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POSITIVITY AND BOUNDEDNESS OF SOLUTIONS OF VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract

The positivity and boundedness of solutions of Volterra linear and nonlinear integro-differential equations are studied. Our approach herein involves the construction of an auxiliary function which, in certain cases, turns out to be the usual resolvent kernel.

1. INTRODUCTION

Friedman [1], Komlenko [3], Levin [4], Rina Ling [5], Miller [6] and Padmavalli [8] among others have studied the positivity and boundedness of solutions of Volterra integral equations and many interesting results have been reported. While nonconvolution kernels are taken into account by one of earlier workers [3], the kernels considered by several others [1,4,5,6,8] are of convolution type. In the case of integral equations of the form

$$x(t) = f(t) - \int_0^t a(t,s)x(s)ds, \quad 0 \leq t < \infty \tag{1.1}$$

one can assert by virtue of Neumann series (see [9]) that if the kernel $a(t,s)$ is negative and the source function $f(t)$ is positive, then the solutions of (1.1) are positive in the interval of their existence. It is interesting to probe the conditions on $f(t)$ in order to obtain the positivity of solutions of integro-differential equation

$$x'(t) = f(t) - \int_0^t a(t,s)x(s)ds, \quad x(0) = x_0 > 0 \tag{1.2}$$

where $0 \leq t < \infty$, especially when the kernel $a(t,s)$ is positive. The objective of this paper is to investigate sufficient conditions for positivity and boundedness of solutions of volterra linear and nonlinear integro-differential equations of both convolution and nonconvolution type. Our approach herein involves the construction of an auxiliary function which, in certain cases, turns out to be the usual resolvent kernel. The advantage of such an approach is to transfer the (i) responsibility of the original kernel to a better kernel and (ii) unwanted behaviour (if any) of the original kernel to the source function, through a proper choice of an auxiliary function. Examples and remarks are provided at appropriate places to illustrate the generality of the conditions and the fruitfulness of the results.

Under suitable conditions on $a(t,s)$ (for example, $a(t,s)$ is absolutely continuous on every compact subset of $R \times R$), the equation (1.2) can be written as

$$x'(t) = f(t) - x_0\int_0^t a(t,s)ds - \int_0^t [\int_s^t a(t,u)du]x'(s)ds. \tag{1.3}$$

Therefore, if $f(t) - x_0\int_0^t a(t,s)ds$ is positive and $\int_0^t a(t,s)ds$ is negative, then in view of Neumann series, one can conclude that $x'(t)$ in (1.3) is positive and hence the soution $x(t)$ of (1.2) is positive as $x_0 > 0$ (such a conclusion is quite obvious if $a(t,s)$ is negative and $f(t)$ is positive). This fact has been extensively used in our subsequent analysis.

2. BASIC LEMMAS

In this section we shall derive some basic results which will provide the necessary leverage to work out the planned trade off between the kernel $a(t,s)$ and the source function $f(t)$.

Lemma 2.1. If $a(t,s)$ is continuous on $0 \leq s \leq t < \infty$ and $f(t)$ is continuous on $0 \leq t < \infty$, then the equation (1.2) is equivalent to

$$y'(t) = h(t) - \int_0^t b(t,s)y(s)ds, \quad y(0) = x_0 > 0, \quad 0 \leq t < \infty, \tag{2.1}$$

where

$$(2.2)\qquad b(t,s) = a(t,s) + \phi_s(t,s) - \int_s^t \phi(t,u)a(u,s)du + \phi(t,t)\int_s^t a(u,s)du,$$

$\phi(t,s)$ being a c^1-function on $0 \le s \le t < \infty$ and

$$(2.3)\qquad h(t) = f(t) + x_0\phi(t,t) - x_0\phi(t,0) + \phi(t,t)\int_0^t f(s)ds - \int_0^t \phi(t,s)f(s)ds.$$

Remark 2.1. If we choose $b(t,s) \equiv 0$ i.e. $\phi(t,s)$ is a solution of (2.2) with $b(t,s) \equiv 0$, then from (2.1), the solution $x(t)$ of (1.2) can be obtained on integrating (2.3) between 0 and t.

Proof of Lemma 2.1. Let $x(t)$ be a solution of (1.2) existing on the interval $0 \le t < \infty$. Consider the identity

$$\int_0^t \phi_s(t,s)x(s)ds = \phi(t,t)x(t) - \phi(t,0)x_0 - \int_0^t \phi(t,s)x'(s)ds.$$

Substituting $x(t)$ and $x'(t)$ from (1.2) and using Fubini's theorem, we get

$$\begin{aligned}(2.4)\qquad \int_0^t \phi_s(t,s)x(s)ds = {} & x_0\phi(t,t) - x_0\phi(t,0) \\ & + \phi(t,t)\int_0^t f(s)ds - \int_0^t \phi(t,s)f(s)ds \\ & - \phi(t,t)\int_0^t \left[\int_\tau^t a(s,\tau)ds\right]x(\tau)d\tau \\ & + \int_0^t \left[\int_\tau^t a(s,\tau)\phi(t,s)ds\right]x(\tau)d\tau.\end{aligned}$$

Therefore, it follows from (1.2), (2.2), (2.3) and (2.4) that

$$\int_0^t b(t,s)x(s)ds = -x'(t) + h(t), \quad x(0) = x_0.$$

This shows that every solution of (1.2) is a solution of (2.1).

For converse, let $y(t)$ be a solution of (2.1). Then by taking $\phi(t,s) \equiv 1$, we have from (2.2) and (2.3), $b(t,s) = a(t,s)$ and $h(t) = f(t)$ and hence $y(t)$ satisfies (1.2).

A result similar to Lemma 2.1 for convolution kernels is given below.

Lemma 2.2. If $a(t)$ and $f(t)$ are continuous on $0 \le t < \infty$, then the following equations are equivalent:

$$(1.2)^* \qquad x'(t) = f(t) - \int_0^t a(t-s)x(s)ds, \quad x(0) = x_0 ,$$

$$(2.1)^* \qquad y'(t) = h(t) - \int_0^t b(t-s)y(s)ds, \quad y(0) = x_0 ,$$

where $0 \le t < \infty$, and

$$(2.5) \qquad b(t) = a(t) + \phi'(t) + \int_0^t \phi(t-\tau)a(\tau)d\tau - \phi(0)\int_0^t a(\tau)d\tau,$$

$\phi \in c^1[0,\infty)$,

$$(2.6) \qquad h(t) = x_0\phi(t) - x_0\phi(0) + \int_0^t (t-\tau)f(\tau)d\tau - \phi(0)\int_0^t f(\tau)d\tau + f(t)$$

The following result is somewhat close to the usual resolvent equations (cf. [2] and [7]) and it is useful in the study of nonlinear Volterra integro-differential equations (see section 4).

Lemma 2.3. If $a(t)$ and $f(t)$ are continuous on $0 \le t < \infty$, then every solution of $x(t)$ of $(1.2)^*$ satisfies

$$(1.2)^{**} \qquad x(t) = \psi(t) - \int_0^t c(t-\tau)x(\tau)d$$

on $0 \le t < \infty$, where

$$(2.7) \qquad c(t) = \phi'(t) + \int_0^t \phi(t-\tau)a(\tau)d\tau,$$

(2.8) $$\psi(t) = x_0\phi(t) + \int_0^t f(s)\phi(t-s)ds,$$

and $\phi \in c^1[0,\infty)$, $\phi(0) = 1$.

Remark 2.2. Choose $\phi(t)$ (for example, $\phi(t) = e^{-t} + te^{-t}$ with $a(t) = e^{-2t}$) such that $c(t) \equiv 0$. Then (2.7) takes the form

$$\phi'(t) = -\int_0^t \phi(t-\tau)a(\tau)d\tau, \quad \phi(0) = 1$$

Thus $\phi(t)$ would agree with the differential resolvent to the kernel $a(t)$ and the equation $(1.2)^{**}$ gives the solution $x(t)$ of (1.2) on $0 \leq t < \infty$ in terms of the resolvent $\phi(t)$ (see Grossman and Miller [2], equations (A) and (4) with $A(t) \equiv 0$).

3. MAIN RESULTS

We shall now study the positivity and boundedness properties of solutions of (1.2) and $(1.2)^*$ through a proper choice of the auxiliary function $\phi(t,s)$ introduced in section 2.

Theorem 3.1. Suppose for $0 \leq s \leq t < \infty$ (i) $a(t,s) > 0$,

(ii) $\lambda(t) = f(t) - x_0 \int_0^t a(t,u)du$ is an increasing function of t and

(iii) $\int_0^t\int_0^t a(t,\tau)d\tau\, ds < 1$ hold. If $x(t)$ is any solution of (1.2) on $[0,\infty)$, then the following estimates are satisfied on $[0,\infty)$:

(a) $x_0 \leq x(t) \leq x_0 + F(t)$

(b) $0 < x'(t) \leq f(t)$

where $F(t) = \int_0^t f(s)ds.$

Corollary 3.1. If, in addition to the assumptions (i), (ii) and (iii) of Theorem 3.1, suppose $f \in L^1[0,\infty)$, then the solution $x(t)$ of (1.2) is positive and bounded on $[0,\infty)$.

Example 3.1. Choose $a(t,s) = e^{-t}/(1+s)^3$ for $0 \le s \le t < \infty$. It can be seen that the conditions (i) and (iii) of Theorem 3.1 are satisfied. If we take $f(t) = 3 - e^{-t}$ and $x_0 = 1$, then the condition (ii) also holds.

Remark 3.1. Let

$$F_1(t) = f(t) - x_0 \int_0^t a(t,u)du$$

and

$$K(t,s) = -\int_s^t a(t,u)du$$

for $0 \le s \le t < \infty$. Then the equation (1.2) reduces, upon integrating by parts (see equation 1.3), to

$$x'(t) = F_1(t) + \int_0^t K(t,s)x'(s)ds \tag{3.1}$$

on $0 \le s \le t < \infty$. In order to obtain an estimate similar to (b) of Theorem 3.1 on the solution $x'(t)$ of (3.1), Komlenko [3, Corollary 2 and remark 4] assumed the condition

$$\frac{d}{dt}\left(\frac{K(t,s)}{F_1(t)}\right) \ge 0. \tag{3.2}$$

If we choose $a(t,s) = e^{-t}/(1+s)^3$ and $f(t) = 3 - e^{-t}$ for $0 \le s \le t < \infty$, and $x_0 = 1$, then

$$\left[\frac{d}{dt}\left(\frac{K(t,s)}{F_1(t)}\right)\right]_{\substack{t=0.001\\ s=0.0001}} \simeq -0.9952$$

Thus the condition 3.2 of Komlenko [3] does not hold. However, all the assumptions of Theorem 3.1 are satisfied (see Example 3.1).

Proof of Theorem 3.1. Take $\phi(t,s) = \int_s^t a(t,u)du$. Then $\phi(t,t) = 0$. Since $a(t,s) > 0$, it follows from (2.2) that

$$b(t,s) = a(t,s) - a(t,s) - \int_s^t a(u,s)\left(\int_u^t a(t,\tau)d\tau\right)du < 0.$$

Further, using (2.2) and (2.3) we obtain

$$h(t) - x_0\int_0^t b(t,u)du = f(t) + x_0\int_0^t [\int_s^t a(u,s)(\int_u^t a(t,\tau)d\tau)du]ds$$

$$- \int_0^t (\int_s^t a(t,u)du)f(s)ds - x_0\int_0^t a(t,u)du$$

$$= f(t) - x_0\int_0^t a(t,u)du$$

$$+ x_0\int_0^t (\int_0^u a(y,s)ds)(\int_u^t a(t,\tau)d\tau)du$$

$$- \int_0^t (\int_s^t a(t,u)du)f(s)ds$$

$$= f(t) - x_0\int_0^t a(t,u)du$$

$$- \int_0^t (\int_s^t a(t,\tau)d\tau)[f(s) - x_0\int_0^s a(s,u)du]ds$$

$$= \lambda(t) - \int_0^t [\int_s^t a(t,\tau)d\tau]\lambda(s)ds.$$

Thus the assumptions (ii) and (iii) yield $h(t) - x_0\int_0^t b(t,s)ds > 0$. Hence, in view of the observations made at the far end of section 1, it is clear that the solution $y(t)$ of (2.1), together with its derivative, is positive for all $t \in [0,\infty)$. Moreover, we have $0 < y'(t) \leq h(t)$ on $0 \leq t < \infty$. Therefore, by invoking Lemma 2.1, we conclude that the inequalities (a) and (b) hold for all $t \in [0,\infty)$ and this completes the proof.

We shall now give a result similar to Theorem 3.1 for convolution equation $(1.2)^*$.

Theorem 3.2. Suppose for $0 \leq t < \infty$ (i) $a(t) > 0$ (ii) $a'(t) < 0$, (iii) $f(t) > 0$, $f'(0) > x_0\, a(0)$, $f''(t) > a(0)\, f(t)$ hold. Then if $x(t)$ is a solution of $(1.2)^*$ on $0 \leq t < \infty$, then we have,

(a) $x_0 \leq x(t) \leq x_0 + F(t)$

(b) $0 < x'(t) \leq f(t)$

where $F(t) = \int_0^t f(s)ds.$

Proof. Take $\phi(t) = -\int_0^t a(\tau)d\tau.$ Then from assumptions (i) and (ii) it is clear that $\phi(0) = 0,$ $\phi'(t) = -a(t) < 0$ and $\phi''(t) = -a'(t) > 0.$ Moreover, from (2.5) and (2.6) we have

$$b(t) = -\int_0^t [\int_0^{t-\tau} a(s)ds]a(\tau)d\tau < 0 \quad \text{for} \quad 0 \leq t < \infty,$$

$$h(0) = f(0) > 0 \quad \text{and}$$

$$h'(t) = x_0\phi'(t) + \int_0^t \phi'(t-\tau)f(\tau)d\tau + f'(t) \tag{3.3}$$

Using the assumption (iii) and (3.3), we obtain

$$h'(0) = x_0\phi'(0) + f'(0) = f'(0) - x_0a(0) > 0$$

and

$$h''(t) = x_0\phi''(t) + \phi'(0)f(t) + \int_0^t \phi''(t-\tau)f(\tau)d\tau + f''(t)$$

$$> f''(t) - a(0)f(t) > 0$$

Thus we have $h(t) > 0$ and $b(t) < 0$ for all $t \in [0,\infty).$ Hence the Lemma 2.2 together with the observation at the end of section 1 yield that the solution $x(t)$ of $(1.2)^*$ along with its derivative $x'(t)$ are positive on $[0,\infty).$ Since $x(t) > 0$ and $a(t) > 0,$ it is clear from $(1.2)^*$ that $0 < x'(t) \leq f(t)$ and hence (a) and (b) are satisfied for all $t \in [0,\infty).$

Remark 3.2. If, in addition to the assumptions of Theorem 3.2, suppose $f(t) \in L^1[0,\infty),$ then the solution $x(t)$ of $(1.2)^*$ is also bounded on $[0,\infty).$

Remark 3.3. While discussing similar properties for solutions of Volterra integral equations of convolution type, Ling [5], Miller [6], and Levin [4] have assumed that $a(t) > 0$ and $a'(t) < 0$. When we integrate $(1.2)^*$ between 0 and t, then the resulting integral equation has the kernel $\int_0^t a(s)ds$ and hence the above assumptions cannot be satisfied for such a kernel. Therefore the results of [4], [5] and [6] are not applicable for $(1.2)^*$. Our endeavor here has been to find conditions such that the auxiliary (or resolvent) kernel $b(t)$ is negative so that the solutions could be positive. More specially, our results assert that the resolvent could be negative without the kernel being negative.

4. NONLINEAR EQUATIONS

Having known the conditions on the kernel $a(t)$ and the source function $f(t)$, for the solution $x(t)$ of the linear convolution equation $(1.2)^*$, we shall now discuss similar properties of solutions of the nonlinear equation

$$(4.1) \qquad x'(t) = f(t) - \int_0^t a(t-s)x(s)ds + G(t,s(t)), \quad x(0) = x_0 > 0,$$

where $0 \leq t < \infty$, under growth conditions on the nonlinear perturbation function $G(t,\phi)$. Assume that

(H_1) $G(t,\phi) \geq 0$ for all $t \geq 0$ and $\phi \geq 0$;

(H_2) there exists a positive continuous function $\lambda(t)$, such that

$$|G(t,\phi)| \leq \phi(t)(1+|\phi|),$$

where $\lim_{t\to\infty} \lambda(t) = 0$, $0 \leq t < \infty$;

(H_3) the resolvent $\phi(t)$ associated with the kernel $a(t)$ is positive for all $t \in [0,\infty)$, and $\phi(t) \in L^1[0,\infty)$.

Theorem 4.1. Suppose the hypothesis (H_1), (H_2) and (H_3) are satisfied. If the solution $y(t)$ of (1.2) with $y(0) = x_0$ is positive and bounded on $[0,\infty)$, then every solution $x(t)$ of (4.1) with $x(0) = x_0$ satisfies the inequality

$$x_0 \leq x(t) < \infty,$$

for all $0 \leq t < \infty$.

Proof. From Lemma 2.3, remark 2.2 and (4.1), it follows that

$$x(t) = y(t) + \int_0^t \phi(t-s)G(s,x(s))ds.$$

Define

(4.2) $$H(s,x) = G(s,x_+) \quad \text{for} \quad s \in [0,\infty), \quad x \in R,$$

where $x_+ = x$ if $x \geq 0$, $x_+ = 0$ if $x < 0$.

Thus $H(s,x) \geq 0$ for $s \in [0,\infty)$; $x \in R$. Let $\rho(t)$ be a solution of

(4.3) $$\rho(t) = y(t) + \int_0^t \phi(t-s)H(s,\rho(s))ds,$$

Therefore, if $y(t)$ is positive on $[0,\infty)$, then (H_1), (H_3), (4.2) and (4.3) imply that $\rho(t)$ is positive on $[0,\infty)$ and hence the solution $x(t)$ of (4.1) is positive on $[0,\infty)$. Moreover, from the boundedness of $y(t)$ on $[0,\infty)$, the Corollary 2.1 of Nohel [7] yields that $x(t)$ is also bounded on $[0,\infty)$. This completes the proof.

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