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GLOBAL ASYMPTOTIC STABILITY OF A CLASS OF SYSTEMS WITH RETARDED ARGUMENT

(Abstract)

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The aim of this investigation is a class of differential systems with retarded argument which occur in the mathematical modelling of the protein synthesis . The methods used here are based on concepts from the stability theory .

We shall consider a class of differential systems with retarded argument of the form

$$\text{(S)}\qquad \begin{aligned} \dot{x}^1(t) &= f(L_n(x^n)) - \beta^1 x^1(t) + g^1(x^1(t)) , \\ \dot{x}^i(t) &= \alpha^i L_{i-1}(x^{i-1}) - \beta^i x^i(t) + g^i(x^i(t)) , \quad i = 2 , \dots , n , \end{aligned}$$

where f , g^i , $i = 1 , 2 , \dots , n$, are decreasing functions on $[0 , \infty)$, $0 \le g^i(0) < \infty$, and the functional L_i satisfy the following hypothesis :

(H) Each L_i , $i = 1 , 2 , \dots , n$, takes continuous functions on a compact , into continuous functions .

The following traditional notation will be adopted : $x_t^i(s) = x^i(t+s)$, for $s \in [-r , 0]$. Of course , this makes sense only in the case $x^i(u)$ is defined on an interval that contains the interval $[t-r , t]$.

It is also assumed that in L_n , x^n appears only with retarded argument , while in L_i , x^i appears both with or without retarded argument , $i = 2 , \dots ,n$. Hence , the initial conditions associated with (S) must be of the form

$$\text{(I)}\qquad x_o^i(s) = \phi^i(s) , \quad i = 1 , 2 , \dots , n , \quad s \in [-r , 0],$$

where ϕ^i are continuous and nonnegative functions .

A result which proves to be very useful in the investigation of the system (S) consists in the fact that it possesses a unique equilibrium position .

The following two theorems generalize results that have been obtained by H. T. Banks and J. M. Mahaffy in [1].

Theorem 1 . *The system (S) has a unique positive solution , defined on the positive halfaxis , and verifying the initial conditions (I) .*

Theorem 2 . *Assume that the (numerical) system*

$$f(\bar{U}_n) + g^1(\bar{U}_1) = \beta^1 \bar{V}_1 ,$$

$$\alpha^i \bar{V}_{i-1} + g^i(\bar{U}_i) = \beta^i \bar{V}_i , \quad i = 2 , \dots , n ,$$

$$f(\bar{V}_n) + g^1(\bar{V}_1) = \beta^1 \bar{U}_1 ,$$

$$\alpha^i \bar{U}_{i-1} + g^i(\bar{V}_i) = \beta^i \bar{U}_i , \quad i = 2 , \dots , n ,$$

has a unique positive solution . Then , the unique equilibrium solution of the system (S) is globally asymptotically stable .

Let us consider now a particular type of system (S) , namely

$$\text{(S1)} \quad \begin{aligned} \dot{x}^1(t) &= a\{1 + k[x^n(t-\tau_1) + \bar{x}^n]^\rho\}^{-1} - b[x^1(t) + \bar{x}^1] , \\ x^i(t) &= \alpha^i x^{i-1}(t - \tau_i) - \beta^i x^i(t) + h^i[x^i(t) + \bar{x}^i] - h^i(\bar{x}^i) , \\ & i = 2 , \dots , n , \end{aligned}$$

where h^i are decreasing functions on the positive halfaxis , and a , b , k , ... are positive constants . By $\bar{x}^i$, i = 1 , 2 , ... , n , one denotes the coordinates of the unique equilibrium solution of the system

$$\dot{z}^1(t) = a\{1 + k[z^n(t-\tau_1)]^\rho\}^{-1} - b\, z^1(t) ,$$

$$\dot{z}^i(t) = \alpha^i z^{i-1}(t-\tau_i) - \beta^i z^i(t) + h^i(z^i(t)) ,$$

$$i = 2 , \dots , n .$$

By means of a transformation used by U. an der Heiden in [2] , the system (S1) can be reduced to the following form :

$$\text{(S2)} \quad \begin{aligned} \dot{x}^1(t) &= a\{1 + k[x^n(t-r) + \bar{x}^n]^\rho\}^{-1} - b[x^1(t) + \bar{x}^1] , \\ \dot{x}^i(t) &= \alpha^i x^{i-1}(t) - \beta^i x^i(t) + h^i[x^i(t) + \bar{x}^i] - h^i(\bar{x}^i) , \quad 2 \le i \le n . \end{aligned}$$

The following initial conditions will be considered in regard to the system (S1) or (S2) :

$$(I') \quad \begin{aligned} & x^i(0) = x^{io} > -\bar{x}^i , \quad i = 1 , 2 , .., n-1 , \\ & x^n(s) = \phi^n(s) > -\bar{x}^n , \quad s \in [-r , 0], \end{aligned}$$

where $\phi^n(s)$ is continuous on $[-r , 0]$.

Theorem 3 . *The system (S1) has a unique solution defined on the positive halfaxis , satisfying the initial conditions (I') . Moreover , this solution is such that*

$$x^i(t) > -\bar{x}^i \quad \textit{for} \quad t \geq 0 , \ i = 1 , 2, ..., n .$$

The next theorem deals with the system (S2) , and establishes a result on ultimate boundedness of the solutions .

Theorem 4 . *If the (numerical) system*

$$\frac{a}{b[1 + k \bar{U}_n^{\rho}]} = \bar{V}_1 ,$$

$$\alpha^i \bar{V}_{i-1} + h^i(\bar{U}_i) = \beta^i \bar{V}_i \quad , \quad i = 2 , ... , n ,$$

$$\frac{a}{b[1 + k \bar{V}_n^{\rho}]} = \bar{U}_1 ,$$

$$\alpha^i \bar{U}_{i-1} + h^i(\bar{V}_i) = {}^i\bar{U}_i \quad , \quad i = 2 , ... , n ,$$

has at least three distinct positive solutions , then there exist some positive constants U_i^* , V_i^* , $i = 1 , 2 , ... , n$, *such that any solution of the system (S2) verifies*

$$U_i^* - \bar{x}^i < x^i(t) < V_i^* - \bar{x}^i , \quad i = 1 , ... , n ,$$

for sufficiently large t .

The Theorems 3 and 4 are generalizations of some results obtained in [3] by J.M.Mahaffy .

It can be also shown that considering appropriate initial conditions , the solution of the system (S2) can be oscillatory .

There is another way to generalize the results obtained in [3].

Let us consider the system with delay

$$\dot{x}^1(t) = \frac{a}{1 + k[x^n(t-r_1) + \bar{x}^n]^\rho} - bx^1(t)$$

$$\dot{x}^i(t) = \alpha^i x^{i-1}(t) - \beta^i x^i(t) \quad , i = 2 , 3 , \dots , n-1 ,$$

$$\dot{x}^n(t) = \alpha^n x^{n-1}(t) - g(x^n(t-r_2)) \quad .$$

In this case , the characteristic equation is essentially different from that of Mahaffy [3] , and we succeeded studying it only in case n = 2 , by means of Rouche's theorem in complex analysis .

In concluding this abstract , let us point out the fact that the systems considered above are shaped on the protein synthesis model based on Jacob - Monod hypothesis concerning gene regulation in prokaryotic cells .

REFERENCES

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