LIBERTAS MATHEMATICA, VOL. 3, 1983

ON NONLINEAR DIFFERENCE APPROXIMATIONS TO NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

Lance D. Drager and William Layton

ABSTRACT

A nonresonance condition upon g is known under which the equation

(E) $$x'(t) + g(x(t),x(t-\tau)) = e(t), \quad -\infty < t < \infty,$$

will have a unique almost periodic solution, for any almost periodic forcing term $e(t)$. In this paper the equation (E) is discretized by a difference scheme. It is shown that the difference approximation to (E) exists and converges to the solution of (E) *without any assumptions upon* g *than the one needed to ensure that* (E) *will have a solution.* First, a detailed analysis of a central difference approximation to (E) is given. Later a general theory of convergence of difference schemes for (E) is developed.

It is shown that if the difference approximation is *stable* and *consistent* for approximating *periodic* solutions of linear, constant coefficient, ordinary differential equations then the scheme will be a convergent approximation for computing almost periodic solutions of the nonlinear functional differential equation (E).

I. INTRODUCTION

Many problems in ecology or mathematical biology are governed by functional differential equations of the general form

(1.1) $$x'(t) + g(x(t),x(t-\tau)) = e(t), \quad -\infty < t < \infty.$$

More generally, equations of this type occur whenever a system being modeled has a delayed response, incubation period or memory of previous states.

In this paper the problem of approximating solutions to such equations by a finite difference approximation is considered. Specifically, the case where $e(t)$ in (1.1) is a known forcing terms that is Besicovitch almost periodic (B^2) in t, and gives rise to a Besicovitch almost periodic solution $x(t)$, will be studied.

The nonlinearity g in (1.1) is assumed to satisfy a nonresonance condition which will be stated precisely in section 3. Loosely speaking, condition (NR) on g states that if the equation (1.1) is linearized by expanding g in a Taylor series and discarding quadratic and higher terms then the associated linearized operator is invertible on B^2. In [8] it was shown that this condition (NR) is sufficient to ensure that (1.1) will have a unique solution $x \in B^2$ for any $e \in B^2$. See also the related papers [4], [5] of the authors for further information.

First a detailed analysis is carried out for the simple central difference approximation to (1.1).

$$(1.2) \qquad \frac{x^h(t+h)-x^h(t-h)}{2h} + g(x^h(t),x^h(t-\tau)) = e(t), \quad -\infty < t < \infty.$$

Here x^h is the approximation to x and h is a parameter (representing a meshwidth) that will tend to zero. It will be shown that if the condition (NR) holds then the nonlinear difference equation (1.2) has a unique solution for h sufficiently small. Convergence of x^h to x will then be established and rates of convergence given.

In section 4 a general convergence theory is given for different schemes for (1.1). An approximation to (1.1) that arises by replacing the derivative by a general type of possibly nonsymmetric differencequotient is analyzed. Based on the careful analysis for the approximation (1.2) natural notions of stability and consistency of the approximation will be suggested. Using these, it will be shown that *if the difference approximation for periodic solutions of linear, constant coefficient, ordinary differential equations is stable and consistent then the resulting one for almost periodic solutions of functional differential equations will be convergent.*

Before the nolinear problem can be considered, the linear equation is analyzed in section 2. C shall denote a generic positive constant that may have differenti values in different places.

II. THE LINEAR CASE

Consider the linear equations given by the operators L and L_h

(2.1) $$Lx(t) \equiv x'(t) + ax(t) + bx(t-\tau) = e(t),$$

(2.2) $$L_h x^h(t) \equiv \frac{x^h(t+h)-x^h(t-h)}{h} + ax^h(t) + bx^h(t-\tau) = e(t)$$

If $e \in B^2(\mathbb{R};\mathbb{R})$ then e possesses an expansion in exponentials

$$e(t) = \sum_{j\in Z} e^{i\lambda_j t}, \quad -\infty < t < \infty,$$

where, since e is real valued, $\lambda_{-j} = -\lambda_j$ are real numbers, $a_j \in C$ and $a_j = \bar{a}_{-j}$, and the Fourier coefficients of e satisfy (see Besicovitch [1])

(2.3) $$||e||^2 \equiv \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |e(t)|^2 dt = \sum_{j\in Z} |e_j|^2 < \infty.$$

The following is known about the operator L (see Layton [8], Drager & Layton [4]).

Proposition 2.1. Let R denote the region $\{a+sb\,|\,-1 \le s \le +1\}$. Then, if $(a,b) \notin R$, (2.1) has a unique solution $x \in B^2$. Furthermore,

$$||x|| \le d^{-1}||e||, \quad d = \min\{|a+b|,|(a-b)|\}$$

so that $||L^{-1}|| \le d^{-1}$. []

Next consider the operator L_h. Expanding

$$x^h(t) = \sum_{j\in Z} x_j e^{i\lambda_j t}$$

gives that $L_h x^h(t)$ has the exponential series

$$L_h x^h(t) = \sum_{j\in Z} m_j^h x_j e^{i\lambda_j t},$$

$$m_j^h = \frac{e^{i\lambda_j h} - e^{-i\lambda_j h}}{2h} + a + be^{-i\lambda_j \tau}.$$

Thus, (2.2) has the formal solution

$$x^h(t) = \sum_{j\in Z} (m_j^h)^{-1} e_j e^{i\lambda_j t}. \tag{2.4}$$

Next, we conduct a *linear* stability analysis on (2.4). Provided h is sufficiently small, it will be shown that the approximation (2.1) is stable, i.e. higher Fourier modes of the solution are not increased.

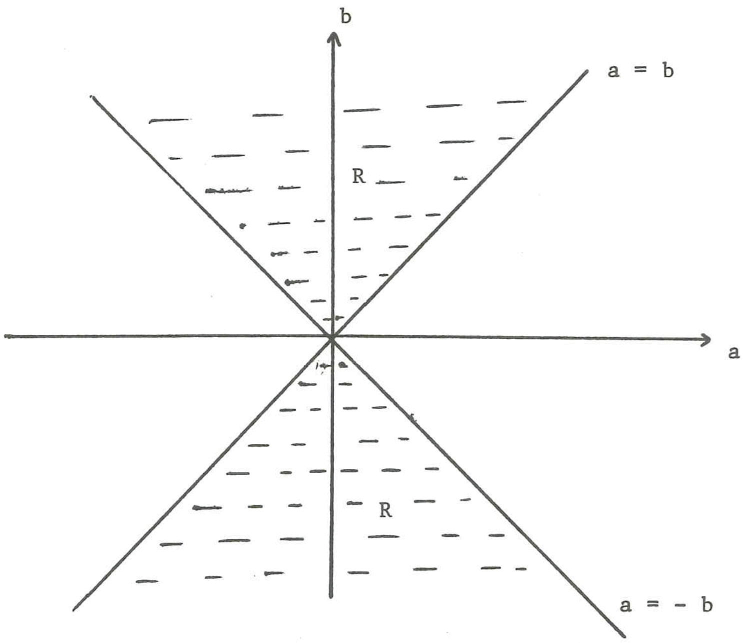

Figure 1. The region $R = \{ (a,b)\ a+sb = 0, -1 \leq s \leq +1 \}$ in the $a - b$ plane.

Proposition 2.2. (Linear Stability). Assume $(a,b) \notin R$ and h is sufficiently small. Then, a unique solution $x^h(t)$ to (2.1) exists in B^2. Furthermore, if d is as in proposition 2.1, x^h satisfies

$$\|x^h\| \le d^{-1}\|e\|,$$

so that $\|L_h^{-1}\| \le d^{-1}$.

Proof. Consider the quantities m_j^h in (2.4):

$$m_j^h = (a+b\cos(\lambda_j\tau)) + i\left(\frac{\sin(\lambda_j h)}{h} - b\sin(\lambda_j\tau)\right). \tag{2.5}$$

For arbitrary λ_j, τ the imaginary part of m_j^h can become arbitrarily small infinitely often. Thus, m_j^h must be bounded by its real part:

$$|m_j^h|^2 \ge (a+b\cos(\lambda_j\tau))^2 \ge \min\{|a+sb| : -1 \le s \le +1\}$$

$$|m_j^h| \ge d, \quad d = \min\{|a+b|, |a-b|\}.$$

Thus, using the Parseval formula (2.3)

$$\|x^h\|^2 = \sum_{j\in Z} |m_j^h|^{-2}|e_j|^2 \le \sum_{j\in Z} d^{-2}|e_j|^2$$

$$\|x^h\| \le d^{-1}\|e\|. \qquad \square$$

The purpose of this paper is to study the nonlinear equation. However, we shall first demonstrate convergence in the linear case. The statement $e \in B^{2,3}$ means that $e, e', e'', e''' \in B^2$, where the derivatives are interpreted in a weak sense appropriate for B^2. See the appendix for more details.

Theorem 2.1. Assume $(a,b) \notin R$, h is small enough for x^h to exists and $e \in B^{2,3}$. Then, the error in the approximation (2.2) to (2.1) satisfies

$$\|x-x^h\| \le Cd^{-4}h^2\|e\|_3,$$

where $d = \min\{|a+b|,|a-b|\}$ is a positive constant.

Proof. The solution to (2.1) is given by

$$x(t) = \sum_{j\in Z} m_j^{-1} e_j e^{i\lambda_j t}, \quad m_j = i\lambda_j + a + be^{-i\lambda_j \tau}.$$

Thus the error $x(t) - x^h(t)$ is given by

$$x(t) - x^h(t) = \sum_{j\in Z} [m_j^{-1} - (m_j^h)^{-1}] e_j e^{i\lambda_j t},$$

so that

$$\|x-x^h\|^2 = \sum_{j\in Z} |m_j^{-1} - (m_j^h)^{-1}|^2 |e_j|^2. \tag{2.6}$$

Next, note that m_j^h can be written as

$$m_j^h = m_j + i\lambda_j [\frac{e^{iz}-e^{-iz}}{2iz} - 1], \quad z = \lambda_j h.$$

A simple Taylor series calculation gives that:

$$m_j^h = m_j + i\lambda_j r(\lambda_j h), \quad r(z) = 0(z^2) \text{ as } |z| \to 0.$$

Thus, provided h is sufficiently small that

$$|\lambda_j r(\lambda_j h)| < d \le |m_j| \tag{2.7}$$

it follows that

$$|(m_j^h)^{-1} - m_j^{-1}| \le C|m_j|^{-2}|\lambda_j|^3 h^2. \tag{2.8}$$

This also implies the weaker estimate

$$|(m_j^h)^{-1}-m_j^{-1}| \leq Cd^{-2}|\lambda_j|^3h^2. \tag{2.9}$$

To complete the proof, decompose the sum (2.6) as

$$\|x-x^h\|^2 = (\sum_{j\in J_1} + \sum_{j\in J_2})|m_j^{-1}-(m_j^h)^{-1}|^2|e_j|^2,$$

where $J_1 = \{j \in Z|\ |\lambda_j r(\lambda_j h)| < d\}$ (i.e. those j's for which (2.7) holds) and $J_2 = Z - J_1$

Consider first the sum over J_1. Using (2.9),

$$\begin{aligned}\sum_{j\in J_1} |m_j^{-1}-(m_j^h)^{-1}|^2|e_j|^2 &\leq Ch^4d^{-4} \sum_{j\in J_1} |\lambda_j|^6|e_j|^2 \\ &\leq Ch^4d^{-4} \sum_{j\in Z} (1+|\lambda_j|^2)^3|e_j|^2 \\ &\leq Cd^{-4}h^4\|e\|_3^2.\end{aligned} \tag{2.10}$$

Next, consider the sum over J_2. Since $r(z) = 0(z^2)$, it follows that for $j \in J_2$

$$|\lambda_j| \geq Ch^{-2/3}d.$$

Thus, since $|m_j|^{-1}$, $|m_j^h|^{-1} \leq d^{-1}$

$$(\sum_{j\in J_2}) \leq 2d^{-2} \sum_{|\lambda_j|\geq Ch^{-2/3}d} |e_j|^2.$$

But, for $|\lambda_j| \geq Ch^{-2/3}d$ and $s \geq 0$

$$(Ch^{-2/3}d)^{-2}(1+|\lambda_j|^2)^2 \geq 1.$$

Therefore, for any s, $0 \leq s < \infty$,

$$\sum_{j\in J_2} |x_j - x_j^h|^2 \le Cd^{-2}(Ch^{-2/3}d)^{-2s} \sum_{|\lambda_j|\ge Ch^{-2/3}d} (1+|\lambda_j|^2)^s |e_j|^2$$

$$(2.11) \qquad \le Cd^{-(2+2s)}h^{4s/3} \sum_{j\in Z} (1+|\lambda_j|^2)^s |e_j|^2$$

$$\le Cd^{-(2+2s)}h^{4s/3} \|e\|_s^2 .$$

Combining (2.10) and (2.11) gives

$$\|x-x^h\|^2 \le Cd^{-4}h^4\|e\|_3^2 + Cd^{-(2+2s)}h^{4s/3}\|e\|_s^2 .$$

Taking the square root of the above inequality and picking $s \ge 0$ such that $2s/3 = 2$ or $s = 3$ gives that

$$\|x-x^h\| \le Cd^{-4}h^2\|e\|_3 ,$$

which is the desired result. □

Remark: Note that d is the distance from the point (a,b) to the region R so that this error estimate becomes less useful as $(a,b) \to R$ (i.e. as $d \to 0$). This reflects the fact that the continuous equation can be ill-posed for $(a,b) \in R$.

III. THE NONLINEAR EQUATION

In this section the difference approximation (1.2) to (1.1) is considered. Because of the work in the previous section in obtaining precise estimates in the linear case it will be possible to prove convergence of x^h to x in the nonlinear case. Since we are not approximating an initial value problem but rather a type of boundary value problem (with "finite energy" radiation type boundary conditions at $\pm\infty$), asymptotic error estimates in the linear case are insufficient to prove convergence in the nonlinear case. The estimate of $\|L_h^{-1}\|$ in proposition 2.2 will be essential in proving convergence.

First the basic results on the continuous equation will be reviewed briefly. The equation (1.1) will have a unique almost periodic solution provided the following nonresonance condition upon g holds.

Conditions (NR): Assume $g \in C^1(\mathbb{R} \times \mathbb{R} : \mathbb{R})$ and that $(g_x(u,v), g_y(u',v'))$: $u,v,u',v' \in \mathbb{R}$ is bounded away from R, and away from infinity.

(NR) is thus sharp when (1.1) reduces to a linear equation, see Drager and Layton [4], Layton [8] for more details.

Theorem 3.1. [Theorem 4.1 of [8]]. Assume (NR) holds, and $e \in B^2$. Then, there is a unique solution $x \in B^2$ to (1.1). □

The condition (NR) can be restated by enclosing the set $\{(g_x(u,v), g_y(u',v')) : u,v,u',v' \in R\}$ in a box whose sides have slope ± 1. This leads to the following restatement of (NR).

Condition (NR)': Assume $g \in C^1$ and that there are numbers p, q, r, s, such that $p < q < 0 < r < s$, $|p-q| = |r-s|$ and either

(i) $p \leq g_y(u,v) - g_x(u',v') \leq q$, $r \leq g_x(u',v') + g_y(u,v) \leq s$

holds for all u, v, u', $v' \in \mathbb{R}$, or

(ii) $p \leq g_x(u',v') + g_y(u,v) \leq q$, $r \leq g_y(u,v) - g_x(u',v') \leq s$

holds for all u, $v \in \mathbb{R}$.

Part (i) of (NR)' above corresponds to the case when $\{(g_x, g_y)\}$ lies to the right of R and part (ii) when $\{(g_x, g_y)\}$ to the left of R. Figure 2 is a sketch of part (i) of the above.

The following technical estimate from the proof of Theorem 4.1 of [8] will also be necessary.

Lemma 3.1. Assume (NR)' also holds. Then, if a and b are chosen to be the center of the square in which (g_x, g_y) lies in (NR)' there follows

$$\alpha \equiv d^{-1} \sup \{ |a - g_x(u,v)| + |b - g_y(u',v')| : u,v,u',v' \in \mathbb{R} \} < 1. \quad \square$$

For example, in case (i) of (NR)' the choice of (a,b) in the previous lemma is given by

$$a = \frac{1}{4}(r + s - p - q), \quad b = \frac{1}{4}(p + q + r + s).$$

Henceforth g will be assumed to satisfy (NR)'.

Theorem 3.2. Assume (NR)' holds and $e \in B^2$. Then there is a unique solution $x^h(t)$ to the difference equation (1.2) in B^2.

Proof. The difference equation (1.2) is rewritten in the following way

$$\frac{x^h(t+h)-x^h(t-h)}{2h} + ax^h(t) + bx^h(t-\tau) = ax^h(t) + bx^h(t-\tau) - g(x^h(t),x^h(t-\tau)) + e(t).$$

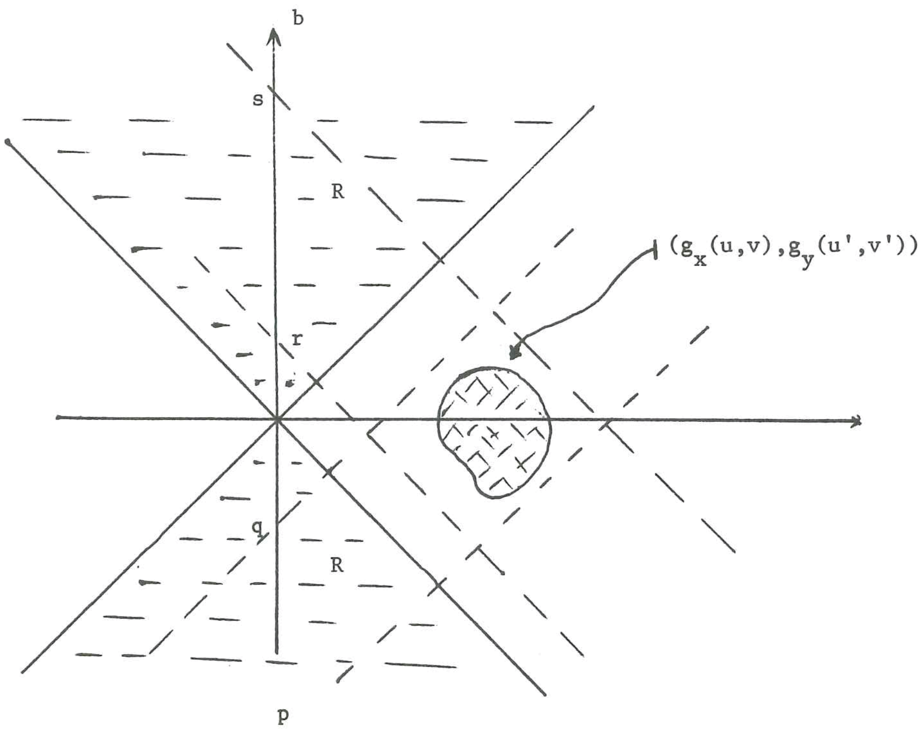

Figure 2. Part (i) of assumption (NR)'. The set $\{(g_x(u,v),g_y(u',v')) : u, v, u', v' \in \mathbb{R}\}$ is contained in a square bounded either to the left or to the right of the region R.

Define $(Nv)(t) = av(t) + bv(t-\tau) - g(v(t),v(t-\tau))$ and $L_h v$ as in section 2. If $(a,b) \notin R$ then (1.2) is equivalent to the fixed point problem

$$x^h(t) = L_h^{-1}[N(x^h(t)) + e(t)] \equiv F_h(x) + L_h^{-1}e.$$

Using the mean value theorem the operator N can be bounded by

$$||N(x)-N(y)|| \leq \sup\{|a-g_x(u,v)| + |b-g(u',v')| : u,v,u',v' \in \mathbb{R}\}||x-y||.$$

Thus F_h satisfies

$$||F_h(x)-F_h(y)|| \leq ||L_h^{-1}|| \quad ||N(x)-N(y)||$$

$$\leq d^{-1}\sup\{|a-g_x(u,v)| + |b-g_y(u',v')| : u,v,u',v' \in \mathbb{R}\}||x-y||.$$

Applying lemma 3.1 gives that for one choice of $(a,b) \notin R$

$$||F_h(x)-F_h(y)|| \leq \alpha||x-y|| \qquad \alpha < 1,$$

and existence - uniqueness of x^h follow from the contraction mapping theorem. □

Next, we give the error estimate.

Theorem 3.3. Assume (NR)' holds, $x \in B^{2,3}$ and $y(t) \equiv g(x(t),x(t-\tau)) \in B^{2,3}$. Then,

$$||x-x^h|| \leq Ch^2\{||x||_3 + ||y||_3\}.$$

Proof. Using the notation of section 2 and the previous theorem, x and x^h satisfy

$$x(t) = L^{-1}[N(x) + e],$$

$$x^h(t) = L_h^{-1}[N(x^h) + e].$$

Thus, $x - x^h$ is given by

$$x - x^h = L_h^{-1}N(x^h) - L_h^{-1}N(x) + (L_h^{-1}-L^{-1})N(x).$$

Applying the convergence result of Theorem 2.1 to the $(L_h^{-1}-L^{-1})N(x)$ term and using the fact that $L_h^{-1}N$ is a contraction gives

$$\|x-x^h\| \leq \alpha\|x-x^h\| + Ch^2\|N(x)\|_{2,3}, \qquad \alpha < 1,$$

from which the theorem follows.

IV. A GENERAL CONVERGENCE THEORY

Let T_h denote the forward shift operator

$$T_h v(t) = v(t+h)$$

and let T_h^j denote powers of T_h, i.e.

$$T_h^j v(t) = v(t+jh), \qquad j \in Z.$$

If $\ell(z)$ is a truncated Laurent series in z

$$\ell(z) = \sum_{j=-N}^{M} \ell_j z^j,$$

then a difference approximation to (1.1) can be written in the general form:

(4.1) $$h^{-1}\ell(T_h)x^h(t) + g(x^h(t),x^h(t-\tau)) = e(t).$$

We shall prove a general convergence theorem for the approximation (4.1). When the difference scheme (4.1) is stable and consistent in the linear case then it will converge for the nonlinear equation (1.1). The precise notions of "stability" and "consistency" will be developed next.

Let L denote the *ordinary* differential operator

$$Lx \equiv x'(t) + ax(t), \qquad a \neq 0.$$

Then, the approximation to $Lx = e^{i\lambda t}$ induced by $\ell(z)$ is given by

$$h^{-1}\ell(T_h)x^h(t) + ax^h(t) = e^{i\lambda t}. \tag{4.2}$$

Complex exponentials are eigenfunctions of the shift operator. Thus, the solution to (4.2) must have the form $x^h(t) = Ce^{i\lambda t}$. Substitution gives

$$C\{h^{-1}\sum_{j=-N}^{M} \ell_j e^{i\lambda(t+jh)} + ae^{i\lambda t}\} = e^{i\lambda t}$$

$$C\{h^{-1}\sum_{j=-N}^{M} \ell_j e^{ijh\lambda} + a\}e^{i\lambda t} = e^{i\lambda t}$$

and thus $C = [a + h^{-1}\sum_{j=-N}^{N} \ell_j e^{ijh\lambda}]^{-1}$.

Hypothesis (H1). Given $a \neq 0$ there is an $h_0 > 0$ such that for $0 \leq h \leq h_0$

$$a + h^{-1}\sum_{j=-N}^{M} \ell_j e^{i\lambda jh} \neq 0$$

for all $\lambda \in \mathbb{R}$.

For stability of the scheme it is necessary that the solution operator to L_h:

$$e^{i\lambda t} \to [a + h^{-1}\sum_{j=-N}^{M} \ell_j e^{ij\lambda h}]^{-1}e^{i\lambda t}$$

not increase the energy of the higher Fourier modes of the solution.

The next hypothesis is a statement of this stability condition.

Hypothesis (H2). Given $a \neq 0$ a real number, there is a nonnegative function $\varepsilon(h) \to 0$ as $h \to 0$ such that for all $\lambda \in \mathbb{R}$

$$|a + h^{-1} \sum_{j=-N}^{M} \ell_j e^{ijh\lambda}|^{-1} \leq (|a|-\varepsilon(h))^{-1}.$$

Remark. This is clearly equivalent to asking that

$$|a + h^{-1} \sum_{j=-N}^{M} \ell_j e^{ijh\lambda}| \geq |a| - \varepsilon(h), \quad h \text{ sufficiently small.}$$

When the approximation (4.2) is consistent, this will always hold for fixed λ since

$$h^{-1} \sum_{j=-N}^{M} \ell_j e^{ijh\lambda} \to \frac{d}{dt} e^{i\lambda t}\Big|_{t=0} = i\lambda$$

so that

$$|a + h^{-1} \sum_{j=-N}^{M} \ell_j e^{ijh\lambda}| = |i\lambda + a + \varepsilon(h,\lambda)|,$$

where $\varepsilon(h,\lambda) \to 0$ as $h \to 0$. The stability hypothesis is that this must occur uniformly in the frequency λ.

This is not a very restrictive condition, we shall show that, for example, the forward Euler method satisfies (H2).

If $x(t) = e^{i\lambda t}$, then

$$Lx(t) = [i\lambda + a]e^{i\lambda t}.$$

If the difference scheme L_h is to be *consistent* with L then $h^{-1} \sum_{j=-N}^{M} \ell_j e^{ijh\lambda}$ must approximate $i\lambda$ well in a neighborhood of zero.

Hypothesis (H3). Assume that there is a function $r(z) = 0(z^k)$ as $|z| \to 0$ for some $k > 0$ such that

$$h^{-1} \sum_{j=-N}^{N} \ell_j e^{ijh\lambda} = i\lambda[1 + r(\lambda h)].$$

The proof of the convergence of the scheme (4.2) to x in the linear case is then identical to the proof of theorem 2.1. We shall state the result without further comment.

Theorem 4.1. Assume $a \neq 0$, $Lx = e$ and $L_h x^h = e$. Assume further that (H1), (H2) and (H3) hold and $e \in B^{2,k+1}$. Then, $x^h \to x$ in B^2 and the error satisfies

$$\|x-x^h\| \le Ch^k \|e\|_{k+1},$$

where C is independent of h, x and e but depends upon a. □

The proof in the nonlinear case goes through with only one minor modification. We shall give a sketch and indicate where it occurs.

Theorem 4.2. Assume (NR)' holds, h is sufficiently small and (H1), (H2) and (H3) hold. Then $x^h(t) \in B^2$ exists uniquely. If $x \in B^{2,k+1}$, $y \equiv g(x(t),x(t-\tau)) \in B^{2,k+1}$, then

$$\|x-x^h\| \le Ch^k\{\|x\|_{k+1} + \|y\|_{k+1}\}.$$

Proof. First note that the square in the second formulation of (NR) can be situated so that it is symmetric about the a-axis. This implies, in particular, that the b-coordinate of its center must be zero.

Next rewrite (4.1) as in section 3

$$x^h(t) = L_h^{-1}[N(x^h(t)) + e(t)] \equiv F_h(t) + L_h^{-1}e.$$

For h sufficiently small, F_h is a contraction. To see this note that (H2) gives the estimate on L_h^{-1}:

$$\|L_h^{-1}\| \le \frac{1}{|a|-\varepsilon(h)}, \quad \varepsilon(h) \to 0 \quad \text{as} \quad h \to 0.$$

Proceeding as in the proof of theorem, consider for $x,y \in B^2$

$$\|F_h(x)-F_h(y)\| \le \frac{1}{|a|-\varepsilon(h)} \sup\{|a-g_x(u,v)|+|b-g_y(u',v')| : u,v,u',v' \in \mathbb{R}\}\|x-y\|.$$

Picking a to be the first coordinate in the (a,b) plane given in lemma 3.1 and noting that the choice of b in lemma 3.1 will be zero gives that

$$\frac{1}{|a|}\sup\{\ldots\} = d^{-1}\sup\{\ldots\} = \alpha < 1.$$

Since $\varepsilon(h) \to 0$, it follows that for h sufficiently small

$$\frac{1}{|a|-\varepsilon(h)}\sup\{\ldots\} = \alpha' < 1.$$

Thus, for $h \le h_0$ (sufficiently small) x^h exists uniqely.

The error estimate follows from the proof of theorem 3.3, mutatis mutandis. $\square$

Thus, *the convergence of a fairly general class of difference approximations to almost periodic solutions of nonlinear functional differential equations can be reduced to the stability and consistency of the schemes on periodic* ($e^{i\lambda t}$) *solutions of a constant coefficient, linear, ordinary differential equation.*

EXAMPLE: The forward Euler method.

Consider the simple forward difference approximation to L:

$$L_h x^h = \frac{x^h(t+h)-x^h(t)}{h} + ax^h(t).$$

If x^h is the solution to $L^h x^h = e^{i\lambda t}$, then a simple computation gives

$$x^h(t) = [(e^{i\lambda h}-1)/h + a]^{-1} e^{i\lambda t}$$

Consider the multiplier $m_\lambda^h = (e^{i\lambda h}-1)/h + a$:

$$\begin{aligned} |m_\lambda^h|^2 &= (h^{-1}\cos(\lambda h)-h^{-1}+a)^2 + h^{-1}\sin(\lambda h))^2 \\ &= h^{-2}\cos^2(\lambda h) + 2(a-h^{-1})(h^{-1}\cos(\lambda h)) + (a-h^{-1})^2 \\ &\qquad + h^{-2}\sin^2(\lambda h) \end{aligned}$$

$$= h^{-2} + 2(a-h^{-1})(h^{-1}\cos(\lambda h)) + (a-h^{-1})^2$$

$$\geq h^{-2} - 2(h^{-1}-a)(h^{-1}) + (h^{-1}-a)^2,$$

for h sufficiently small that $h^{-1} - a > 0$. Thus,

$$|m_\lambda^h|^2 \geq (h^{-1}-(h^{-1}-a))^2 = a^2.$$

Thus, (H1), and (H2) hold ((H2) with $\varepsilon(h) \equiv 0$). (H3) holds since a Taylor's series argument yields

$$\frac{e^{i\lambda h}-1}{h} = i\lambda + i\lambda\, O(\lambda h).$$

V. REMARKS AND CONCLUSIONS

In this paper difference approximations to nonlinear functional differential equations have been analyzed in the almost periodic case. It is shown that the difference approximation will converge to the true solution *with no extra assumption on the nonlinearity other than the one needed for the existence of a solution*. Convergence is shown for a wide class of difference schemes.

Aside from the obvious problem of approximating the solutions of nonlinear differential equations, the theorems stated herein can be thought of as giving a theory of nonlinear recursion relations, or (with a change in emphasis in the presentation) a constructive existence theorem for the nonlinear differential equation.

The extension of this material systems of equations of the same form as (1.1) is almost immediate under the added hypothesis that $g \in C^1(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$ is a gradient operator with respect to each of its variables separately (see the last section of [8] for more details). Also, higher order equations can be treated provided the program of [8] can be carried out on the continuous equation.

Let us remark that the regularity required of $x(t)$ can be reduced to smoothness conditions on g and the forcing term $e(t)$. This was done in the periodic case in [3], corresponding results can be expected in the almost periodic case.

Since the existence of $x^h(t)$ was proven using a contraction argument, the difference equations giving $x^h(t)$ could be solved iteratively by the modified Picard iteration used in the proof of the theorem.

VI. APPENDIX: ALMOST PERIODIC FUNCTIONS AND DIFFERENTIAL EQUATIONS

The theory of almost periodic functions was begun by the Danish mathematician Harald Bohr. The space of uniformly almost periodic functions introduced by Bohr can be characterized in several ways, for example:

Definition. $f:\mathbb{R} \to \mathbb{R}$ is uniformly almost periodic, written $f \in AP(\mathbb{R};\mathbb{R})$, if, for any $\varepsilon > 0$, there is a real trigonometric polynomial $T_\varepsilon(t)$ such that

$$|f(t)-T_\varepsilon(t)| < \varepsilon, \quad -\infty < t < \infty.$$

In brief, the space $AP(\mathbb{R};\mathbb{R})$ can be characterized as the closure of the real trigonometric polynomials

$$T(t) = \sum_{s=-n}^{n} a_s e^{i\lambda_s t}, \quad a_s = \bar{a}_{-s} \in C, \quad \lambda_{-s} = -\lambda_s \in \mathbb{R}, \tag{6.1}$$

where $-\infty < t < \infty$, in the uniform norm.

There is also an equivalent pointwise definition of the space $AP(\mathbb{R};\mathbb{R})$ given in terms of "ε-translation numbers" of the almost periodic function $f(t)$, see Corduneanu [2] for more information.

This definition has been generalized in several directions. The one used in this paper was introduced by Besicovitch [1] who proved that if another norm is chosen the resulting space of almost periodic functions becomes a Hilbert space with a continuum of orthonormal basis functions $\{e^{i\lambda t} | \lambda \in \mathbb{R}\}$ in which the Parseval formula holds.

The space of Besicovitch almost periodic functions, $B^2(\mathbb{R};\mathbb{R})$, is the closure of the set $\mathcal{T}$ of real trigonometric polynomials (6.1) in the norm

$$||f||^2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 dt.$$

As usual, functions $\eta \in B^2$ whose difference has norm zero are identified. An alternate definition of B^2 is as follows. B^2 consists of all real trigonometric series

$$f(t) = \sum_{j=-\infty}^{\infty} a_j e^{i\lambda_j t}, \quad a_{-j} = \bar{a}_j, \quad \lambda_{-j} = -\lambda_j, \tag{6.2}$$

for which the B^2 norm is finite:

$$\|f\|^2 = \sum_{j=-\infty}^{\infty} |a_j|^2 < \infty.$$

In Layton and Drager [4], a one parameter family of such spaces, $B^{2,s}(\mathbb{R},\mathbb{R})$ is introduced to study the solutions of differential equations. Using the Fourier series definition (6.2) of B^2, $B^{2,s}$ can be defined in the following way. Write

$$f(t) = \Sigma f_j e^{i\lambda_j t} \in B^{2,s}(\mathbb{R};\mathbb{R})$$

(where $\lambda_{-j} = -\lambda_j$, $f_{-j} = \bar{f}_j$) provided

$$\|f\|_s^2 \equiv \sum_{j=-\infty}^{\infty} (1+|\lambda_j|^2)^s |f_j|^2 < \infty.$$

$B^{2,s}$ can be characterized in terms of weak-B^2-derivatives of elements in B^2, see [4] for more details. When the functions involved are periodic $\|\cdot\|_s$ reduces to the norm on the Sobolev space W_2^s.

The almost periodic theory of differential equations was investigated soon after almost periodic functions were introduced when many people realized the intimate connection between almost periodic functions and many problems of mathematical physics and, in particular, to certain stability problems in celestial mechanics. For the basic theory of ordinary and partial differential equations, see the book [2].

The theory of periodic and almost periodic solutions of *functional* differential equations was treated in the fundamental paper [6] of Hale. In the nonlinear case, he proved a result along the following lines. If the linear equation

$$x'(t) + Ax(t) = e(t)$$

is uniformly asymptotically stable and is $f(x,t)$ is a nonlinear term that is almost periodic in t uniformly in x, and Lipschitz in x, then the nonlinear equation

$$x'(t) + Ax(t) = \varepsilon f(tmx_t)$$

will have a unique uniformly almost periodic solution for ε sufficiently small. See also the book [7] by Hale.

In the paper [8] the second author considered this problem of existence of almost periodic solutions to such nonlinear functional differential equations when the forcing term is Besicovitch almost periodic. Since the space B^2 is a Hilbert space, a nonresonance type condition (condition (NR) in section 3) was found under which such solutions could be shown to exist. Thus, using the extra Hilbert space structure, the small parameter restriction mentioned above was avoided. In a later paper [5], Drager and Layton showed that the weak B^2 solutions obtained in [8] are in fact C^k $(k \geq 1)$ classical solutions when the forcing term is continuous.

REFERENCES

[1] Besicovitch, A. S., *Almost Periodic Functions*, Cambridge Univ. Press, Cambridge (1932).

[2] Corduneanu, C., *Almost Periodic Functions*, Wiley Interscience, New York (1968).

[3] Drager, L., Layton, W., "On the smoothness of the solution operator for certain nonlinear functional differential equations in the class of periodic functions," in preparation.

[4] Drager, L., Layton, W., "Parameter dependent solutions to functional differential equations subject to almost periodic forcing," in preparation.

[5] Drager, L., Layton, W., "Periodic, almost periodic and bounded solutions of functional differential equations," in preparation.

[6] Hale, J., "Periodic and almost periodic solutions of functional differential equations," Arch. Rat. Mech. Anal., v. 15, 289-304 (1964).

[7] Hale, J., *Theory of Functional Differential Equations*, Springer-Verlag, New York (1977).

[8] Layton, W., "Existence of almost periodic solutions to delay differential equations with Lipschitz nonlinearity," to appear.