{W}$ are identically zero. This fact is due to the infinite number of crossing the boundaries, associated each time with a decrease by a factor α of the distribution. This will not be the case for the problem (1)-(3') where the (free) trajectory hits the boundary only once, hence $\lim_{t\to\infty} V(t) \neq 0$.

For finishing the proof, we verify that the injection j is isometric. Indeed, taking into account (15)-(16) and the fact that in (16) the f's have two by two disjoint supports, we get

$$\begin{aligned}
\|\underline{V}\underline{f}\|^2_{\underline{H}} &= (\underline{V}(0)\underline{f},\ \underline{V}(0)\underline{f})_{\underline{H}} = \int_0^\infty ds[V(s)f,\ V(s)f] \\
&= \int_0^\infty ds \int_{-1}^{1} d\sigma(\mu) \Big| \sum_{m=-\infty}^{\infty} \alpha^{m}(\mu) f((2m-1)a-\mu s,\mu)\Big|^2 \\
&= \int_0^1 \mu d\mu(\alpha^{-2}(\mu)-1) \sum_{m=1}^{\infty} \alpha(\mu)^{2m} \int_{-a}^{a} \frac{dx}{d\mu} |f(x,\mu)|^2 - \int_{-1}^{0} \mu d\mu(\alpha^{-2}(\mu)-1) \\
&\qquad \sum_{m=-\infty}^{0} \alpha^{2m}(\mu) \int_{-a}^{a} \frac{dx}{d\mu} |f(x,\mu)|^2 = \int_{-1}^{1}\int_{-a}^{a} |f(x,\mu)|^2 dx d\mu = \|f\|^2_H .
\end{aligned} \tag{20}$$

The projection from $\underline{H}$ to H is j^* defined by

$$(jf,g)_{\underline{H}} = (\underline{f},g)_{\underline{H}} = (f,g)_H = (f,j^*g)_H \tag{21}$$

This will give the action of j^* as

$$f = j^*\underline{f} = \begin{cases} (\alpha^{-2}(\mu)-1) \displaystyle\sum_{m=1}^{\infty} \alpha^m(\mu)\underline{f}\Big(\frac{(2m-1)a-x}{\mu},\mu\Big) & \mu > 0 \\ (1-\alpha^{-2}(\mu)) \displaystyle\sum_{m=-\infty}^{0} \alpha^m(\mu)\underline{f}\Big(\frac{(2m-1)a-x}{\mu},\mu\Big) & \mu < 0 \end{cases} \tag{22}$$

and one can easily verify that $j^*j = I_H$ and $jj^* = I_{\underline{H}}$.

Now, we can use the dilation (17) for getting the interesting information about the spectrum of $T + J \equiv A, \sigma(A)$. In doing so, we use the following result [6]:

Let S_λ be the integral operator

$$S_\lambda = J(\lambda-T)^{-1}J. \tag{23}$$

Then, whenever $\lambda \in \sigma(A) \cap \{\lambda | \mathrm{Re}\ \lambda > -1\}$, $1 \in \sigma(S_\lambda)$ and vice versa. Moreover, if the (generalized) eigenvector f solves the eigenvalue problem $(\lambda-A)f_\lambda = 0$, then $\emptyset_\lambda = Jf_\lambda$ solves the eigenvalue problem $(1-S_\lambda)\emptyset_\lambda = 0$ and vice versa (in this case, $f_\lambda = (\lambda-T)^{-1}\emptyset_\lambda$).

We are therefore interested in

$$J(\lambda-T)^{-1}Jf = J\int_0^\infty e^{-\lambda t}V(t)Jfdt = J\int_0^\infty e^{-\lambda t}j^*\underline{V}(t)jJfdt$$

or

$$Ji^*(\lambda-T)jJf = Jf, \tag{24}$$

where $\underline{T}$ is the infinitesimal generator of the translations (17). Now, applying the Fourier transform $\underline{F}$ in $\underline{H}$, leads to

$$\underline{F}jJj^*\underline{F}^{-1} \cdot \underline{F}(\lambda-\underline{T})\underline{F}^{-1} \cdot \underline{F} \cdot jJf = \underline{F} \cdot jJf \qquad f \in H \tag{25}$$

where $\underline{F}(\lambda-\underline{T})\underline{F}^{-1}$ has now the very simple form $(\lambda-ik)^{-1}$.

For f independent on $\mu(f = Jf)$, we have

$$(jf)(t,\mu) = f(t,\mu) = \begin{cases} 0 \\ \sum\limits_{m=-\infty}^{\infty} \alpha(\mu)^m f((2m-1)a-\mu t) & t \geq 0 \end{cases} \tag{26}$$

$$\underline{\tilde{f}}(k,\mu) = (\underline{F}jf)(k,\mu) = \frac{1}{\mu}\,\frac{e^{\frac{ika}{\mu}}}{\alpha(\mu) - e^{\frac{ika}{\mu}}}\,\tilde{f}(-\frac{k}{\mu}) \equiv g(k,\mu)\underline{\tilde{f}}(-\frac{k}{\mu}) \tag{27}$$

Then, using (22)

$$j^*(\underline{F}^{-1} \cdot \underline{F}(\lambda-\underline{T})^{-1}\underline{F}^{-1} \cdot Fjf)(x,\mu) = j^*(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iks}(\lambda-ik)^{-1}g(k,\mu)\tilde{f}(-\frac{k}{\mu})dk)$$

$$= \frac{1}{2\pi}\begin{cases} (\alpha^{-2}(\mu)-1)\sum\limits_{m=1}^{\infty}\alpha(\mu)^m\int_{\infty}^{\infty} e^{-\frac{ik}{\mu}[(2m-1)a-x]}(\lambda-ik)^{-1}g(k,\mu)\tilde{f}(-\frac{k}{\mu})dk & \mu > 0 \\ (1-\alpha^{-2}(\mu))\sum\limits_{m=-\infty}^{\infty}\alpha(\mu)^m\int_{\infty}^{\infty} e^{-\frac{ik}{\mu}[(2m-1)a-x]}(\lambda-ik)^{-1}g(k,\mu)\tilde{f}(-\frac{k}{\mu})dk & \mu < 0 \end{cases} \tag{28}$$

Applying J, we obtain

$$\frac{1}{2}\,\frac{1}{2\pi}\,\Big\{\int_0^1 d\mu'(\alpha^{-2}(\mu')-1)\sum_{m=1}^{\infty}\alpha^{m}(\mu')\int_{-\infty}^{\infty} e^{-\frac{ik'}{\mu'}[(2m-1)a-x]}(\lambda-ik')^{-1}g(k',\mu')\underline{\tilde{f}}(-\frac{k'}{\mu'})dk'$$

(29)

$$+\int_{-1}^{0} d\mu'(1-\alpha^{-2}(\mu'))\sum_{m=-\infty}\alpha^{m}(\mu')\int_{-\infty}^{\infty} e^{-\frac{ik'}{\mu'}[(2m-1)a-x]}(\lambda-ik')^{-1}g(k',\mu')\tilde{f}(-\frac{k'}{\mu'})dk'\Big\}$$

and finally applying $\underline{F}j$ which acts according to (27) we get, after a rather lengthy computation, the left hand side of (25). After all, Eq. (25) reads

$$(30)\qquad \frac{1}{2\pi}\int_{-1}^{1} d\sigma(\mu')\int_{-\infty}^{\infty} dk'(\lambda-ik')^{-1}\,\frac{\sin(\frac{k'}{\mu'}-\frac{k}{\mu})a}{\frac{k'}{\mu'}-\frac{k}{\mu}}\,|g(k',\mu')|^{2}\tilde{f}(-\frac{k'}{\mu'}) = \tilde{f}(-\frac{k}{\mu})$$

Now, introducing $\kappa \equiv \frac{k}{\mu}$, $\kappa' \equiv \frac{k'}{\mu'}$ and taking into account the definition of g (Eq. 27), we have

$$(31)\qquad |g(k',\mu')|^{2} = \frac{1}{\mu'^{2}}\;\frac{1}{1+\alpha^{2}(\mu')-2\alpha(\mu')\cos 2\kappa' a}$$

For $\alpha(\mu)$ given by (4), we can perform the integration upon μ' and get

$$(32)\qquad \tilde{f}(-\kappa) = \frac{1}{2\pi}\int_{\infty}^{\infty} d\kappa'\,\frac{\alpha^{2}-1}{1+\alpha^{2}-2\alpha\cos 2\kappa' a}\cdot\frac{2}{\kappa'}\arctan\frac{\kappa'}{\lambda}\cdot\frac{\sin(\kappa'-\kappa)a}{\kappa'-\kappa}\cdot\tilde{f}(-\kappa')$$

which yields the reality of the eigenvalues λ. Indeed, for $f \in JH$, $\frac{\sin(\kappa-\kappa')a}{\pi(\kappa-\kappa')}$ is a reproducible kernel and therefore

$$(33)\qquad 0 \leq (f,f) = \int_{-\infty}^{\infty}\frac{\alpha^{2}-1}{1+\alpha^{2}-2\cos 2\kappa' a}\;\frac{1}{\kappa'}\arctan\frac{\kappa'}{\lambda}|f(-\kappa')|^{2}d\kappa'$$

But $\frac{1}{\kappa'}\arctan\frac{\kappa'}{\lambda}$ is Herglotz as a function of the complex variable λ (i.e. $\operatorname{Im}\frac{1}{\kappa'}\arctan\frac{\kappa'}{\lambda} \gtrless 0$ for $\operatorname{Im}\lambda \lessgtr 0$), therefore (33) cannot be satisfied unless λ is real. From (33) one sees also that $||S_{\lambda}|| < \frac{1}{\lambda}$ which is less than 1 for $\lambda > 1$, implying (due to the relation between λ and the eigenvalue 1 of S_{λ}) that the eigenvalues of $T + J$ lie on the segment $(0,1]$ or, equivalently, the eigenvalues of the original operator $T - 1 + J$ lie on $(-1,0]$).

This finishes the proof for the problem (1)-(3).

The semiinfinite medium problem (1)-(3') can be treated similarly with the only difference that the operator Q in (14) is now different from zero. Using the same notation as above (as no confusion is possible), the non-unitary group generated in H' by T under BC (3') reads

$$(V(t)f)(x,\mu) = f(x-\mu t,\mu) + \alpha^{-1}(\mu)f(-x+\mu t,-\mu). \tag{34}$$

Further, we have

$$(V^*(t)f)(x,\mu) = f(x+\mu t,\mu) + \alpha(\mu)f(-x-\mu t,-\mu) \tag{35}$$

and then according to (14) the operator Q defined by

$$(V^*V)^{\frac{1}{2}}f \equiv Qf = \begin{cases} f & \mu < 0 \\ \alpha f & \mu > 0 \end{cases} \tag{36}$$

intertwins $V(t)$ with the unitary $\underline{W}(t)$ on $\underline{K} \equiv \overline{QH'}$ (i.e. realizes what j and j^* do on $\underline{H}$). It is easy to verify that

$$[f,f] = \int_0^1 \mu d\mu(1-\alpha^2)|f(0,\mu)|^2 \geq 0 \tag{37}$$

$$jf = \underline{f}(t,\mu) = \begin{cases} 0 & t < 0 \\ (V(t)f)(0,\ ) & t \geq 0 \end{cases} \qquad j:H' \hookrightarrow \underline{H} \tag{38}$$

On $\underline{K}$ the evolution is given by $\underline{W}(t) = QV(t)Q^{-1}$ which acts as

$$\underline{W}(t)f(x,\mu) = f(x-\mu t,\mu) + f(-x+\mu t,-\mu) \tag{39}$$

i.e. the unitary evolution given by purely specularly reflecting BC. Having the action of the extended semigroups in $\underline{H} \oplus \underline{K}$, everything can be computed as before, and we shall not insist on this any further. Before concluding, we want to mention only that the dilation theory might be used as a powerful tool in getting a general spectral decomposition for the (nonnormal) transport operator A which now can be obtained only for very special BC and using explicit calculations [7,8]. We have in mind especially the case $\alpha = 0$, when the free evolutions given by the original problems are actual semigroups (no $V(-t)$ can be defined) but for which drastic simplifications occur, due to a naive embedding of the problems in the space $L_2((-\infty,\infty) \times [-1,1])$ [3,6].

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[4] In fact, $V(t)$ is a group, but the inverse operators $V(-t)$ are not contractions anymore. $V(-t)$ is given by

$$(V(-t)f)(x,\mu) = \chi_{[-a,a]}(x) \sum_{m=-\infty}^{\infty} \alpha(\mu)^m f(x+2am+\mu t,\mu)$$

and its type is $(-\log \alpha)/2a > 0$ (see Ref. 2).

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