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To the memory of
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FIXED POINT THEOREMS FOR SEQUENCES OF MAPPINGS

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Generally, the study in fixed point theory has developed in the following directions: 1) generalization of conditions which imply existence of fixed points; 2) the study of uniqueness; 3) topological study of the set of fixed points in the case of nonuniqueness; 4) the study of fixed points for a mapping f using the sequence of iterates $\{f^n(x)\}_{n=0}^{\infty}$ 5) the study of fixed point for a mapping f using a sequence of mappings $\{f_n\}_{n=0}^{\infty}$ convergent to f, and 6) the study of common fixed points for families of mappings [18].

Many results on the existence of common fixed points for families of mappings use a commutativity assumption [2], [4], [8-10], [13], [20].

Our main purpose in this note is the study of the existence of common fixed points for a sequence of mappings, on a complete metric space, without commutativity assumption.

We obtain two general fixed point theorems which contain as particular cases many well-known fixed point theorems.

Some results on the existence of common fixed points for noncommutative familied of mappings can be found in [1], [18], but our results are different.

Theorem 1. Let (E,d) be a complete metric space and let $\{f_n\}_{n\in N}$ be a sequence of maps, $f_n:E \to E$, verifying the following assumptions:

1) There exist two constants $0 \leq a,b$ such that $a + b < 1$ and for each $x, y \in E$,

$$d(f_i(f_{i-1}(x)),f_j(f_{j-1}(y))) \leq a\, d(f_{i-1}(x),f_{j-1}(y)) + b\, d(x,y); \quad \forall i,j > 1.$$

2) There exist a constant $0 < \beta < 1$ and a function $\chi:[0,+\infty[\to[0,+\infty[$, verifying $\lim_{r\to 0_+} \chi(r) = 0$, and such that for every $i \in N$ there exists $n_i \in N$ such that, for each $j > n_i$ and each $x, y \in E$,

$$d(f_j(x),f_i(y)) \leq \chi(d(x,f_j(x))) + \beta\, d(y,f_i(y))$$

Then, there exists a unique common fixed point $x^* \in E$ for the sequence $\{f_n\}_{n\in N}$, and for each $x_0 \in E$ the sequence $\{x_n\}_{n\in N}$ defined by $x_n = f_n(x_{n-1})$, $\forall\, n \in N$, converges to x^*.

Proof. First, we will prove that the sequence $\{x_n\}_{n\in N}$ defined by, $x_n = f_n(x_{n-1})$; $\forall n \in N$, where x_0 is an arbitrary point of E, is a Cauchy sequence.

Indeed, we denote

$$K = \max \{d(x_0,f_1(x_0)),d(f_1(x_0),f_2(x_1))\},$$

and using the definition of the sequence $\{x_n\}_{n\in N}$ and assumption 1) we obtain the following inequalities:

$$(i_1) \quad d(f_2(x_1),f_3(x_2)) \leq a\, d(f_1(x_0),f_2(x_1)) + b\, d(x_0,f_1(x_0)) \leq K(a + b),$$

$$(i_2) \quad d(f_3(x_2),f_4(x_3)) \leq a\, d(f_2(x_1),f_3(x_2)) + b\, d(f_1(x_0),f_2(x_1)) \leq a\, K(a+b) + b\, K = K[a(a+b)+b] \leq K(a+b)$$

$$(i_3) \quad d(f_4(x_3),f_5(x_4)) \leq a\, d(f_3(x_2),f_4(x_3)) + b\, d(f_2(x_1),f_3(x_2)) \leq a\, K(a+b) + b\, K(a+b) = K(a+b)^2.$$

Now, by induction we have the general formula

$$(i_4) \quad d(f_{2m}(x_{2m-1}),f_{2m+1}(x_{2m})) \leq K(a+b)^m, \quad \forall\, m \in N.$$

From (i_2) and (i_4), and again by induction we obtain

$$(i_5) \quad d(f_{2m+1}(x_{2m}),f_{2m+2}(x_{2m+1})) \leq K(a+b)^m, \quad \forall\, m \in N.$$

Consider now, $m, n \in N$ arbitrary but such that, $m < n$. In this case, from (i_4) and (i_5) we have

$$d(f_m(x_{m-1}),f_n(x_{n-1})) \leq d(f_m(x_{m-1}),f_{m+1}(x_m)) + d(f_{m+1}(x_m),f_{m+2}(x_{m+1}))$$

$$+ \dots + d(f_{n-1}(x_{n-2}),f_n(x_{n-1})) \leq K(a+b)^{[\frac{m}{2}]} \frac{1}{1-(a+b)}$$

The last formula implies that the sequence $\{x_n\}_{n\in N}$ is a Cauchy sequence and since E is a complete metric space then there exists $x^* = \lim_{n\to\infty} x_n$.

We will prove now that x^* is a common fixed point for the sequence $\{f_n\}_{n\in N}$.

Let f_i be an arbitrary map of the sequence $\{f_n\}_{n\in N}$.

From assumption 2) there is a $n_i \in N$ such that for each $n > n_i$ we have,

$$d(x^*,f_i(x^*)) \leq d(x^*,x_n) + d(x_n,f_i(x^*))$$

$$= d(x^*,x_n) + d(f_n(x_{n-1}),f_i(x^*))$$

$$\leq d(x^*,x_n) + \chi(d(x_{n-1},f_n(x_{n-1}))) + \beta\, d(x^*,f_i(x^*))$$

which implies,

$$(i_6) \quad d(x^*,f_i(x^*)) \leq \frac{1}{1-\beta} [d(x^*,x_n) + \chi(d(x_{n-1},f_n(x_{n-1})))]$$

Letting $n \to \infty$ in the formula (i_6) and using the property of the function χ, we obtain $d(x^*,f_i(x^*)) = 0$, and finally $f_i(x^*) = x^*$ for every $i \in N$.

To finish, we observe that x^* is a unique common fixed point for the sequence $\{f_n\}_{n\in N}$.

Indeed, we suppose that there exists another common fixed point v for the sequence $\{f_n\}_{n\in N}$.

Choosing an index $i \in N$ from assumption 2) there is a $n_i \in N$ such that for each $j > n_i$ we have

$$d(x^*,v) = d(f_j(x^*),f_i(v)) \leq \chi(d(x^*,f_j(x^*)) + \beta d(v,f_i(v))$$

which implies $v = x^*$ and the proof is finished.

Remark 1. If the inequality of the assumption 2) is verified with $0 < \beta < \frac{1}{2}$

and $\chi(r) = \beta r$, for each i and j, then we obtain the fixed point theorem proved by Rassias [14].

In this case, the assumption 1) is not used in the proof.

Remark 2. If in theorem 1 we have $f_n = f$, for every $n \in N$, then we obtain the fixed point theorem proved by Istratescu [7].

Remark 3. The theorem 1 can be considered also as a generalization of Kannan's fixed point theorem, or of certain generalizations of Kannan's fixed point theorem [3], [6], [12], [15-17], [18], [21].

For the next theorem we consider a complete metric space (E,d), and we suppose that there exist three functions ϕ, Ψ, χ from $[0,+\infty[$ into $[0,+\infty[$, which satisfy the assumptions [5]:

1) $\phi(r) < r$ if $r > 0$;
2) there exists $\lim\limits_{r \to r_0^+} \phi(r) \leq \phi(r_0)$; $\forall r_0 \in [0,+\infty[$;
3) $\psi(0) = \chi(0) = 0$;
4) $\chi(r) < r$ if $r > 0$;
5) ψ and χ are continuous functions at $r = 0$.

Theorem 2. Let $\{f_n\}_{n \in N}$ be a sequence of maps, $f_n : E \to E$, verifying the assumptions:

A) $$d(f_i(x), f_j(y)) \leq \phi(d(x,y)) + \psi(d(f_i(x),x)) + \chi(d(f_j(y),y)),$$
$$\forall x,y \in E, \quad \forall i,j \in N.$$

B) For each $x_0 \in E$ the sequence $\{x_n\}_{n \in N}$ defined by $(*) : x_n = f_n(x_{n-1})$, $\forall n \in N$ verifies, $\lim\limits_{n \to \infty} d(f_n(x_{n-1}), f_{n-1}(x_{n-2})) = 0$.

Then there exists a unique common fixed point $x^* \in E$ for the sequence $\{f_n\}_{n \in N}$, and for each $x_0 \in E$ the sequence $\{x_n\}_{n \in N}$ defined by (*) converges to x^*.

Proof.

a) The sequence $\{x_n\}$ is a Cauchy sequence for each $x_0 \in E$.

To prove this affirmation we suppose that there exists $x_0 \in E$, such that $\{x_n\}_{n \in N}$ (where $x_n = f_n(x_{n-1})$, $\forall n \in N$) is not a Cauchy sequence.

As in the paper [16], there exist $\varepsilon > 0$, $\{m_j\}_{j\in N}$, $\{n_j\}_{j\in N}$ which satisfy:

i_1) $m_j > n_j$, $\forall j \in N$;

i_2) $\lim_{j\to\infty} n_j = +\infty$;

i_3) $d(f_{m_j}(x_{m_j-1}), f_{n_j}(x_{n_j-1})) \geq \varepsilon$;

i_4) $d(f_{m_j-1}(x_{m_j-2}), f_{n_j}(x_{n_j-1})) < \varepsilon$.

In this case we have,

$$\begin{aligned}\varepsilon &\leq d(f_{m_j}(x_{m_j-1}), f_{n_j}(x_{n_j-1}))\\ &\leq d(f_{m_j}(x_{m_j-1}), f_{m_j-1}(x_{m_j-2})) + d(f_{m_j-1}(x_{m_j-2}), f_{n_j}(x_{n_j-1}))\\ &\leq \varepsilon + d(f_{m_j}(x_{m_j-1}), f_{m_j-1}(x_{m_j-2})),\end{aligned}$$

which implies, letting $j \to +\infty$ (and using B)),

$$(\theta_1) \quad \lim_{j\to\infty} d(f_{m_j}(x_{m_j-1}), f_{n_j}(x_{n_j-1})) = \varepsilon.$$

On the other hand,

$$\begin{aligned}d(f_{m_j}(x_{m_j-1}), f_{n_j}(x_{n_j-1})) &\leq d(f_{m_j}(x_{m_j-1}), f_{m_j+1}(x_{m_j}))\\ &+ d(f_{n_j}(x_{n_j-1}), f_{n_j+1}(x_{n_j})) + d(f_{m_j+1}(x_{m_j}), f_{n_j+1}(x_{n_j}))\\ &\leq d(f_{m_j}(x_{m_j-1}), f_{m_j+1}(x_{m_j})) + d(f_{n_j}(x_{n_j-1}), f_{n_j+1}(x_{n_j}))\\ &+ \phi(d(x_{m_j}, x_{n_j})) + \psi(d(f_{m_j+1}(x_{m_j}), x_{m_j})) + \chi(d(f_{n_j+1}(x_{n_j}), x_{n_j}))),\end{aligned}$$

which implies

$$d(f_{m_j}(x_{m_j-1}),f_{n_j}(x_{n_j-1})) - \phi(d(f_{m_j}(x_{m_j-1}),f_{n_j}(x_{n_j-1}))$$

$$\leq d(f_{m_j}(x_{m_j-1}),f_{m_j+1}(x_{m_j})) + d(f_{n_j}(x_{n_j-1}),f_{n_j+1}(x_{n_j}))$$

$$+ \psi(d(f_{m_j+1}(x_{m_j}),x_{m_j})) + \chi(d(f_{m_j+1}(x_{n_j}),x_{n_j})).$$

Letting $j \to +\infty$ in the last inequality, and using the properties of the functions ϕ, ψ and χ, we obtain

(θ_2) $$\varepsilon - \lim_{j\to\infty} \phi(d(f_{m_j}(x_{m_j-1}),f_{n_j}(x_{n_j-1}))) \leq 0,$$ which is impossible.

(See (θ_1) and the properties 1) and 2) of the function ϕ).

Hence the sequence $\{x_n\}_{n\in N}$ is a Cauchy sequence, and there exists $x^* = \lim_{n\to\infty} x_n$.

b) The point x^* is a common fixed point for the sequence $\{f_n\}_{n\in N}$.

Indeed, if $n_0 \in N$ is fixed we have,

$$d(x^*,f_{n_0}(x^*)) \leq d(x^*,f_n(x_{n-1})) + d(f_n(x_{n-1}),f_{n+1}(x_n))$$

$$+ d(f_{n+1}(x_n),f_{n_0}(x^*))$$

$$\leq d(x^*,f_n(x_{n-1})) + d(f_n(x_{n-1}),f_{n+1}(x_n))$$

$$+ \phi(d(x_n,x^*)) + \psi(d(f_{n+1}(x_n),x_n)) + \chi(d(f_{n_0}(x^*),x^*)).$$

Letting $n \to +\infty$ and using the properties of the function ϕ we obtain,

(θ_3) $$0 \leq d(x^*,f_{n_0}(x^*)) \leq \chi(d(f_{n_0}(x^*),x^*)).$$

If we suppose that $d(f_{n_0}(x^*),x^*) > 0$ then from the property 4) of the function χ and (θ_3) we obtain $d(x^*,f_{n_0}(x^*)) < d(f_{n_0}(x^*),x^*)$ which is impossible.

Then $f_{n_0}(x^*) = x^*$ for each $n_0 \in N$.

c) The point x^* is a unique common fixed point for the sequence $\{f_n\}_{n \in N}$. To prove this, we suppose that there exists another common fixed point v. Since,

$$d(x^*,v) = d(f_i(x^*),f_j(v)) \leq \phi(d(x^*,v))$$

$$+ \psi(d(f_i(x^*),x^*)) + X(d(f_j(v),v)) = \phi(d(x^*,v)),$$

using the property 1) of the function ϕ we obtain $d(x^*,v) = 0$, and the proof is finished.

Remark. If in the theorem 2 we have $f_n = f$, for every $n \in N$, then we obtain the fixed point theorem proved by Emmanuele [5].

Several well-known fixed point theorems are particular cases of Emmanuele's theorem.

Hence, our theorem 2 is a generalization for sequences of maps of some well-known fixed point theorems [5].

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