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To the memory of Miron Nicolescu

SOME ALMOST PERIODICITY CRITERIA FOR ORDINARY DIFFERENTIAL EQUATIONS

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In their paper [1],H. Bohr and O. Neugebauer considered for the first time the almost periodicity problem of bounded solutions of ordinary differential equations/systems.Dealing (basically) with the finite-dimensional linear system $x' = Ax + f(t)$,where A is a constant matrix,and f(t) is an almost periodic vector valued function,Bohr and Neugebauer have proved the following classical result:Any bounded solution on the whole real axis is necessarily almost periodic.The proof of this result is rather elementary,and besides the above quoted paper it can be found in several monographs and textbooks.See,for instance,[7] ,[16] ,[20] .

The research direction initiated by Bohr and Neugebauer turned out to be a very fruitful one,and a good deal of results has been obtained particularly during the last three decades.Most of the efforts are now concentrated on the almost periodicity properties of solutions to partial differential equations,to different other types of functional equations,or to abstract differential equations.In general,more sophisticated types of equations than the ordinary ones drew the attention of many researchers.Nevertheless,the almost periodicity problem of bounded solutions for ordinary differential equations is still a challenge for many researchers in this field.The list of references at the end of this paper is an illustration in this regard.

Without any attempt to discuss this topic in full generality,we would like to emphasize a few features of the research work conducted in this field.

J. Favard [15] has undertaken a systematic investigation of linear systems with almost periodic coefficients,obtaining several significant results.For the first time,Favard recognized the significance of the property of "separation" of solutions of almost periodic (finite-dimensional) systems,as well as the role of the "hull" of such a system.Later on [2] ,L.Amerio extended such properties to the case of almost periodic nonlinear systems.

Amerio has provided also a criterion of existence for almost periodic solutions, in which a basic assumption is the existence of a semitrajectory of the almost periodic system x' = f(t,x) that belongs to a compact set in R^n. Results of this kind have been also obtained by W. A. Coppel [6], whose results involve the asymptotic almost periodicity of the semitrajectory.

The case of linear equations in Banach spaces, with constant or almost periodic operators, has been investigated by several authors. Yu. L. Daleckii and M. G. Krein [14] are mainly concerned with the equation x' = Ax + f(t), where A is a linear stationary operator on a Banach space E, while f is an almost periodic map from R into E. Spectral conditions are indicated, under which the existence of almost periodic solutions is assured. In their book [24], J.L.Massera and J.J. Schäffer obtain almost periodicity criteria for linear equations with nonstationary operators x'(t) = A(t)x(t) + f(t), where both A(t) and f(t) are almost periodic. S.Zaidman [34] , [35] has been particularly concerned with almost periodicity in abstract equations, obtaining several results which can be applied to both ordinary and partial differential equations. In [13], C.Corduneanu and J.A.Goldstein deal with some classes of nonlinear abstract equations, finding some almost periodicity criteria. The common feature of the research work conducted in this area is the constant use of functional analytic methods.

M.A.Krasnoselskii and his coworkers [20] have extensively used the method of integral equations. The basic idea relies on the use of certain Green's type functions, which naturally appear in the inversion of some differential operators with almost periodic coefficients. By means of the Green's function, the study of nonlinear almost periodic equations can be reduced to the investigation of bounded solutions (on the whole real axis) to nonlinear integral equations. This approach conducted to interesting results which seem to be of a rather different nature than those obtained by the use of other available methods.

Another valuable method in the study of almost periodic differential equations is based on the construction of the Fourier series. For earlier results obtained in this regard, see I.G.Malkin [23]. More recent contributions have been brought by W.A.Coppel [6]. We think that this method has yet a good potential for further application.

Finally, another method that has been used in investgating almost periodicity properties for ordinary differential equations is the Liapunov's function method. Most of the significant results obtained by means of this method are included in Yoshizawa's book [33], this author being a main contributor to the topic.

The results we shall give in this paper are also based on the use of some Liapunov's type functions.The new feature of the results consists in the fact that we rely on certain differential inequalities [8], [9] ,that seem to be particularly adequate when the unknown function is bounded on the entire real (as a potential almost periodic solution always is).It should be also noticed that our approach provides connections in between the almost periods of the right hand side of the equation/system,and those of the solution.It is also proper to stress the fact that we do not rely on the "hull" of the almost periodic system. The conditions imposed to the almost periodic system for obtaining the almost periodicity of the bounded solution lead immediately to the uniqueness of such a solution (within the set of those having trajectories in a given compact set). Let us mention that several authors (see [5] , [28], [31],as well as the references in those papers) have studied similar problems to those dealt with in this paper,using inequalities which are related to the ones we shall use. These authors choose the Hilbert/Banach space as the framework,but the hypotheses of continuity leave little possibility of applying their results to partial differential equations.The paper [5] has some potential applications to infinite dimensional problems.

QUALITATIVE DIFFERENTIAL INEQUALITIES

The method we shall apply in deriving new criteria of almost periodicity of bounded solutions for ordinary differential equations is the <u>comparison method</u> , i.e.,the simultaneous use of Liapunov"s functions and differential inequalities. Since the almost periodic functions (Bohr) are bounded on the real axis,the differential inequalities involved in discussion will usually hold on the entire real line. No initial conditions will be required,excepting the case when the inequalities are restructed to a half axis.We already dealt with such kind of inequalities in [8] and [9],being mainly concerned with the almost periodicity of solutions to partial differential equations.In order to include the almost periodicity in Stepanov's sense,we considered in [10] and [11] some qualitative inequalities which are generalized in this section (Lemmas 3 and 4). For the reader's convenience,we shall state here the results obtained by the author in [8] and [9].

Lemma 1

Let x(t) be a differentiable map from R into R_+,such that

(1) $$x'(t) \geq \omega(x(t)) \ , \ t \in R \ ,$$

with ω continuous from R_+ into R,and satisfying the condition $\omega(x) > 0$ when $x > M > 0$. If $x(t)$ is bounded on the real line,then it must verify

(2) $$x(t) \leq M \ , \ t \in R.$$

The proof of this Lemma is provided in[9] .More precisely,in order to obtain the statement ,one has to change t into -t in [9] .

It should be also noticed that instead of the derivative of $x(t)$,one can use the upper Dini derivative at the right.

Lemma 2

Let $x(t)$ be continuously differentiable of the second order,from R into R_+, and such that

(3) $$x''(t) \geq \omega(x(t)) \ , \ t \in R \ ,$$

with ω satisfying the assumptions of Lemma 1. Then the conclusion of Lemma 1 is valid,i.e.,(2) holds true.

The proof of Lemma 2 is given in [8] as well as the proof of the following

Corollary

Let $x(t)$ be a map from R_+ into itself,continuously differentiable of the second order,and such that (3) holds true on R_+.If $x(t)$ is bounded,and in addition $x(0) = 0$, or $x'(0) - hx(0) = 0$, $h > 0$,then the estimate (2) is valid on R_+.

While Lemmas 1 and 2 proved to be useful tools in establishing almost periodicity of bounded solutions to equations/systems which enjoy almost periodicity properties in Bohr's sense [8], [9], somewhat different results are necessary when the right hand side of the equation/system is almost periodic in a weaker sense (say,for instance,in the sense of Stepanov).It has been noticed in [10] and [11], that some differential inequalities,involving functions which are not necessarily bounded on R,can be dealt with in view of obtaining boundedness and estimates for their solutions.

Let us first consider the differential inequality

(4) $$x'(t) \geq kx(t) - f(t)\lambda(x(t)) \ , \ t \in R \ ,$$

where $k > 0$ is a constant,f is a locally integrable map from R into R_+ ,such that

(5) $$\sup \int_t^{t+1} f(s)ds = \|f\|_M < \infty \ , \ t \in R \ .$$

Furthermore, we shall assume that $\lambda(r)$ is a map of R_+ into itself, continuous and nondecreasing, such that $\lambda(r) = 0$ implies $r = 0$, while

$$\mu(r) = r\{\lambda(r)\}^{-1} , \quad r > 0 , \quad \mu(0) = 0, \tag{6}$$

belongs to the class K, i.e., vanishes at the origin, is strictly increasing for $r > 0$, is continuous and tends to infinity with r. It is obvious that the inverse of such a function always exists, and belongs also to the class K.

Lemma 3

Let $x(t)$ be a differentiable map from R into R_+, such that (4) holds true, where k, f and λ verify the above stated conditions. If $x(t)$ is bounded on R, then necessarily

$$\sup x(t) \leq \mu^{-1}(K\|f\|_M) \quad , \ t \in R, \tag{7}$$

where K is a constant depending on k only.

Proof. If one multiplies both sides of (4) by $\exp(kt)$, and integrates from t_o to t ($t_o < t$), one obtains

$$x(t) \leq x(t_o)\exp\{-k(t - t_o)\} + \int_{t_o}^{t}\lambda(x(s))f(s)\exp\{-k(t - s)\}ds . \tag{8}$$

Since $x(t)$ is bounded, (8) yields for $t_o \to -\infty$ and fixed t :

$$x(t) \leq \lambda(m) \int_{-\infty}^{t} f(s)\exp\{-k(t - s)\}ds , \tag{9}$$

where $m = \sup x(t)$, $t \in R$. Taking the supremum (with respect to $t \in R$) in both sides of (9), one obtains after dividing by $\lambda(m)$;

$$\mu(m) \leq K\|f\|_M , \tag{10}$$

which is obviously equivalent to (7). (See [24] for the definition and the properties of the norm in the space M).

Remark 1

The results obtained by the author in 10 and 11 can be derived also from Lemma 3, choosing $\lambda(r) = 1$, and $\lambda(r) = \sqrt{r}$.

Remark 2

Instead of (4), one could obviously deal with the equivalent inequality

$$x'(t) \leq -kx(t) + f(t)\lambda(x(t)) \quad , \ t \in R , \tag{4'}$$

with k, f and λ as above.

Remark 3

If we restrict the inequality (4) to the half-axis R_+ , one obtains easily the same estimate (7) for any bounded solution (on R_+) of (4):

(10') $$\sup x(t) \le \mu^{-1}(K\|f\|_M) , \quad t \in R_+,$$

where $\|f\|_M$ is defined by (5), with $t \in R_+$ instead of $t \in R$.

Of course , (4') restricted to R_+ leads to a different estimate,which involves $x_o = x(0)$.We leave to the reader the task to get this estimate,following an approach to that used above.

We shall state now a lemma which deals with second order differential inequalities on the real axis.

Lemma 4

Let x(t) be a map from R into R_+,twice differentiable,and such that

(11) $$x''(t) \ge k^2x(t) - f(t)\lambda(x(t)) , \quad t \in R ,$$

If x(t) is bounded on R , and k , f and verify the conditions stated in Lemma 3, then x(t) satisfies

(12) $$\sup x(t) \le \mu^{-1}(K\|f\|_M) , \quad t \in R ,$$

with $K > 0$ depending of k only.

Proof. The inequality (11) can be rewritten in the form

(13) $$(x' - kx)' + k(x' - kx) \ge -f(t)\lambda(x(t)) , \quad t \in R ,$$

which leads to the first order inequality

(14) $$\{x'(t) - kx(t)\}\exp(kt) \ge \{x'(t_o) - kx(t_o)\}\exp(kt_o) - \int_{t_o}^{t} \lambda(x(s))f(s)\exp(ks)ds , \quad t > t_o.$$

The boundedness of x(t) implies the existence of a sequence t_n , $n \ge 1$, such that $t_n \downarrow -\infty$, and $x'(t_n) \to 0$. Let now $t_o \downarrow -\infty$ on such a sequence in (14). One obtains the inequality

(15) $$x'(t) - kx(t) \ge - \int_{-\infty}^{t}\lambda(x(s))f(s)\exp\{-k(t - s)\}ds ,$$

holds true for any $t \in R$. If we denote as in Lemma 3 $\sup x(t) = m$, $t \in R$, then (15) leads to

(16) $$m \le \lambda(m)K\|f\|_M ,$$

from which the inequality (12) follows immediately (the constant K in (12) does not equate that occuring in Lemma 3).

Remark

If the inequality (11) holds only on R_+ , then estimates for the bounded solutions on R_+ can be obtained using the same approach as above. In particular , if we know that $x(0) = 0$, then (12) keeps its validity.

ALMOST PERIODICITY RESULTS

We shall apply now the Lemmas from the preceding section in order to obtain almost periodicity criteria for the systems

(17) $$x'(t) = f(t,x(t)) \ , \ t \in R \ ,$$

(18) $$x''(t) = f(t,x(t)) \ , \ t \in R \ ,$$

where x , $f \in R^n$, and $f(t,x)$ is almost periodic in t , uniformly with respect to x in any compact subset of R^n . The almost periodicity of $f(t,x)$ will mean either Bohr's almost periodicity , or the almost periodicity in Stepanov's sense. The almost periodicity of the solution will be always meant in Bohr's sense.

Let us formulate now the conditions that will be required for $f(t,x)$ by the criteria of almost periodicity we are going to establish in this section.

1) The map f , from $R \times R^n$ into R^n, is continuous,and such that there exist two functions $\phi(r)$ and $\psi(r)$, from R_+ into itself , with

(19) $$\langle f(t,x) - f(t,y) \ , \ x - y \rangle \geq \phi(|x - y|) - \psi(|x - y|) \ ;$$

2) The functions $\phi(r)$ and $\psi(r)$ satisfy

(20) $$\phi(r) = o(r) \text{ as } r \downarrow 0,$$

(21) $$\liminf \{ r^{-1} [\phi(r) - \psi(r)] \} > 0 \ , \text{ as } r \uparrow \infty \ ,$$

while the greatest root of the equation

(22) $$\phi(r) = \psi(r) + \varepsilon r \ , \ \varepsilon > 0 \ ,$$

say $r(\varepsilon)$, is such that

(23) $$\lim r(\varepsilon) = 0 \text{ as } \varepsilon \downarrow 0 \ ;$$

3) The map $t \quad f(t,\cdot)$, from R into R^n , is (Bohr) almost periodic, uniformly with respect to the second argument in any bounded set of R^n

We can state now the following result.

Theorem 1

Let $x(t)$ be a bounded (on R) solution of the system (17), in which f satisfies the conditions 1), 2) and 3). Then $x(t)$ is (Bohr) almost periodic. If f is periodic in t, of period T, the so is $x(t)$.

Proof. Let $\tau \in R$ be a translation number, and denote

$$y(t) = |x(t+\tau) - x(t)|^2 , \quad t \in R. \tag{24}$$

From (17), and the analogous equation obtained by changing t in $t+\tau$, one derives after subtraction and scalar multiplication by $x(t+\tau) - x(t)$:

$$y' \geq 2\{\phi(\sqrt{y}) - \psi(\sqrt{y}) - \varepsilon\sqrt{y}\}, \quad t \in R . \tag{25}$$

The number ε in (25) is defined by

$$\varepsilon = \sup \{|f(t+\tau,x(t)) - f(t,x(t))|;\ t \in R\}, \tag{26}$$

and is certainly finite because of the assumtion 3) on $f(t,x)$. Of course, we can choose ε as small as we want, provided τ is chosen among the ε-almost periods of the almost periodic function $f(t,x)$.

Now, we can apply the result given in Lemma 1 to the differential inequality (25). Indeed, on behalf of (21) one has $\phi(r) - \psi(r) - \varepsilon r > 0$, provided r is taken large enough. On the other hand, because of the assumption (20) and condition $\psi(r) \geq 0$, one sees that $\phi(r) - \psi(r) - \varepsilon r < 0$ for small values of r. Consequently, taking into account the boundedness of $y(t)$ on R, (25) implies

$$\{y(t)\}^{\frac{1}{2}} = |x(t+\tau) - x(t)| < r(\varepsilon), \quad t \in R , \tag{27}$$

and condition (23) guarantees the (Bohr) almost periodicity of $x(t)$.

The periodic case can be dealt with in the same manner, taking $\tau = T$, and noticing the fact that (25) holds true for any $\varepsilon > 0$.

Corollary

A noteworthy special case of Theorem 1 is obtained for $\phi(r) = mr^2$, and $\psi(r)$ identically equal zero ($m > 0$). It means that $f(t,x)$ is strictly monotone in x, a case that has been dealt with by several authors. For the finite - dimensional case the papers [19] and [29] are relevant. They contain further references related to this topic. The case of equations in a Banach/Hilbert space has been considered, under monotonicity assumption, in [5], [26], [27], [28], [31]. The paper [13] is also dedicated to equations in Banach spaces, and uses the same approach.

Next almost periodicity result we are going to prove deals with second order differential systems . Lemma 2 of the preceding section is providing the necessary tool in carrying out the proof.

Theorem 2

Let x(t) be a map from R into R^n , twice differentiable , bounded on R , and satisfying the system (18) . We assume that f(t,x) verifies the conditions 1) , 2) and 3) stated above . Then x(t) is (Bohr) almost periodic . If f(t,x) is periodic in t , of period T , then so is x(t) .

Proof. We shall proceed exactly as in proof of Theorem 1 , keeping the notations used there . This time , we shall rely on the simple identity $yy'' = (yy')' - (y')^2$, in order to obtain the inequality

$$y'' \geq 2\{\phi(\sqrt{y}\,) - \psi(\sqrt{y}\,) - \varepsilon\sqrt{y}\} , \quad t \in R , \tag{28}$$

where ε is defined by (26) . The Lemma 2 is now applicable to the inequality (28), which leads to

$$\{y(t)\}^{\frac{1}{2}} = |\, x(t + \tau) - x(t)| \quad < r(\varepsilon) , \ t \in R . \tag{29}$$

On behalf of our assumptions , (29) implies the almost periodicity of the solution x(t) .

Again , the case of periodic f(t,x) is covered by the considerations exposed at the end of the proof of Theorem 1 .

Corollary

If one chooses ψ(r) to be identically zero , and $\phi(r) = mr^2$, i.e. , condition (19) takes the form of a monotonicity condition for f(t,x) ,

$$\langle f(t,x) - f(t,y) , x - y \rangle \geq m|x - y|^2 , \quad m > 0 . \tag{30}$$

In order to be able to deal with Stepanov's almost periodicity of the right hand side of the differential systems under investigation , the use of Lemmas 3 and 4 is required . Nevertheless , we still assume the continuity of f(t,x) , in order to secure the existence of continuously differentiable solutions.

Theorem 3

Consider the system (17) , under the following assumptions:

a) the map f(t,x) , from $R \times R^n$ into R^n is continuous, and such that (3) holds true ;

b) the map $t \to f(t,\cdot)$, from R into R^n ,is almost periodic in Stepanov's sense (with p = 1) , uniformly with respect to the second argument in any compact set of R^n .

Then , if x(t) is a bounded (on R) solution of (17) , x(t) is (Bohr) almost periodic . Moreover , the almost periodic solution of (17) is unique .

Proof. Let $\tau \in R$ be a fixed number , and consider again the function y(t) defined by (24) , where x(t) stands for the bounded (on R) solution of (17) whose existence is guaranteed by our hypotheses . Proceeding again as in the proof of Theorem 1 , one obtains the following inequality:

(31) $$y'(t) \geq 2\{my(t) - g(t)\sqrt{y(t)}\} , \quad t \in R ,$$

where one denotes

(32) $$g(t) = |f(t + \tau,x(t)) - f(t,x(t))| , \quad t \in R .$$

Since $g \in M$, the Lemma 3 is applicable , and (31) leads to the estimate

(33) $$\{y(t)\}^{\frac{1}{2}} \equiv |x(t + \varepsilon) - x(t)| \leq K|g|_M ,$$

where the constant K depends only of m . From (32) it is obvious that

(34) $$|g|_M = \sup \int_t^{t+1} |f(s + \tau,x(s)) - f(s,x(s))|ds , \quad t \in R ,$$

can be done arbitrarily small , provided the translation number τ is chosen among the almost periods of f (notice the boundedness of x(t)). Consequently , any $K^{-1}\varepsilon$-almost period of f is an ε-almost period for the solution x(t).

The uniqueness of an almost periodic solution to (17) can be obtained again by means of Lemma 3 , if one takes into account the fact that $y(t) = |x(t)- \bar{x}(t)|^2$, here x(t) and $\bar{x}(t)$ stand for two bounded solutions of (17), verifies the inequa- $y' \geq 2my - \varepsilon$, for every positive ε .

Remark 1

Results like Theorem 3 can be found in the literature,even in the case of equations in Banach/Hilbert spaces : [18] , [22] , [26], [28]. In most cases , the conditions imposed are somewhat stronger , either for the proof of existence of solution , or in order to compensate for the noncompactness of bounded sets in such spaces . The proof we provided above seems to be new , even though the result appears to be familiar to the reader.

Remark 2

The existence problem for almost periodic solutions of (17) , under the assumptions of Theorem 3 , is a rather simple problem if we rely on the Amerio's result that guarantees the existence of a bounded solution (on R) for any almost periodic system for which there exists a bounded half-trajectory. Indeed , the comparison equation attached to (17) is obtainable from (31) , in which the sign $\geq$ is replaced by the sign = . If $g \in M$, then the comparison equation has bounded solutions on both positive and negative half-axis (see [1] for recent contributions to this topic. This implies the existence of bounded semitrajectories for the system (17).

Theorem 4

Consider the differential system (18) , under the assumptions a) and b) of Theorem 3 . Then , if x(t) is a bounded (on R) solution of (18) , x(t) is (Bohr) almost periodic . Moreover , there exists only one almost periodic solution of this system .

Proof. We shall proceed again as in the proof of Theorem 2 . One derives the following differential inequality :

(35) $$y''(t) \geq 2\{my(t) - g(t)\sqrt{y(t)}\} , \quad t \in R ,$$

where y(t) is defined by (24) , and g(t) is defined by (32) .Since y(t) is obviously bounded on R , from (35) and Lemma 4 we obtain

(36) $$\{y(t)\}^{\frac{1}{2}} = |x(t + \tau) - x(t)| \leq K\|g\|_M , \quad t \in R ,$$

with K depending on m only . In Lemma 4 of the preceding section one has to take $\lambda(r) = \sqrt{r}$, in order to obtain (36) .

The (Bohr) almost periodicity of x(t) follows now from (36) , if one takes into account the almost periodicity (Stepanov) of f(t,x) .

The uniqueness of the almost periodic solution is an immediate consequence of Lemma 2 . Indeed , if $x_1(t)$ and $x_2(t)$ are almost periodic solutions of (18) , then their difference y(t) verifies the inequality

(37) $$y'' \geq 2my , \quad t \in R ,$$

which is strengthened if one subtract an arbitrary $\varepsilon > 0$ from its right side.

Remark

Theorem 4 is somewhat more general than the result obtained in [1] by R. Aftabizadeh , where f(t,x) is supposed to be of gradient type and almost periodic in Bohr's sense . This author deals also with the existence problem .

LIAPUNOV FUNCTIONS , AND FURTHER RESULTS

The reader who is familiar with the Liapunov's functions technique in the theory of differential equations has certainly noticed that all the results in the preceding section are , in fact , obtained by means of the Liapunov function $V(t,x,y) = |x - y|^2$, whose derivatives are evaluated with respect to the systems

(38) $$x' = f(t,x) \ , \ y' = f(t + \tau,y) \ ,$$

(39) $$x'' = f(t,x) \ , \ y'' = f(t + \tau,y) \ ,$$

where τ is an arbitrary real number , to be chosen among the almost periods of the function $f(t,x)$.

Several criteria of almost periodicity of solutions , based on the existence or construction of a convenient Liapunov function , are given in Yoshizawa [33]. Of course , such criteria are usually related to those concerning stability properties of solutions (for which reason the Liapunov functions have been devised) and - to the best of our knowledge - all such criteria involve only the first derivative of the Liapunov function .

On the other hand , in stability theory several authors [21] , [25] , [32] have dealt with inequalities or properties of the second derivatives of the Liapunov functions , obtaining new stability criteria .

As we shall see in this section , Lemmas 2 and 4 provide the necessary tools in connection with the use of Liapunov functions satisfying certain second order differential inequalities , in view of finding new criteria of almost periodicity of bounded solutions . Actually , Theorems 2 and 4 of the preceding section constitute good illustrations of such kind of new criteria .

Let us now state some almost periodicity criteria for bounded solutions of the system (17) , involving a rather general type of Liapunov function , and satisfying differential inequalities like those discussed in the Lemmas proven in the second section of this paper .

A few notations will be necessary in order to formulate the basic results of this section . They are mainly those used in [33].

First , we shall use a system derived from the system (17) , namely

(40) $$x' = f(t,x) \ , \ y' = f(t,y) \ .$$

This system differs from the system (38) , but it allows the statement of the assumptions on the Liapunov function in adequate form . It is worth pointing out that the system (38) involves the real parameter τ, which implies the fact we have to deal with a family of systems , rather than a fixed one .

Let $V(t,x,y)$ be a map from $R \times B^* \times B^*$ into R_+ , where B^* denotes a bounded neighborhood of the ball $B = \{x \mid x \in R^n , |x| \le r \}$. We shall assume the following conditions hold true for the function $V(t,x,y)$:

1) $V(t,x,y) \in C^{(1)}$ in $R \times B^* \times B^*$, and its restriction to $R \times B \times B$ is bounded , together with the restrictions of V_x and V_y (the gradients of V in respect to x and y) ;

2) there exists a continuous map λ from $[0,A)$ into R_+ , for some $A > 0$, such that $\lambda(0) = 0$, $\lambda(r) > 0$ for $r > 0$, and

(41) $$\lambda(|x - y|) \le V(t,x,y) \quad , \quad t \in R \quad , \quad x , y \in B^* ;$$

3) $V(t,x,y)$ satisfies the differential inequality

(42) $$V'_{(40)}(t,x,y) \ge \omega(V(t,x,y)) ,$$

in the whole domain of definition of the function $V(t,x,y)$, where ω is a continuous map from some interval $[0,\alpha)$ into R , and such that the gratest positive root , say $r = r(\varepsilon)$, of the scalar equation

(43) $$\omega(r) - \varepsilon = 0 , \ \varepsilon > 0 ,$$

verifies

(44) $$\lim r(\varepsilon) = 0 , \text{ as } \varepsilon \downarrow 0 ,$$

while $\omega(r) - \varepsilon > 0$ for $r > r(\varepsilon)$.

We shall notice the fact that under our assumptions one has

(45) $$V'_{(40)}(t,x,y) = V_t + \langle V_x , f(t,x) \rangle + \langle V_y , f(t,y) \rangle .$$

Let us also notice that , without loss of generality , the function $\lambda(r)$ in (41) can be assumed strictly increasing . Indeed , if $\lambda(r)$ is not strictly increasing , then $\lambda_o(r) = \int_o^r \exp(-t)\lambda(t)dt$ is , while $\lambda_o(r) < \lambda(r)$.

Theorem 5

Consider the system (17) , with f(t,x) almost periodic in t , uniformly with respect to $x \in B$. Assume there exists a Liapunov function $V(t,x,y)$ satisfying the conditions 1) , 2) and 3) , specified above . If the system (17) has s aemi-trajectory $\{ x_o(t) \mid t \ge t_o \} \subset B$, for some $t_o \in R$, then there exists a unique almost periodic solution x(t) of the system (17) , such that $\{ x(t) \mid t \in R\} \subset B$.

Proof. As pointed out in the previous section , the existence of the almost periodic solution x(t) follows from a well known result of Amerio [2]. See also [7], [16] , [33]. We need only to prove that x(t) is an almost periodic solution . Indeed , if $\tau \in R$ is an arbitrary (translation) number , then x(t) and $y(t) = x(t + \tau)$ satisfy the system (38) . The last equation in (38) can be rewritten in the form

(46) $$y'(t) = f(t,y(t)) + [f(t + \tau,y(t)) - f(t,y(t))] .$$

This equation appears as a perturbation of the second equation in the system (40). The relationship between the derivatives of the function V(t,x,y) taken with respect to the systems (40) and (38) respectively is

(47) $$V'_{(40)}(t,x,y) \leq V'_{(38)}(t,x,y) + L\varepsilon ,$$

where L is a constant dominating $|V_x|$ and $|V_y|$ in $R \times B \times B$, and

(48) $$\varepsilon = \sup |f(t + \tau,y) - f(t,y)| , t \in R , y \in B .$$

The supremum in (48) is alwys finite , because of the almost periodicity of f(t,x) and of the compactness of the set B [7]. Of course , the number ε can be arbitrarily small if τ is chosen among the almost periods of f(t,x).

From the inequalities (42) and (47) one derives

(49) $$V'_{(38)}(t,x,y) \geq \omega(V(t,x,y)) - L\varepsilon ,$$

which constitutes the basic inequality . Indeed , since V(t,x,y) is bounded on the set $R \times B \times B$, the Lemma 1 applies , and we obtain

(50) $$V(t,x(t),x(t + \tau)) \leq r(L\varepsilon) , t \in R ,$$

rovided $\tau \in R$ is an ε - almost period of f(t,x) in $R \times B$. Hence , taking into account properties 2) and 3) in Theorem 5 , one obtains

(51) $$\lambda(|x(t + \tau) - x(t)|) \leq r(L\varepsilon) , t \in R ,$$

from which the almost periodicity of x(t) follows immediately .

To prove the uniqueness of the almost periodic solution for the system (17) we will use again Lemma 1 . Indeed , if we assume the system (17) has two almost periodic solutions in $R \times B$, say x(t) and y(t) , then (42) yields

(52) $$V'_{(40)}(t,x,y) \geq \omega(V(t,x,y)) - \varepsilon , \varepsilon > 0 .$$

The Lemma 1 provides now the following inequality :

(53) $$\lambda(|x(t) - y(t)|) \le V(t,x(t),y(t)) < r(\varepsilon) , \quad t \in R ,$$

from which the uniqueness follows immediately , taking into account the fact that $\varepsilon > 0$ is arbitrary . This ends the proof of Theorem 5 .

We shall compare now the result of Theorem 5 , with a similar one given in Yoshizawa [33]. A feature of Yoshizawa's result consists in the fact that the Liapunov function $V(t,x,y)$ is defined only on R_+ . But this does not prevent us to apply the result in Theorem 5 , since a rather elementary procedure allows the extension of $V(t,x,y)$ from $R_+ \times B^* \times B^*$, to $R \times B^* \times B^*$.

Let us formulate first the result of Yoshizawa , slightly modifying some of the conditions involved . In particular , we do not require the upper estimate $V(t,x,y) \le \mu(|x - y|)$, replacing it by the boundedness of $V(t,x,\cdot)$. Instead , we will impose somewhat more restrictive conditions in regard to the growth of $V(t,x,y)$ in t , as well as in regard to the existence of $V'(t,x,y)$.

Theorem (19.1 in [33])

Consider the system (17) and assume that $f(t,x)$ is almost periodic from $R \quad B^*$ into R^n , uniformly with respect to x in any compact part of B^* . Let $V(t,x,y)$ be a Liapunov type function , satisfying the following conditions :

i) there exists a positive definite function $\lambda(r)$, such that

$$\lambda(|x - y|) \le V(t,x,y) \le V_0 < +\infty ,$$

in the whole domain of definition of $V(t,x,y)$, $V_0 = \text{const.}$;

ii) there exists a constant $L > 0$, such that

$$|V(t,x,y) - V(\bar{t},\bar{x},\bar{y})| \le L(|t - \bar{t}| + |x - \bar{x}| + |y - \bar{y}|) ;$$

iii) the derivative of $V(t,x,y)$, with respect to the system (40) ,defined as in [33] , verifies the inequality

$$V'_{(40)}(t,x,y) \ge \alpha V(t,x,y) , \quad \alpha > 0 ,$$

and is supposed to exist uniformly with respect to $t \in R$.

If the system (17) has a semitrajectory that belongs to B , then there exists a unique almost periodic solution of (17) in B .

<u>Proof</u>. Before we derive the proof of the theorem stated above from the Theorem 5 , we shall notice that the change of sign that appears in iii) does not

mean anything else, but the fact that the first order differential inequality on R is equivalent to its dual inequality (the one obtained by the change of the variable t into $-t$).

We are going to prove that $V(t,x,y)$ can be extended to the whole real axis with respect to the variable t, such that conditions i), ii), and iii) keep their validity. Indeed, let us denote $V_m(t,x,y) = V(t + t_m,x,y)$, $t \geq -t_m$, $x, y \in B^*$, for $m = 1, 2, 3, \ldots$, where $\{t_m\}$ is a monotone sequence of positive numbers that tends to $+\infty$, and such that

(54) $$\lim f(t + t_m,x) = f(t,x) \quad \text{as} \quad m \to \infty,$$

uniformly on $R \times B$. See, for instance, [16] in connection to the existence of such a sequence $\{t_m\}$. Let us fix now $t < 0$, and notice the fact that all the terms in the function sequence $\{V_m(t,x,y)\}$ are defined at t, starting with a sufficiently large rank. Moreover, on behalf of the assumptions i) and ii), one sees that this function sequence is uniformly bounded and equicontinuous on any compact part of $B^* \times B^*$. Therefore, without loss of generality, we can assume $\{V_m(t,x,y)\}$ is uniformly convergent on any compact part of $B^* \times B^*$. Let us denote now, for each fixed $t < 0$,

(55) $$V(t,x,y) = \lim V_m(t,x,y), \quad m \to \infty.$$

It is obvious that conditions i) and ii) are preserved for the extended function defined by (55). It remains only to prove that condition iii) holds true for the negative t's. More exactly, one has to prove that

(56) $$\limsup_{h \downarrow 0} \frac{V(t + h,x + hf(t,x),y + hf(t,y)) - V(t,x,y)}{h} \geq \alpha V(t,x,y).$$

We already know that (56) holds true for $t > 0$, and the limit exists uniformly with respect to $t \in R_+$. Therefore, we can find a sequence $\{h_k\}$, $h_k \downarrow 0$, for which

(57) $$\frac{V(t + t_m + h_k,x + h_k f(t + t_m,x),y + h_k f(t + t_m,y)) - V(t + t_m,x,y)}{h_k} > \alpha V(t+ t_m,x,y) - \eta,$$

where $\eta > 0$ is arbitrarily chosen (it does not depend on m, k or t). But (54) and ii) obviously imply

$$(58)\quad \frac{V(t+t_m+h_k,x+h_kf(t+t_m,x),y+h_kf(t+t_m,y)) - V(t+t_m+h_k,x+h_kf(t,x),y+h_kf(t,y))}{h_k}$$

$$\leq 2L\sup|f(t+t_m,x) - f(t,x)| ,$$

where the supremum is taken with respect to $(t,x) \in R \times B$. From (57) and (58) one obtains

$$(59)\quad 2L\sup|f(t+t_m,x) - f(t,x)| + \frac{V(t+t_m+h_k,x+h_kf(t,x),y+h_kf(t,y)) - V(t+t_m,x,y)}{h_k}$$

$$> \alpha V(t+t_m,x,y) - \eta .$$

On the other hand , (59) can be rewritten in the form

$$(60)\quad 2L\sup|f(t+t_m,x) - f(t,x)| + \frac{V_m(t+h_k,x+h_kf(t,x),y+h_kf(t,y)) - V_m(t,x,y)}{h_k}$$

$$> \alpha V_m(t,x,y) - \eta .$$

Let us keep now k fixed , and let m tend to ∞ . Then (60) implies

$$(61)\quad \frac{V(t+h_k,x+h_kf(t,x),y+h_kf(t,y)) - V(t,x,y)}{h_k} \geq \alpha V(t,x,y) - \eta .$$

Now , let us make k tend to infinity in (61) . One derives

$$(62)\quad V'_{(40)}(t,x,y) \geq \alpha V(t,x,y) - \eta ,$$

and since $\eta > 0$ is arbitrary , (62) implies the condition iii) for $V(t,x,y)$. It should be pointed out that the existence of the upper right derivative for the function $V(t,x,y)$ is guaranteed by the fact it satisfies condition ii) .

This ends the proof of Yoshizawa's theorem , in the variant stated above.

Remark 1

The proof shows , actually , that the existence of a Liapunov's type function on a half-axis implies the existence of a similar function which is defined on the entire real axis . This fact appears to be quite natural , if we keep in mind the fact that the system (17) is almost periodic , and that any almost periodic function is completely determined when given on a half-axis [7] .

Remark 2

In Theorem 1 we did require $V(t,x,y)$ to be of class $C^{(1)}$. Theorem 1 is valid with $V(t,x,y)$ possessing only Dini's derivative , like in [33].

Let us consider again the second order differential system (18) , in view of obtaining a result of almost periodicity , using a Liapunov function which is of a more general form than the one used in Theorem 4 of the preceding section .

First , let us point out that we need to estimate the second derivative of the Liapunov function with respect to the system

(63) $$x'' = f(t,x) \ , \quad y'' = f(t,y) \ ,$$

which does not involve the derivatives of the first order . Normally , we look for Liapunov functions (to such systems) involving both variables x and y , as well as their first order derivatives . In what follows , we shall be concerned with the construction of a Liapunov function for (63) , having the special form

(64) $$V(t,x,y) = \lambda(|x - y|^2) \ ,$$

where $\lambda(r)$ is a convex function of class K (see the second section of this paper for the definition of this class) . The advantage of seeking such Liapunov functions will follow from the fact that estimates can be found for the second derivative $V''(t,x,y)$, independent of the first derivatives x' and y' .

Indeed , if we assume $\lambda(r)$ to be continuously differentiable of the second order , then V'' can be expressed in the form

(65) $$V''(t,x,y) = 2\lambda'(|x - y|^2)|x' - y'|^2 + 2\lambda'(|x - y|^2)\langle x - y, x'' - y''\rangle + 4\lambda''(|x - y|^2)\langle x - y, x' - y'\rangle^2 \ .$$

Since the first and third terms in the right hand side of (65) are nonnegative (the second , due to the convexity of λ) , one obtains the following inequality in case we estimate the second derivative with respect to (63) :

(66) $$V''_{(63)}(t,x,y) \geq 2\lambda'(|x - y|^2)\langle x - y, f(t,x) - f(t,y)\rangle \ .$$

We shall make now an assumption on $f(t,x)$, which is somewhat less restrictive than (30) . Namely , let us assume that

(67) $$\langle x - y, f(t,x) - f(t,y)\rangle \geq g(|x - y|^2) \ ,$$

where $g(r)$ is a function of class K , such that

(68) $$\int_{0+} [g(u)]^{-1} du = +\infty \ .$$

From (66) and (67) one derives the inequality

(69) $$V''_{(63)}(t,x,y) \geq 2\lambda'(|x - y|^2) g(|x - y|^2) \ ,$$

which can be rewritten in the form

(70) $$V''_{(63)}(t,x,y) \geq V(t,x,y) ,$$

provided $\lambda(r)$ is chosen such that

(71) $$2\,\lambda'(r)g(r) = \lambda(r) ,$$

or , equivalently ,

(72) $$\lambda(r) = \exp\{\tfrac{1}{2}\int^{r}[g(u)]^{-1}du\} ,$$

where the lower limit in the integral can be taken at any positive number . Since (68) holds true , one finds out that defining $\lambda(0) = 0$ the resulting $\lambda(r)$ will be in the class K . Of course , $\lambda(r)$ does not necessarily verify the convexity assumption , under which we have obtained the inequality (66) . Actually , this property must be imposed on $g(r)$, together with (68) . Moreover , the existence of the second derivative of $\lambda(r)$ has to be assured , which means one more restriction for $g(r)$. Nevertheless , it can be easily seen that $g(r) = mr$, with sufficiently small $m > 0$, verifies all three conditions required for $g(r)$.

If one takes into account (66) and (70) , one obtains the following inequality :

(73) $$V''_{(39)}(t,x,y) \geq V(t,x,y) - 2K\bar{g}(t)|x - y| ,$$

where $K > 0$ is a constant that dominates $\lambda'(r)$ on a certain interval $[0,A]$, and $\bar{g}(t)$ is defined by (32) . The inequality (73) has the form required by the Lemma 4 , and consequently we can derive an estimate similar to (12) . It remains only to provide for a function $\mu(r)$,like the one defined by (6) , such that it belongs to the class K. It might be also useful to notice that a function and its inverse belong simultaneously to the class K .

Taking into account (64) and (73) , the condition corresponding to (6) in Lemma 4 takes the form

(74) $$r^{-1}\exp\{\int^{r}[g(u)]^{-1}du\} \in K ,$$

which can be easily checked . In particular , it is true for $g(r) = mr$, $m > 0$, provided one chooses $m < 1$.

The discussion conducted above leads to a theorem which constitutes a variant of Theorem 4 in this paper . It also proves the potential of the comparison method in dealing with qualitative problems.

Theorem 6

Consider the system (18) under the following assumptions on f(t,x) :

a) the map $t \to f(t,\cdot)$, from R into R^n , is (Stepanov) almost periodic , unoformly with respect to the second argument in any compact set of R^n ;

b) the map $(t,x) \to f(t,x)$, from $R \times R^n$ into R^n is continuous , and such that condition (67) holds true ;

c) the function g(r) , occuring in (67) , is subject to (68) , (74) , and such that (72) defines a convex function (i.e. , there exists a second derivative to $\lambda(r)$, excepting perhaps at r = 0+ ,which is nonnegative) .

Then , any bounded (on R) solution of (18) is (Bohr) almost periodic .

Remark

The proof of Theorem 6 has bee carried out above . We only want to precise the way K must be chosen in (73) . As mentioned above , K must serve as an upper bound for $\lambda'(r)$ on some interval [0,A]. If we are given a bounded solution x(t) of (18) , say $|x(t)| \geq M$ on R , then A must be at least $4M^2$. This is clear from the formula (64) .

In concluding this paper , we take the opportunity to point out that by the use of similar inequalities to those discussed in the second section , but restricted to a half-axis , one can prove the asymptotic almost periodicity of the solutions bounded on that half-axis .

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