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To the memory of
Miron Nicolescu

PARABOLIC VARIATIONAL INEQUALITIES WITH STRONG NONLINEARITIES

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The paper is concerned with the existence of a weak solution of variational inequalities associated with nonlinear operators of the form

$$\frac{\partial}{\partial t} + p(t) + g(t)$$

in an open cylinder $Q =]0,T[\times \Omega$ corresponding to any smooth bounded domain in $\mathbb{R}^N$ and to any fixed $T > 0$. The procedure takes into account partial differential operators in the generalized divergence structure

$$p(t)u = \sum_{|\alpha| \leqq m} (-1)^{|\alpha|} D^{\alpha} p_{\alpha}(t,x,u,\dots,D^m u)$$

and

$$g(t)u = \sum_{|\alpha| \leqq m-1} (-1)^{|\alpha|} D^{\alpha} g_{\alpha}(t,x,u,\dots,D^{m-1} u)$$

where $m \geqq 1$ is an integer. In that case, $p(t)$ and $g(t)$ act on functions with values in hilbertian Sobolev spaces $H^m(\Omega)$ and $H^{m-1}(\Omega)$, respectively.

Among the previous investigations of variational problems connected to these operators, two essential situations in regards to the lower perturbations g can be distinguished, The first deals with non-monotone bounded terms [7], while the second, called *strongly nonlinear* case, considers monotone perturbations with unrestricted growth. Strongly nonlinear parabolic problems were studied by means of a variant Aubin's lemma in [2] or with the aid of a special Galerkin approximation in [6].

By using a realization L of $\frac{\partial}{\partial t}$ in $L^2(Q)$, compactness arguments were extended in [3] to variational inequalities containing zero-order nonlinear perturbations. In this context, we first present a Hilbert space form of the Brézis-Browder method and then, assuming that g is a potential operator and $L + g$ is maximal monotone, we apply compactness criteria to variational inequalities with strongly nonlinear perturbations involving derivatives of u up to order $m-1$.

Let X_0, X_1, X_2 be three real Hilbert spaces such that the imbeddings of X_2 into X_1 and X_1 into X_0 are compact and continuous, respectively. We identify X_0 with its dual space to have

$$X_2 \subseteq X_1 \subseteq X_0 \subseteq X_{-1} \subseteq X_{-2}$$

where X_{-1} and X_{-2} are the dual spaces of X_1 and X_2 respectively, and denote by $\langle\cdot,\cdot\rangle$ the duality pairing both in $X_2 \times X_{-2}$ and $X_1 \times X_{-1}$ as well as the inner product in X_0.

Moreover, for a finite set $T > 0$ let $H_\iota = L^2(0,T;X_\iota)$, $\iota = 0,\pm1,\pm2$, be the L^2 - spaces of functions on $(0,T)$ with values in X_ι. We also have

$$H_2 \subseteq H_1 \subseteq H_0 \subseteq H_{-1} \subseteq H_{-2}$$

and write

$$(\cdot,\cdot) = \int_0^T \langle\cdot,\cdot\rangle \, dt$$

with $(\cdot,\cdot)$ for corresponding dualities in $H_2 \times H_{-2}$, $H_1 \times H_{-1}$ and H_0.

Lemma 1. *To every pair* δ, $\rho > 0$ *there corresponds a constant* $\mu = \mu(\delta,\rho) > 0$ *such that*

$$||v-w||_{H_1} \leqq \delta + \mu||v-w||_{H_0}$$

for all $v, w \in H_2$ *with* $||v||_{H_2}, ||w||_{H_2} \leqq \rho$.

Proof. Ehrling's inequality ([8],p.90) reads as follows: For any $\varepsilon > 0$ there is $C(\varepsilon) > 0$ such that

$$||z||_{X_1} \leqq \varepsilon||z||_{X_2} + C(\varepsilon)\,||x||_{X_0} \qquad \forall\, z \in X_2.$$

Choose $\varepsilon = \frac{\delta}{2\rho}$ and integrate over $(0,T)$ to obtain the lemma. □

An operator $S:H_0 \to H_0$ is *monotone* if

$$(Sv-Sw,v-w) \geq 0 \quad \text{for all} \quad v,w \in D(S) \subseteq H_0,$$

and is *maximal monotone* if it admits no properly monotone extension in $H_0 \times H_0$. In particular, let $\phi: H_0 \to]-\infty, +\infty]$ be a convex lower-semi-continuous function. Assume that ϕ is *proper*, i.e., not merely the constant function $+\infty$, and denote by $D(\phi) = \{v \in H_0 | \phi(V) < +\infty\}$ the *effective domain* of ϕ. Its *subdifferential* (generally multi-valued) defined by

$$\partial\phi(v) = \{h \in H_0 | (h, w-v) \leqq \phi(w) - \phi(v) \qquad \forall\, w \in H_0\}$$

is a simple example of a maximal monotone operator. The elements $h \in \partial\phi(v)$ are called *subgradients* of ϕ at v. Moreover, if ϕ is Gâteaux differentiable, then $\partial\phi(v) = \nabla\phi(v) = \{\phi'(v)\}$ is a *potential* operator.

A bounded mapping carries bounded sets into bounded sets. By " $\to$ " and " $\rightharpoonup$ " we indicate strong and weak convergence, respectively. A bounded operator $P: H_2 \to H_{-2}$ is said to be (H_2, H_1) - *pseudo-monotone* if for every sequence $\{v_n\} \subset H_2$ such that $v_n \rightharpoonup v$ in H_2, $v_n \to V$ in H_1 and $\limsup (Pv_n, v_n - v) \leqq 0$ it follows that

$$(Pv, v-w) \leqq \liminf\, (Pv_n, v_n - w) \qquad \forall\, w \in H_2.$$

In contrast to the elliptic case, where it follows by the compact imbedding of X_2 into X_1, the condition $v_n \to v$ in H_1 must explicitly be specified here. In general, the classes of pseudo-monotone mappings do not only enlarge that of maximal monotone operators, but they also provide a natural setting for variational inequalities.

Let $L: H_0 \to H_0$ be a maximal monotone operator. Then the range $R(I+\lambda L) = H_0$ for all $\lambda > 0$ and so the *resolvent* $J_\lambda = (I+\lambda L)^{-1}$ acts on H_0. In addition, we suppose that:

I_1) J_λ maps H_2 into H_2 and $\|J_\lambda v\|_{H_2} \leqq \|v\|_{H_2}$ for all v in H_2 and $\lambda > 0$;

I_2) For any pair (B_1, B_2) of positive constants, the set

$$\{v \in H_2 \cap D(L) \mid \|v\|_{H_2} \leqq B_1, \quad \|Lv\|_{H_0} \leqq B_2\}$$

is strongly relatively compact in H_1.

We associate with L and J_λ the *Yosida approximant*

$$L_\lambda = \frac{1}{\lambda}(I - J_\lambda): H_0 \to H_0 \quad \text{for all} \quad \lambda > 0.$$

This is also maximal monotone and $LJ_\lambda v = L_\lambda v$ for all v in H_0.

The core of our treatment is based on the following form of the Brézis-Browder compactness criterion for solutions of parabolic variational inequalities [3], which should have a broad range of applications also to other boundary problems.

Proposition 1. *Let* $L:H_0 \to H_0$ *be a maximal monotone operator satisfying* $I_1) - I_2)$, $\phi:H_0 \to]-\infty,+\infty]$ *a proper convex lower-semicontinuous function and* $C > 0$ *a constant such that*

(1) $\phi(J v) \leqq \phi(v) + C\lambda$ *for all* $v \in D(\phi)$ *and* $\lambda > 0$.

If $\{v_n\}$ *is a bounded sequence in* $D(L) \cap D(\phi)$ *so that*

$$(LJ_\lambda v_n, J_\lambda v_n - v_n) + \phi(J_\lambda v_n) - \phi(v_n) \geqq -k \qquad \forall\ \lambda > 0$$

for a constant k *independent of* n, *then* $\{v_n\}$ *is contained in a strongly relatively compact set in* H_1.

Proof. The hypothesis $I_1)$ and condition (1) imply $\|J_\lambda v_n\|_{H_2} \leqq M$ and

$$(LJ_\lambda v_n, J_\lambda v_n - v_n) \leqq -C\lambda - k.$$

Since $(J_\lambda - I)v_n = -\lambda L_\lambda v_n = -LJ_\lambda v_n$, we have

$$\|LJ_\lambda v_n\|_{H_0} < M_\lambda \quad \text{with} \quad M_\lambda^2 = C + k\lambda^{-1}.$$

For each fixed $\lambda > 0$, the sequence $\{J_\lambda v_n\}$ is contained in the set

$$\{w \in H_2 \cap D(L) \mid \|w\|_{H_2} \leqq M, \quad \|Lw\|_{H_0} < M_\lambda\}$$

and, by assumption $I_2)$, it is strongly relatively compact in H_1.

Taking $\rho = M$ in lemma 1, we get

$$\|J_\lambda v_n - v_n\|_{H_1} \leqq \delta + \mu\|J_\lambda v_n - v_n\|_{H_0}$$

uniformly in n. Since $J_\lambda v_n \to v_n$ in H_0 as $\lambda \to 0$, (see e.g., [8], p. 129), the second term becomes small enough for a suitable choice of $\lambda > 0$. We infer that the sequence $\{v_n\}$ lies in a strongly relatively compact set of H_1. □

Assumption (1) is in fact a maximality condition for the monotone sum $L + \partial\phi$, ([8], p. 138).

This compactness principle allows us to insert an (H_2,H_1) - pseudo-monotone perturbation in the following existence result:

Theorem 1. *Let* $P:H_2 \to H_{-2}$ *be an* (H_2,H_1) *- pseudo-monotone operator and* L,ϕ,C *as in proposition* 1. *In addition,* L *is linear and* $\phi(0) = 0$. *For a given* $f \in H_{-2}$ *assume that* P *is coercive in the sense*

(2) $$(Pv,v) + \phi(v) > (f,v)$$

for all v *outside a closed ball in* H_{-2} *with center at the origin and radius* R. *Then there exists at least one solution* $u \in D(\phi)$ *of the variational inequality*

(3) $$(Lv,v-u) + (Pu,v-u) + \phi(v) - \phi(u) \geq (f,v-u) \qquad \forall v \in H_{-2} \cap D(L)$$

Moreover, if P *is strictly monotone or* P *is monotone and* ϕ *is strictly convex, then* u *is uniquely determined by* f.

Proof. First, for every natural n, we notice that the convex closed set

$$K_n = \{v \in H_2 \cap D(L) \mid \|v\|_{H_2} \leq n, \ \|Lv\|_{H_0} \leqq n\}$$

is strongly relatively compact in H_1, by assumption I_2). Since the restriction of L to K_n is a bounded maximal monotone operator, we can successively apply the classical variational inequality arguments (Brouwer's fixed point theorem, compactness and pseudo-monotonicity, Mosco's trick, (see e.g., [9],Part 2,p. 25)), to deduce the existence of a $u_n \in K_n \cap D(\phi)$ such that

$$(Lu_n + Pu_n,v-u_n) + \phi(v) - \phi(u_n) \geqq (f,v-u_n) \qquad \forall v \in K_n.$$

Because L is monotone,

(4) $$(Lv,v-u_n) + Pu_n,v-u_n) + \phi(v) - \phi(u_n) \geqq (f,v-u_n) \qquad \forall v \in K_n,$$

and the origin lies in K_n,

$$(Pu_n,u_n) + \phi(u_n) \leqq (f,u_n)$$

then, in view of (2), we have $\|u_n\|_{H_2} \leqq R$ for all $n \in \mathbb{N}$.

On the other hand, since $LJ_\lambda v = v - J_\lambda v$, the monotonicity relation gives $\|LJ_\lambda v\|_{H_0} \leqq \|Lv\|_{H_0}$. This, together with I_1), shows the invariance of K_n under J_λ for all $\lambda > 0$.

Now take $v = J_\lambda u_n$ in (4) and get

$$(LJ_\lambda u_n, J_\lambda u_n - u_n) + \phi(J_\lambda u_n) - \phi(u_n) \geqq (f - Pu_n, J_\lambda u_n - u_n) \geqq -k \qquad \forall \lambda > 0,$$

in which k is a constant independent of n. Using proposition 1, $\{u_n\}$ is strongly compact in H_1. Then, there exists a $u \in D(\phi)$ and we may assume, at least for a subsequence, that $u_n \to u$ in H_1, $u_n \rightharpoonup u$ in H_2 and $Pu_n \rightharpoonup Pu$ in H_{-2}. It remains to prove that

$$\limsup\, (Pu_n, u_n - u) \leqq 0.$$

Each $v \in H_2 \cap D(L)$ lies in some K_j and thus, setting $n \geqq j$ in (4),

(5) $$\limsup\, (Pu_n, u_n - v) \leqq (Lv, v-u) - (f, v-u) + \phi(v) - \phi(u)$$

and so

$$\limsup\, (Pu_n, u_n - u) \leqq (h-f, v-u) + (Lv, v-u) + \phi(v) - \phi(u).$$

The left-hand side being independent of v, replace v by $J_\lambda u$ and note, in virtue of I_1, that $\{J_\lambda u\}$ remains bounded in H_2 for all $\lambda > 0$. Since $J_\lambda u \to u$ in H_0, then $\{J_\lambda u\}$ converges weakly to u in H_2 as $\lambda \to 0$. Because $(LJ_\lambda u, J_\lambda u - u) = -\lambda \|L_\lambda u\|^2_{H_0} \leqq 0$ and $\phi(J_\lambda u) - \phi(u) \leqq C\lambda$, we infer that

$$\limsup_{n \to \infty}\, (Pu_n, u_n - u) \leqq \liminf_{\lambda \to 0}\, [(h-f, J_\lambda u - u) + C\lambda] = 0.$$

As P is (H_2, H_1) - pseudo-monotone, we have

$$(Pu, u-v) \leqq \liminf\, (Pu_n, u_n - v) \qquad \forall v \in H_2 \cap D(\phi),$$

and thus $Pu = h$. These relationships and (5) prove that $u \in D(\phi)$ satisfies the variational inequality (3).

Finally, let u_1 and u_2 be two solutions of (3). Set $v = \frac{1}{2}(u_1 + u_2)$ in the corresponding inequalities for u_1 and u_2. Addition gives

$$\frac{1}{2}(Pu_1 - Pu_2, u_2 - u_1) + 2[\phi(\frac{u_1 + u_2}{2}) - \phi(u_1) - \phi(u_2)] \geqq 0,$$

and the uniqueness follows. □

The accretiveness assumption I_1) for the monotone operator L restricts us, despite of [3], to a Hilbert space setting.

Now we return to the divergence form of perturbations and put into evidence assumptions which force $p(t)$ and $g(t)$ to belong to the classes of nonlinear operators defined above. Let us denote the space derivatives $D^\alpha = D_1^{\alpha_1}\dots D_N^{\alpha_N}$ where $D_j = \frac{\partial}{\partial x_j}$ and $\alpha = (\alpha_1,\dots,\alpha_N)$ is a multi-index of non-negative integers with $|\alpha| = \alpha_1 + \dots + \alpha_N$. Consider the Sobolev space $H^m(\Omega)$ of real functions defined on Ω, whose distributional derivatives of order $\leqq m$ belong to $L^2(\Omega)$. $H^m(\Omega)$ is a Hilbert space with respect to the inner product

$$<v,w>_m = \sum_{|\alpha| \leqq m} <D^\alpha v, D^\alpha w>_0$$

where $<\cdot,\cdot>_0$ is the pairing in $L^2(\Omega)$. Let $\|\cdot\|_m$ be the corresponding norm.

On the other hand, let s_m be the number of multi-indices α with $|\alpha| \leqq m$. For $\xi = \{\xi_\alpha \,|\, |\alpha| \leqq m\} \in \mathbb{R}^{s_m}$ separate the highest order terms from lower-order ones, writing $\xi = (\zeta,\eta)$ where $\zeta = \{\zeta_\alpha \,|\, |\alpha| = m\}$ and $\eta = \{\eta_\alpha \,|\, |\alpha| \leqq m-1\}$, $\eta \in \mathbb{R}^{s_{m-1}}$. $|\bullet|$, indexed sometimes by the dimension, stands for the Euclidean norm.

The following hypotheses are imposed on the first perturbation:

II_1) $p: Q \times \mathbb{R}^{s_m} \to \mathbb{R}^{s_m}$ satisfies the *Carathéodory conditions*, i.e., it is measurable in $(t,x) \in Q$ for each $\xi \in \mathbb{R}^{s_m}$ and continuous in ξ for almost all $(t,x) \in Q$, and grows at most linearly in ξ, that is,

$$|p(t,x,\xi)|_{s_m} \leqq c_1(|\xi|_{s_m} + h_1(t,x)) \qquad \forall\ (t,x) \in Q,\quad \xi \in \mathbb{R}^{s_m},$$

where $c_1 > 0$ and $h_1 \in L^2(Q)$;

II_2) For almost all $(t,x) \in Q$, all $\eta \in \mathbb{R}^{s_{m-1}}$ and any pair of distinct elements ξ and ξ' in $\mathbb{R}^{s_m}-\mathbb{R}^{s_{m-1}}$

$$\sum_{|\alpha| = m} [p_\alpha(t,x,\eta,\xi) - p_\alpha(t,x,\eta,\xi')](\xi_\alpha - \xi'_\alpha) > 0 ;$$

II_3) There are a constant $c_2 > 0$ and a function $h_2 \in L^2(Q)$ so that

$$\sum_{|\alpha| \leq m} p_\alpha(t,x,\xi)\xi_\alpha \geq c_2|\xi|^2_{s_m} + h_2(t,x) ,$$

for all (t,x) in Q and ξ in $\mathbb{R}^{s_m}$.

Denote further $\xi(v) = \{D^\alpha v \,|\, |\alpha| \leq m\} = \nabla^m v$ and $\eta(v) = \{D^\alpha v \,|\, |\alpha| \leq m-1\} = \nabla^{m-1} v$, specify the Hilbert spaces

$$H_2 = L^2(0,T;H^m(\Omega)),\ H_1 = L^2(0,T;H^{m-1}(\Omega)),\ H_0 = L^2(0,T;L^2(\Omega)) = L^2(Q)$$

and write $(\cdot,\cdot)$ for the usual duality

$$(v,w) = \int_0^T \langle v(t),w(t)\rangle\, dt .$$

The semilinear form associated with the operator $p(t)$

$$a(v,w) = \sum_{|\alpha| \le m} \int_0^T \langle p_\alpha(\cdot,\cdot,\nabla^m v), D^\alpha w\rangle_0 dt$$

is well-defined for v, w in H_2 and thus the bounded linear functional $w \to a(v,w)$ induces a mapping $P:H_2 \to H_{-2}$ by the rule

$$(Pv,w) = a(v,w) \quad \text{for all} \quad w \in H_2 .$$

An operator $P:H_2 \to H_{-2}$ is said to be *coercive* if

$$\frac{(Pv,v)}{\|v\|_{H_2}} \to \infty \quad \text{as} \quad \|v\|_{H_2} \to \infty .$$

Proposition 2. *Under assumptions* II), $P:H_2 \to H_{-2}$ *is a coercive* (H_2,H_1)-*pseudo-monotone operator.*

Proof. On the basis of II_1), we may write the inequalities

$$\|Pv\|_{H_{-2}} = \sup_{\|v\|_{H_2} \le 1} |(Pv,w)| \le \Big[\int_0^T \sum_{|\alpha| \le} \int_\Omega |p_\alpha(t,x,\nabla^m v)|^2 dxdt\Big]^{\frac{1}{2}}$$

$$\le c_3(\|v\|_{H_2} + \|h_1\|_{H_0}),$$

which imply the boundedness of the operator P.

Let us choose a sequence $\{v_n\}$ in H_2 such that $v_n \rightharpoonup v$ in H_2, $v_n \to v$ in H_1 and $\limsup\, (Pv_n,v_n-v) \le 0$. For (H_2,H_1)-pseudo-monotonicity of P it suffices to show that $Pv_n \to Pv$ in H_{-2}. Since

(6) $$\limsup\, (Pv_n-Pv,v_n-v) = \limsup\, (Pv_n,v_n-v) \le 0,$$

we write the sequence of integrands

$$A_n(t,x) = \sum_{|\alpha| \leqq m} [p_\alpha(t,x,\eta_n,\xi_n) - p_\alpha(t,x,\eta,\xi)](D^\alpha v_n(t,x) - D^\alpha v(t,x)),$$

where $\eta_n = \eta(v_n)(t,x)$ and $\xi_n = \xi(v_n)(t,x)$, in the form

$$A_n(t,x) = \sum_{|\alpha| \leqq m} p_\alpha(t,x,\eta_n,\xi_n)D^\alpha v_n - \Phi_n(t,x)$$

with

$$\Phi_n(t,x) = \sum_{|\alpha| \leqq m} [p_\alpha(t,x,\eta_n,\xi_n)D^\alpha v + p_\alpha(t,x,\eta,\xi)(D^\alpha v_n - D^\alpha v)].$$

By hypothesis II_1), we derive the estimate

$$|\Phi_n(t,x)| \leqq c_4(|\eta_n| + |\xi_n| + h_1(t,x))(|\eta| + |\xi|) +$$
$$+ c_5(|\eta_n| + |\xi_n| + h_1(t,x))(|\eta_n| + |\xi_n| + |\eta| + |\xi|)$$

where c_4 and c_5 are positive constants, from which the equi-integrability of the sequence $\{\Phi_n(t,x)\}$ in $L^1(Q)$ follows.

If we apply Young's inequality

$$ab \leqq \frac{1}{2}(\varepsilon a^2 + \varepsilon^{-1}b^2) \quad \text{with } a, b \geqq 0 \text{ and } \varepsilon > 0,$$

then to each $\varepsilon > 0$ there correspond a number $k(\varepsilon) > 0$ and a function $h_3 \in L^1(Q)$ such that

$$|\Phi_n(t,x)| \leqq \varepsilon|\xi_n|^2 - k(\varepsilon)[|\eta_n|^2 + |\eta|^2 + |\xi| + h_3(t,x)].$$

Choosing $\varepsilon = \frac{1}{2} c_2$ and $h_4 = h_2 + k(\varepsilon)h_3$, the hypothesis II_3) yields

(7) $$A_n(t,x) \geqq \varepsilon|\xi_n|^2 + k(\varepsilon)[|\eta_n|^2 + |\eta|^2 + |\xi|^2] - h_4(t,x),$$

where the last two terms are bounded almost everywhere in Q. Hence the sequence $\{\xi(v_n)(t,x)\}$ remains bounded on the complement of a certain subset ω_1 in Q of measure zero. Fixing a pair $(t,s) \in Q-\omega_1$ and passing eventually to a subsequence, $\{\xi(v_n)(t,x)\}$ converges to $\bar{\xi}$.

On the other hand, because $\{\eta(v_n)\}$ converges uniformly to $\eta(v)$ in Q, the assumption (6) really implies

$$\sum_{|\alpha| = m} [p_\alpha(t,x,\eta(v)(t,x),\bar{\xi}) - p_\alpha(t,x,\eta(v)(t,x),\xi(v)(t,x)](\bar{\xi}-D^\alpha v(t,x)) \leqq 0$$

on the complement of another subset ω_2 in Q of measure zero. Using the full force of hypothesis II_2), we deduce $\bar{\xi} = \xi(v)(t,x)$ for a given (t,x) in $Q-(\omega_1 \cup \omega_2)$. Thus any convergent subsequence of the bounded sequence of vectors $\{\xi(v_n)(t,x)\}$ converges to $\xi(v)(t,x)$. We have therefore proved a.e. in Q the convergence of the original sequence to $\xi(v)(t,x)$ as well as that of $\{p_\alpha(t,x,\xi(v_n)(t,x))\}$ to $p_\alpha(t,x,\xi(v)(t,x))$ for all $|\alpha| \leqq m$. In virtue of same hypothesis II_1), the sequence $\{p_\alpha(\cdot,\cdot,\xi(v_n))\}$ is bounded in $L^2(Q)$ for all $|\alpha| \leqq m$. The Lebesque dominated theorem guarantees that

$$p_\alpha(\cdot,\cdot,\xi(v_n)) \to p_\alpha(\cdot,\cdot,\xi(v)) \quad \text{in} \quad L^2(Q)$$

and so a fortiori

$$\liminf\, (\mathcal{P}v,v_n-w) = (\mathcal{P}v,v-w) \qquad \forall\, w \in \mathcal{H}_2.$$

Finally, the coerciveness of $\mathcal{P}$ is a direct consequence of II_3). □

Upon the lower-order perturbing term, we assume:

III_1) $g:Q \times \mathbb{R}^{s_{m-1}} \to \mathbb{R}^{s_{m-1}}$ satisfies the Carathéodory conditions and the map $t \to G(v,w) = \sum_{|\alpha| \leqq m-1} \langle g_\alpha(t,x,\nabla^{m-1}v),D^\alpha w\rangle_0$ is measurable on $]0,T[$ for all v, w in $\mathcal{H}_2$;

III_2) There is a convex Gâteaux differentiable function $F:\mathbb{R}^{s_{m-1}} \to [0,+\infty]$ with $F(0) = 0$ such that

$$g(t,x,\eta) = \nabla_\eta F(\eta), \quad \text{i.e.,} \quad g_\alpha = \frac{\partial F}{\partial \eta_\alpha}$$

III_3) $\sup_{|\eta|_{s_{m-1}}=r} F(\eta)$ defines an $L^1(Q)$-function for some fixed $r > 0$.

By hypothesis III_3) and the subgradient inequality, we deduce

$$G(v,v) = \sum_{|\alpha| \leqq m-1} \langle g_\alpha(t,x,\nabla^{m-1}v),D^\alpha v\rangle_0 \geqq 0$$

for all $v \in H_1$ and almost all $(t,x) \in Q$, that is, g satisfies a sign condition.

As in the elliptic case ([4], p. 175), when g is continuously differentiable in η for almost all $(t,x) \in Q$, the hypothesis III_2) is fulfilled if

$$\frac{\partial g_\alpha}{\partial \eta_\beta} = \frac{\partial g_\beta}{\partial \eta_\alpha} \qquad \forall \alpha,\beta = 0,\dots,s_{m-1}.$$

The condition III_3) can be regarded as a natural extension of Webb's corresponding hypothesis for the elliptic equation [10].

The assumption III_2) allows us to write the Gâteaux differential

$$DF(v,w) = \sum_{|\alpha| \leq m-1} \int_Q g_\alpha(t,x,\nabla^{m-1} v)\, D^\alpha w \, dxdt$$

and the primitive

$$F(v) = \int_0^1 d\tau \sum_{|\alpha| \leq m-1} \int_Q g_\alpha(t,x,\tau\nabla^{m-1} v) D^\alpha v \, dxdt$$

for every v, w in H_1.

Because no growth restriction on $g(t,x,\cdot)$ is made, this perturbation does not induce a mapping in $H_1 \times H_{-1}$, and then we introduce the truncation

$$g^{(j)}(t,x,v) = \begin{cases} g(t,x,v) & \text{if } |g(t,x,v)|_{s_{m-1}} \leq j, \\ j \dfrac{g(t,x,v)}{|g(t,x,v)|} & \text{otherwise} \end{cases}$$

and $F^{(j)}(v)$ its corresponding potential, for each $j \in \mathbb{N}$.

Let us consider now the restriction of $\frac{\partial}{\partial t}$ to H_0, i.e., a linear operator $L: H_0 \to H_0$ with the graph

$$D(L) = \left\{ v \in L^2(Q) \;\middle|\; \frac{\partial v}{\partial t} \in L^2(Q),\ v(0,x) = 0 \right\}$$

Since $D(L) \subset C^1[0,T];L^2(\Omega))$, the condition $v(0,x) = 0$ makes sense and, in addition, L is densely defined and maximal monotone in $H_0 \times H_0$ (see e.g., [1], p. 65). For all $\lambda > 0$, let J_λ be its resolvent.

An elementary computation gives

$$\int_0^T \|(I+\lambda L)w\|_m dt \geqq \int_0^T \|w\|_m dt$$

for all $w \in C^1([0,T];H^m(\Omega))$ with $w(0,x) = 0$. Taking $w = J_\lambda v$ and the closure in H_2 of $\{v \in C^1([0,T];H^m(\Omega)) \mid v(0,x) = 0\}$, the hypothesis $I_1)$ for this L is checked.

In the elliptic case, as we mentioned already above, the selection of a strongly convergent and thus of an almost everywhere convergent sequence from a bounded set is generally possible in view of the Sobolev-Rellich theorem. A similar rôle is played for parabolic equations by Aubin's compactness lemma (see e.g., [9], Part 2, p. 50). In our particular setting, it asserts: A subset of $D(L)$ which is bounded in the graph norm $\|v\|_{H_2} + \|Lv\|_{H_0}$ is strongly realtively compact in H_1. Thus L satsifies also the hypothesis $I_2)$.

Let K be a closed convex set containing the origin in $H^m(\Omega)$ and consider the indicator function

$$\gamma(v) = \begin{cases} 0 & \text{if } v(t) \text{ lies in } K \text{ a.e.,} \\ +\infty & \text{otherwise,} \end{cases}$$

with the domain $D(\gamma) = \{v \in H_2 \mid \gamma(v) = 0\}$.

Theorem 2. *Let* f *be a given element in* H_{-2} *and assume that hypotheses* II) - III) *hold. Then there exists a* $u \in D(\gamma) \cap C([0,T]:L^2(\Omega))$ *with* $u(0,x) = 0$, $g(\cdot,\cdot,u) \in L^1(Q)$ *and* $G(u,u) \in L^1(0,T)$ *which is a solution of the variational inequality*

$$(8)\quad \begin{aligned} &(Lv,v-u) + (Pu,v-u) + \sum_{|\alpha| \leqq m-1} \int_Q g_\alpha(t,x,\nabla^{m-1}u) D^\alpha(v-u)\,dxdt \\ &\geqq (f,v-u) \qquad \forall v \in D(\gamma) \cap D(L) \cap L^\infty(Q). \end{aligned}$$

Proof. For every natural j, the sum $P + g^{(j)}$ is a bounded coercive (H_2,H_1)-pseudo-monotone operator and, in virtue of theorem 1, there exists a solution $u_j \in D(\gamma)$ of the variational inequality

$$(9)\qquad \begin{aligned}&(Lv,v-u_j) + (Pu_j,v-u) + \sum_{|\alpha|\leqq m-1}\int_Q g_\alpha^{(j)}(t,x,\nabla^{m-1}u_j)D^\alpha(v-u_j)dxdt\\ &\qquad + \gamma(v) - \gamma(u_j) \geqq (f,v-u_j) \qquad \forall v\in H_2\cap D(L).\end{aligned}$$

With each $v\in H_2$ such that $F(v)<\infty$, let there be associated the function $\phi(v)=\gamma(v)+F(v)$. By I_1, $\gamma(J_\lambda v)\leqq\gamma(v)$ and, using the subgradient inequality, we have

$$F(J_\lambda v) - F(v) \leqq \sum_{|\alpha|\leqq m-1}(g_\alpha(J_\lambda v),D^\alpha(J_\lambda v-v)) \leqq \|g(J_\lambda v)\|_{H_{-1}}\|J_\lambda v-v\|_{H_1},$$

where, for simplicity, $g_\alpha(v)=g_\alpha(t,x,\nabla^{m-1}v)$. Hence ϕ fulfills the assumption (1) with $C=0$.

Because of the subgradient inequality, (9) can be written as

$$(10)\qquad (Lv,v-u_j) + (Pu_j,v-u_j) + \phi(v) - \phi(u_j) \geqq (f,v-u_j) \qquad \forall v\in H_2\cap D(L).$$

We can proceed as in the proof of theorem 1, to infer that the sequence $\{u_j\}$ is bounded in H_2 and strongly compact in H_1. Let us denote

$$G^{(j)}(u_j,u_j) = \sum_{|\alpha|\leqq m-1}\int_Q g_\alpha^{(j)}(u_j)D^\alpha u_j dxdt.$$

Substitute $v=0$ in (10) and obtain

$$(Pu_j,u_j) + G^{(j)}(u_j,u_j) \leqq \|f\|_{H_{-2}}\|u_j\|_{H_2}$$

Since P is coercive, the sequence $\{G^{(j)}(u_j,u_j)\}$ remains uniformly bounded. Therefore, we may assume, at least for a subsequence, that $u_j\rightharpoonup u$ in H_2 and $Pu_j\rightharpoonup h$ in H_{-2} as well as $u_j\to u$ in H_1 and $\{u_j(t,x)\}$ converges to $u(t,x)$ for almost all $(t,x)\in Q$. Moreover, in view of the Carathéodory condition, $\{g^{(j)}(u_j)\}$ and $\{G^{(j)}(u_j,u_j)\}$ converge to $g(u)$ and $G(u,u)$ a.e. in Q, respectively. Then Fatou's lemma guarantees that $G^{(j)}(u_j,u_j)\to G(u,u)$ in $L^1(Q)$.

Furthermore, (as in [5], lemma 3), for every α, we consider s_{m-1}-tuple

$$\chi_\alpha(r,g,t,x,u) = \begin{cases} r \text{ sign } g_\alpha(t,x,\nabla^{m-1}u) & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

Then the subgradient inequality and $F(u) \geqq 0$ yield

$$|g_\alpha^{(j)}(u_j)| \leqq \frac{1}{r}[G^{(j)}(u_j,u_j) + F(\chi)] \quad \text{with} \quad |\chi|_{s_{m-1}} = r$$

Hypothesis III_1) also ensures that $g^{(j)}(u_j) \to g(u)$ in $L^1(Q)$.

Because P is (H_2,H_1)-pseudo-monotone, it follows, by the same argument as in the proof of theorem 1, that $Pu = h$.

Now, the passing from (10) to the variational inequality (8) is rigorously justified.

Finally, take $v = J_\lambda u$ in (8) and get

$$||L_\lambda u||_{H_0} + (Pu,L_\lambda u) \leqq ||f||_{H_{-2}} ||L_\lambda u||_{H_2},$$

because $F(J_\lambda u) \leqq F(u)$, and hence $\{L_\lambda u\}$ is bounded in H_0 for all $\lambda > 0$. Since $||J_\lambda u-u||_{H_0} = \lambda ||L_\lambda u||_{H_0}$ and so $J_\lambda u \to u$ in H_0 as $\lambda \to 0$, we derive that $u \in D(L)$ and $u \in C([0,T];L^2(\Omega))$. The proof of the theorem is thereby completed. □

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