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To the Memory of Miron Nicolescu,
my teacher and guide from 1951-1959

SOLUTIONS OF ABSTRACT DIFFERENTIAL EQUATIONS WITH MINIMAL UNIFORM NORM

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In this Note we give a Banach space version of Theorem 5.1 in [1, p. 77].

1. Let X be a Banach space and $\mathbb{R}$ the real line. Then consider a continuous function $f = f(x,t)$, de $S \times \mathbb{R}$ dans X where S is a ball in X. Next, consider functions $x(t)$, $\mathbb{R} \to S$, such that $x'(t) = f(x(t),t)$, $t \in \mathbb{R}$. If $R_{x(\cdot)} = \{x(t),\ t \in \mathbb{R}\}$ is a relatively compact set in X we speak about a "compact solution". Any compact solution is obviously bounded on $\mathbb{R}$. We shall prove the following

Theorem. Let us assume that there exists a closed ball $S_1 \subseteq S$ such that

$$\sup_{\substack{x \in S_1 \\ t \in \mathbb{R}}} \|f(x,t)\| = L < \infty,$$

and let $x_0(t)$, $R \to X$, be a compact solution of $x_0'(t) = f(x_0(t),t)$, whose range $R_{x_0}(\cdot)$ is contained in S_1. Then, there exists a compact solution $y(t)$, $\mathbb{R} \to S_1$, of $y'(t) = f(y(t),t)$, having the minimal property: $\sup_{t \in \mathbb{R}} \|y(t)\| \leq \sup_{t \in \mathbb{R}} \|x(t)\|$ for any solution $x(t)$, $\mathbb{R} \to S$ of the same differential equation such that $R_{x(\cdot)} \subset \bar{R}_{x_0(\cdot)} = K$.

Proof. Let us define a real positive number λ by the relation

$$\lambda = \inf\{\sup_{t \in \mathbb{R}} \|x(t)\|,\ x(t) \in K\ \ \forall t \in \mathbb{R},\ x'(t) = f(x(t),t),\ t \in \mathbb{R}\}.$$

Then, a sequence $\lambda_n \geq 0$ is given by

$$\lambda_n = \inf\{\sup_{|t| \leq n} \|x(t)\|,\ x(t) \in K\ \ \forall t \in \mathbb{R},\ x'(t) = f(x(t),t),\ t \in \mathbb{R}\}.$$

Accordingly, $\forall n = 1,2,\ldots,\ \exists x_n(t) \in K\ \ \forall t \in \mathbb{R},\ \ x_n'(t) = f(x_n(t),t),\ \ t \in \mathbb{R}$.

and

$$\lambda_n \le \sup_{|t|\le n} \|x_n(t)\| < \lambda_n + \frac{1}{n}, \quad n = 1,2,\ldots$$

For any fixed $t_0 \in \mathbb{R}$, the set $\{x_n(t_0)\}$ is contained in the compact K; furthermore, $\forall t_1, t_2 \in \mathbb{R}$, it is

$$x_n(t_1) - x_n(t_2) = \int_{t_1}^{t_2} f(x_n(s),s)ds \quad \text{and} \quad \|x_n(t_1)-x_n(t_2)\| \le |t_2-t_1| \cdot L.$$

Therefore, a subsequence $\{x_{n_k}(t)\}$ $(n_1 < n_2,\ldots,\to\infty)$, can be extracted, such that $\lim_{k\to\infty} x_{n_k}(t) = y(t)$ exists, uniformly on compact subsets of $\mathbb{R}$. Thus, $y(t)$ is a continuous function, $\mathbb{R} \to K$.

Remark now that $f(x,t)$ is uniformly continuous on $K \times [-T,T]$ for any $T > 0$. This implies that $\lim_{k\to\infty} f(x_{n_k}(s),s) = f(y(s),s)$ uniformly on compact sets of $\mathbb{R}$.

Now, it is also

$$x_{n_k}(t) = x_{n_k}(0) + \int_0^t f(x_{n_k}(s),s)ds.$$

As $k \to \infty$ it follows:

$$y(t) = y(0) + \int_0^t f(y(s),s)ds,$$

and because of the continuity of y, $f(y(s),s)$ is also continuous, $y(t)$ has a derivative, and $y'(t) = f(y(t),t)$, $t \in \mathbb{R}$.

It remains to establish the equality: $\sup_{t\in\mathbb{R}} \|y(t)\| = \lambda$. First it is $\lambda_n \le \lambda$ $\forall n = 1,2,\ldots$ Remark that if $x(t) \in K$, $\forall t \in \mathbb{R}$ and $x'(t) = f(x(t),t)$ then

$$\sup_{|t|\le n} \|x(t)\| \le \sup \|x(t)\|.$$

Consider two sets of positive numbers:

$$A = \{\sup_{t\in\mathbb{R}} \|x(t)\|,\ x(t) \in K \quad \forall t \in \mathbb{R},\quad x' = f(x,t)\},$$

$$B_n = \{ \sup_{|t|\le n} \|x(t)\|,\quad x(t) \in K,\ \forall t \in \mathbb{R} \quad \text{and} \quad x' = f(x,t)\}$$

Then $\inf B_n \le \inf A$ ($\forall n = 1,2,\ldots$).

Consequently:

$$\inf\{ \sup_{|t|\le n} \|x(t)\|,\ x(t) \in K,\quad t \in \mathbb{R},\ x' = f(x,t)\} = \lambda_n \quad \text{is} \quad \le \lambda,\quad \forall n = 1,2\ldots$$

Next we shall use estimate:

$$\sup_{|t|\le n_k} \|x_{n_k}(t)\| < \lambda_{n_k} + \frac{1}{n_k} .$$

Take any $t_0 \in \mathbb{R}$ and then k_0 such that $|t_0| < n_k$ for $k \ge k_0$. It follows that

$$\|x_{n_k}(t_0)\| \le \lambda_{n_k} + \frac{1}{n_k} \le \lambda + \frac{1}{n_k} \quad \text{for} \quad k \ge k_0.$$

as $k \to \infty$ we obtain: $\|y(t_0)\| \le \lambda$. Thus $\sup_{t\in\mathbb{R}} \|y(t)\| \le \lambda$. However $y(t) \in K$ $\forall t \in \mathbb{R}$ and $y'(t) = f(y(t),t)$ implies, by definition of λ, that $\lambda \le \sup_{t\in\mathbb{R}} \|y(t)\|$. Hence $\sup_{t\in\mathbb{R}} \|y(t)\| = \lambda$, q.e.d.

REFERENCES

[1] Fink, A. M., *Almost Periodic Differential Equations*, Springer-Verlag, Berlin-Heidelberg-New York (1974).