LIBERTAS MATHEMATICA VOL. 4, 1984

CONFORMAL COSYMPLECTIC MANIFOLDS ENDOWED WITH A PSEUDO-SASAKIAN STRUCTURE

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INTRODUCTION

Let $\tilde{M}(\xi,\tilde{\eta},\tilde{\Omega},\tilde{g})$ be a (2m+1)-dimensional Riemannian or pseudo-Riemannian C^{∞}-manifold. The structure tensor fields $(\xi,\tilde{\eta},\tilde{\Omega},\tilde{g})$ are defined as follows:

1) ξ is the canonical vector field (or Reeb's vector);

2) $\tilde{\eta}$ is the dual form of ξ, or $\tilde{\eta} = \xi^{\flat}$ ($\flat$: *musical isomorphism* with respect to the metric tensor $\tilde{g}$)

3) $\tilde{\Omega}$ is the canonical 2-form of rank 2m, and $\xi = \ker(\tilde{\Omega})$.

If $\xi = \text{grad}\ \tilde{f}$ and $d\tilde{\Omega} = c\tilde{f}\eta \wedge \tilde{\Omega}$ (c = const.) we say that the treble $(\tilde{\Omega},\tilde{\eta},f)$ defines a *conformal cosymplectic* structure (abr. c.c-structure) and $\tilde{M}$ is called a *conformal cosymplectic manifold*. In addition, we assume that $\tilde{g}$ defines a pseudo-Riemannian metric of *inertia index* m+1, and $\tilde{M}$ admits the *para-complex* operator U [1] $(U^2 = +1)$ as (1,1)-structure tensor field. Then if for any vector fields $\tilde{X}$, $\tilde{Y}$ on $\tilde{M}$, we have

$$\text{(a)}\qquad \begin{cases} U\tilde{X} = \tilde{X} - \tilde{\eta}(\tilde{X}),\ \tilde{g}(U\tilde{X},U\tilde{Y}) = -\tilde{g}(\tilde{X},\tilde{Y}) + \tilde{\eta}(\tilde{X})\tilde{\eta}(\tilde{Y}), \\ \tilde{g}(\tilde{X},\xi) = \tilde{\eta}(X),\ U\xi = 0,\ \tilde{\Omega}(X,Y) = -\tilde{g}(\tilde{X},U\tilde{Y}), \\ \tilde{\nabla}_{\tilde{X}}\xi = \tilde{f}(-\tilde{X} + \tilde{\eta}(\tilde{X})\xi); \end{cases}$$

we say that the c.c-manifold M under consideration is endowed with a *pseudo-Sasakian structure* [2].

It is shown that any such manifold is foliated by para Kählerian hypersphere M_0, orthogonal to ξ and if $\tilde{M}$ is a space-form $\tilde{M}(c)$, then M_0 is an *Eisnstein* submanifold. In addition: (i) the scalar field $\tilde{f}$ is an *eigenfunction* of the *harmonic* operator; (ii) the Lie derivative of $\tilde{\Omega}$ with respect to ξ is *exterior recurrent* (iii) if $\forall X \in D_{\tilde{\eta}}$ ($D_{\tilde{\eta}}$: distribution annihilated by $\tilde{\eta}$), then necessary and sufficient condition that X be an *infinesimal automorphism* of the c.c-structure $(\tilde{\Omega},\tilde{\eta},\tilde{f})$ is that it's dual form with respect to $\tilde{\Omega}$ be *exterior recurrent*, with $-2\tilde{f}\tilde{\eta}$ as recurrence for m [3]; (iv) any globally defined (2p+1)-form $\tilde{\Theta}_p$ $(p < m)$ defined by $\tilde{\Theta}_p = 2L^p\tilde{u}$ $(\tilde{u} = (i-\tilde{f}^2)\tilde{\eta})$ is: 1° closed; 2° ξ-conformal invariant; 3° has exterior recureent Lie derivative with respect to ξ.

In section 3° we consider the proper immersion of a ξ-vertical CR submanifold M in $\tilde{M}$. If $D^{\perp}$ is the vertical distribution on M, then $D^{\perp}$ is always involution and the necessary and sufficient condition that M be *minimal* is that it be $D^{\perp}$-*minimal* [4]. Further, the necessary and sufficient condition that M be foliated is that the simple unit form which corresponds to $D^{\perp}_p$, be exterior recurrent. In this case if M^{T} is the maximal integral manifold tangent to the horizontal distribution D, then M^{T} is *minimal* in $\tilde{M}$ and is endowed with a *symplectic structure* (M^{T},Ω). With respect to this structure one defines two *polarizations*.

In section 4° we discuss some properties of the improper immersion $x:M_c \to \tilde{M}$ where M_c is a *coisotropic* hypersurface. It is shown that M_c is a ξ-vertical CR submanifold of $\tilde{M}$ whose vertical distribution $D^{\perp}$ is always involutive. In addition one proves that:

1° M_c is *almost minimal* and $D^{\perp}$-*totally geodesic*;

2° the restriction $\Omega = \tilde{\Omega}/M_c$ is $D^{\perp}$-*conformal invariant*.

1° PRELIMINARIES

Let $\tilde{M}(U,\xi,\tilde{\eta},\tilde{\Omega},\tilde{g})$ be a (2m+1)-dimensional pseudo-Riemannian C^{∞}-manifold of *inertia index* m+1. The quintuplet $(U,\xi,\tilde{\eta},\tilde{\Omega},\tilde{g})$ defines the structure tensors of $\tilde{M}$ as follows:

(i) U is the *para-complex* operators of square +1 [1] and it defines a *para-f-structure* [2] on $\tilde{M}$;

(ii) ξ is the canonical vector field and is time-like;

(iii) $\tilde{\eta} = \xi^{\flat}$ where $\flat: E \to E^{*}$ defines the *musical isomorphism* with respect to $\tilde{g}$ [6];

(iv) $\tilde{\Omega}$ is a canonical 2-form of rank 2m $(\text{Ker}\ \tilde{\Omega}:\tilde{\Omega}(\xi) = 0)$ which defines a *conformal cosymplectic structure* on $\tilde{M}$ (abr. c.c-structure), i.e., $d\tilde{\Omega} = c\tilde{f}\tilde{\eta} \wedge \tilde{\Omega}$; $\tilde{f} \in C^{\infty}(\tilde{M})$; c = const;

(v) $\tilde{g}$ is the fundamental metric tensor of $\tilde{M}$.

Let $\tilde{\nabla}$ and $\tilde{Z},\tilde{Z}^1$ be the *covariant, derivative operator* and any vector fields on $\tilde{M}$. Then referring to [2] if one has

$$(1.1)\qquad \begin{cases} U^2\tilde{Z} = \tilde{Z}-\tilde{\eta}(\tilde{Z})\xi,\ \tilde{g}(U\tilde{Z},U\tilde{Z}^1) = -\tilde{g}(\tilde{Z},\tilde{Z}^1)+\tilde{\eta}(\tilde{Z})\tilde{\eta}(\tilde{Z}^1) \\ \tilde{g}(\tilde{Z},\xi) = \tilde{\eta}(\tilde{Z}),\ U\xi = 0,\ \tilde{\Omega}(\tilde{Z},\tilde{Z}^1) = -\tilde{g}(\tilde{Z},U\tilde{Z}^1) \\ d\tilde{\eta} = 0,\ \tilde{\nabla}_{\tilde{Z}}\xi = \tilde{f}(-\tilde{Z}+\tilde{\eta}(\tilde{Z})\xi), \end{cases}$$

we say that the conformal cosymplectic pseudo-Riemannian manifold under consideration is endowed with *pseudo-Sasakian structure* $(U,\xi,\tilde{\eta},\tilde{\Omega},\tilde{g},\tilde{f})$. It is easy to see by the last equation (1.1), that one may move to

$$(1.2)\qquad \xi = \text{grad}\ \tilde{f} \Longleftrightarrow \tilde{\eta} = d\tilde{f}.$$

<u>Remark</u>: Since one has

$$(1.3)\qquad U\xi = 0,\ U^2\tilde{Z} = \tilde{Z} - \tilde{\eta}(\tilde{Z})\xi,\ \tilde{\eta}(\xi) = 1,\ \tilde{\eta}(U\tilde{Z}) = 0,$$

the manifold $M(U,\xi,\tilde{\eta},\tilde{\Omega},\tilde{g},\tilde{f})$ can be also considered as a *paracontact* manifold in the sense [7]. Let $\mathcal{O}(\tilde{M})$ be the bundle of orthonormal frames over $\tilde{M}$ and let $\mathcal{O} \in \mathcal{O}(M)$ be an element of $\mathcal{O}(\tilde{M})$ (i.e., a *moving frame* on $\tilde{M}$).

Set $\mathcal{O} = \{e_a, e_{a^*}, \xi, a=1\ldots m;\ a^*=a+m\}$ (since $e_{a^*} = Ue_a$, $U\xi = 0$, $Ue_{a^*} = e_a$, one may say that the vector basis $\mathcal{O}$ is a *U-vectorial basis*).

If $\mathcal{O}^* = \{\tilde{\omega}^a, \tilde{\omega}^{a^*}, \tilde{\eta}\}$ is the dual basis, the *line element* $d\tilde{p}$ (i.e., the canonical vectorial 1-form on $\tilde{M}$) is given by [2].

$$(1.4)\qquad d\tilde{p} = \tilde{\omega}^a \otimes e_a = \tilde{\omega}^{a^*} \otimes e_{a^*} + \tilde{\eta} \otimes \xi \Rightarrow$$
$$\Rightarrow \langle d\tilde{p},d\tilde{p}\rangle = \tilde{g} = \textstyle\sum_a(\tilde{\omega}^a \otimes \tilde{\omega}^a - \tilde{\omega}^{a^*} \otimes \tilde{\omega}^{a^*}) + \tilde{\eta} \otimes \tilde{\eta}.$$

Next, set

$$(1.5)\qquad \tilde{\omega}^A_B = \tilde{\gamma}^A_{BC}\,\tilde{\omega}^C;\quad \tilde{\gamma}^A_{BC} \in C^\infty(\tilde{M})$$

for the connection forms on $\mathcal{O}(\tilde{M})$ $(A,B,C,D = 0,1,\ldots,2m)$; $\tilde{\gamma}^A_{BC}$: connection coefficients), Then by (1.1) and (1.4), Cartan structural equations, are:

$$(1.6)\qquad \begin{cases} \tilde{\nabla}e_a = \tilde{a}^b_a \otimes c_b - \tilde{\omega}^{b^*}_a \otimes e_{b^*} + \tilde{f}\tilde{\omega}^a \otimes \xi, \\ \tilde{\nabla}e_{a^*} = \tilde{\omega}^b_{a^*} \otimes c_b - \tilde{\omega}^{b^*}_{a^*} \otimes e_{b^*} + \tilde{f}\tilde{\omega}^{a^*} \otimes \xi, \\ \tilde{\nabla}\xi = -\tilde{f}U^2d\tilde{p}; \end{cases}$$

$$(1.7)\qquad \begin{cases} d\tilde{\omega}^a = \tilde{\omega}^b \wedge \tilde{\omega}^a_b - \tilde{\omega}^{b*} \wedge \tilde{\omega}^a_b + \tilde{f}\omega^a \wedge \tilde{\eta}, \\ d\tilde{\omega}^{a*} = \tilde{\omega}^b \wedge \tilde{\omega}^{a*}_b - \tilde{\omega}^{b*} \wedge \tilde{\omega}^{a*}_{b*} + \tilde{f}\tilde{\omega}^{a*} \wedge \tilde{\eta}, \\ d\tilde{\eta} = 0; \end{cases}$$

and

$$(1.8)\qquad \begin{cases} d\tilde{\omega}^a_b = \tilde{\Omega}^a_b + \tilde{\omega}^c_b \wedge \tilde{\omega}^a_c - \tilde{\omega}^{c*}_b \wedge \tilde{\omega}^a_{c*} + \tilde{f}^2\tilde{\omega}^a \wedge \tilde{\omega}^b, \\ d\tilde{\omega}^{a*}_{b*} = \tilde{\Omega}^{a*}_{b*} + \tilde{\omega}^c_{b*} \wedge \tilde{\omega}^{a*}_c - \tilde{\omega}^{c*}_{b*} \wedge \tilde{\omega}^{a*}_{c*} + \tilde{f}^2\tilde{\omega}^{a*} \wedge \tilde{\omega}^{b*}, \\ d\tilde{\omega}^{b*}_a = \tilde{\Omega}^{b*}_a + \tilde{\omega}^c_a \wedge \tilde{\omega}^{b*}_c - \tilde{\omega}^{c*}_a \wedge \tilde{\omega}^{b*}_{c*} + \tilde{f}^2\tilde{\omega}^{b*} \wedge \tilde{\omega}^a, \end{cases}$$

where $\tilde{\Omega}^A_B = \frac{1}{2} R^A_{BCD}\, \tilde{\omega}^{c^a} \wedge \tilde{\omega}^D$ are the *curvature forms* and R^A_{BCD} is the *curvature tensor.*

Since $\mathcal{O}$ is an orthonormal frame, one has $\tilde{\omega}^A_B + \tilde{\omega}^B_C = 0$. In addition by (1.1) and (1.6) one has

$$(1.9)\qquad \tilde{\omega}^a_b + \tilde{\omega}^{a*}_{b*} = 0, \quad \tilde{\omega}^a_{b*} + \tilde{\omega}^{a*}_b = 0$$

and also

$$(1.10)\qquad \tilde{\Omega} = \sum_a \tilde{\omega}^a \wedge \tilde{\omega}^{a*}.$$

Finally by (1.7) and (1.9), exterior derivative of (1.10) gives

$$(1.11)\qquad d\tilde{\Omega} = -2\tilde{f}\tilde{\eta} \wedge \tilde{\Omega}.$$

2° C.C.-MANIFOLD ENDOWED WITH A PSEUDO-SASKIAN STRUCTURE

Let $T_{\tilde{p}}(\tilde{M})$ be the tangent space of $\tilde{M}$ at $\forall\, \tilde{p} \in \tilde{M}$. We agree to denote by $D_{\tilde{\eta}}$, the distribution of $T_{\tilde{p}}(\tilde{M})$, which is annihilated by $\tilde{\eta}$, i.e.:

$$D\tilde{\eta} = \{\forall\, \tilde{X} \in T_{\tilde{p}}(\tilde{M});\ \tilde{\eta}(\tilde{X}) = 0\}.$$

Referring to (1.11) and the last equation (1.7) we quickly see that $D_{\tilde{\eta}}$ is both a *symplectic* vector space and an involutive distribution.

Therefore we may say that $\tilde{M}$ is foliated by *symplectic hypersurface* M_0 orthogonal to the canonical vector field ξ. Moreover, let p_0 be any point of M_0 and g_0, f_0 the restrictions on M_0 of $\tilde{g}$, $\tilde{f}$. Since the second fundamental quadratic form ℓ_ξ of M_0 is given by $\ell_\xi = \langle dp_0, \nabla\xi \rangle$, one readily finds by the last equation (1.6) that: $\ell_\xi = f_0 g_0$.

Hence, according to a well-known definition, M_0 is an *umbilical* hypersurface of $\tilde{M}$. But by (1.2) it follows f_0 = const. Therefore since all the principal curvatures of M_0, are equal to the same constant, we conclude that M_0 is a *hypersphere* of $\tilde{M}$.

Since by (1.11) the netric tensor g_0 defines a *neutral* [15] metric on the symplectic manifold M_0, we may say (taking into account of the restriction on M_0 of equations (1.1)) that M_0 gives an illustration of a *para-Kählerian* [1] hypersphere.

Moreover, since as is known $III = \langle\nabla\xi,\nabla\xi\rangle$ represents the *third quadratic fundamental* form of M_0, one finds by the last equation (1.6) $III = f_0^2 g_0$. Therefore III being also conformed to g_0, if $\tilde{M}(C)$ is a space form, it follows according to [8], that M_0 is an *Einstein submanifold*.

Let now $\tilde{\sigma} = \tilde{\omega}^1 \wedge \ldots \wedge \tilde{\omega}^{2m} \wedge \tilde{\eta}$, be the volume element of $\tilde{M}$. With the help of (1.1) and (1.7) one readily finds

(2.1) $$L_\xi \tilde{\sigma} = -2m\tilde{f}\,\tilde{\sigma} \Rightarrow \operatorname{div}\xi = -2m\tilde{f}$$

(L: Lie derivative).

Denoting like usual by $\tilde{\Delta} = d\circ\delta + \delta\circ d$ the *harmonic* operator on $\tilde{M}$, a short calculation gives with the help of (1.2)

(2.2) $$\tilde{\Delta}\tilde{f} = \delta d\tilde{f} = -2m\tilde{f}$$

(δ: *co-differential* operator). Hence $\tilde{f}$ is an *eigenfunction* of the harmonic operator.

On the other hand from (1.10) and (1.11) one gets instantly:

(2.3) $$L_\xi \tilde{\Omega} = -2\tilde{f}\tilde{\Omega}.$$

Taking now exterior derivative of (2.3) one quickly obtains with the help of (1.11) and (1.2).

(2.4) $$dL_\xi \tilde{\Omega} = \left(\frac{2-\tilde{f}^2}{\tilde{f}}\right)\tilde{\eta} \wedge L_\xi \tilde{\Omega}.$$

From the above, we may say that:

(i) $\tilde{\Omega}$ is ξ-*conformal invariant*;

(ii) the Lie derivative $L_\xi \tilde{\Omega}$ is *exterior recurrent* [9].

Clearly the *recurrence* 1-form in (2.4) is a gradient and this is in accordance with the fact that rank $L_\xi \Omega > 4$ [3]. Let now $\mu: T(\tilde{M}) \to T^*(\tilde{M})$ be the fibre isomorphism defined by $\mu: \tilde{Z} \to i_{\tilde{Z}}\tilde{\Omega}$; $\forall \tilde{Z} \in T(\tilde{M})$ and let $\tilde{X}$ be any vector field of $D_{\tilde{\eta}}$.

By (1.1) and (1.10) one finds

$$\mu(\tilde{X}) = (U\tilde{X})^{\flat} \in \Lambda^1(\tilde{M}). \tag{2.5}$$

Setting

$$\mu(\tilde{X}) = \tilde{\omega}, \tag{2.6}$$

one readily finds by (1.11) and the last equation (1.7)

$$\begin{cases} L_{\tilde{X}}\tilde{\Omega} = d\tilde{\omega} + 2\tilde{f}\ \tilde{\eta} \wedge \tilde{\omega} \\ L_{\tilde{X}}\tilde{\eta} = 0 \end{cases} \tag{2.7}$$

First of all, exterior derivative of the first equation (2.7), gives

$$d(L_{\tilde{X}}\tilde{\Omega}) = -2\tilde{f}\ \tilde{\eta} \wedge L_{\tilde{X}}\tilde{\Omega}.$$

Hence the Lie derivative $L_{\tilde{X}}\tilde{\Omega}$ of $\tilde{\Omega}$ with respect to any vector field $\tilde{X} \in D_{\tilde{\eta}}$ is exterior recurrent, with the gradient $-2\tilde{f}\ \tilde{\eta}$ as recurrence 1-form.

Further, we derive from (2.7) that the necessary and sufficient condition that $\tilde{X}$ be an *infinitesimal automorphism* of both $\tilde{\Omega}$ and $\tilde{\eta}$ is that $\tilde{\omega}$ be exterior recurrent with the same recurrence 1-form as that of $L_{\tilde{X}}\tilde{\Omega}$. We say in this case that $\tilde{X}$ is an *infinitesimal automorphism* of the c.c-structure define by the treble $(\tilde{\Omega};\tilde{\eta},\tilde{f})$.

If $\tilde{Z}$ is now any vector field on $\tilde{M}$, and Ric. means the Ricci tensor one has with respect to an orthonormal basis $\{e_A\}$ the formula of K. Yano [10].

$$\begin{aligned} \operatorname{div}(\tilde{\nabla}_{\tilde{Z}}\tilde{Z}) - \operatorname{div}(\operatorname{div}\tilde{Z})\tilde{Z} &= \operatorname{Ric}(\tilde{Z}) \\ &+ \textstyle\sum_{A,B}\tilde{g}(\tilde{\nabla}_{e_A}\tilde{Z},e_B)\tilde{g}(e_A,\tilde{\nabla}_{e_B}\tilde{Z}) - (\operatorname{div}\tilde{Z})^2. \end{aligned} \tag{2.8}$$

Now with the help of (1.1), (1.4) and (2.1) and setting $\tilde{Z} = \xi$ in (2.8), a short computation gives:

$$\operatorname{Ric}(\xi) = 2m(1-\tilde{f}^2). \tag{2.9}$$

In the following we agree to call $\tilde{\Omega}^0_a$, $\tilde{\Omega}^0_{a*}$ the *vertical curvature* 2-forms associated with the connection $\tilde{\nabla}$. Making use of (1.6), (1.8) and (1.9) we obtain after some calculation

$$\begin{cases} \tilde{\Omega}^0_a = (1-\tilde{f}^2)\tilde{\eta} \wedge \tilde{\omega}^a, \\ \tilde{\Omega}^0_{a*} = (1-\tilde{f}^2)\tilde{\eta} \wedge \tilde{\omega}^{a*}. \end{cases} \tag{2.10}$$

In order to simplify, the form $(1-\tilde{f}^2)\tilde{\eta}$ which vanishes nowhere, will be denoted by $\tilde{u}$ (clearly by (1.2) $\tilde{u}$ is an exact form).

Next let L be the (1.1)-operator defined for any p-form $\tilde{\alpha} \in \Lambda^p(\tilde{M})$ by: $L\tilde{\alpha} = \tilde{\alpha} \wedge \tilde{\Omega}$. Then we derive from (2.10) the global 3-form $\tilde{\Theta}_1$ expressed by

$$\tilde{\Theta}_1 = \sum(\tilde{\Omega}^0_a \wedge \tilde{\omega}^{a*} - \tilde{\Omega}^0_{a*} \wedge \tilde{\omega}^a) = 2L\tilde{u}. \tag{2.11}$$

Referring to (1.1) one readily finds $d\tilde{\Theta}_1 = 0$, and be a simple argument one has also $d\tilde{\Theta}_p = 0$, for any (2p+1)-form $\tilde{\Theta}_p = 2L^p\tilde{u}$, $(p < m)$. Since $\tilde{u} = (u-\tilde{f})\tilde{\eta}$ one quickly finds by (1.11)

$$L_\xi\tilde{\Theta}_p = \frac{2\tilde{f}(1-p(\tilde{f}^2-1))}{\tilde{f}^2-1}\tilde{\Theta}_p. \tag{2.12}$$

Moreover if we set $\frac{2\tilde{f}(1-p(\tilde{f}^2-1))}{\tilde{f}^2-1} = \tilde{l}$ one finds with the help of (1.2)

$$dL_\xi\tilde{\Theta}_p = \frac{d\tilde{l}}{\tilde{l}} \wedge L_\xi\tilde{\Theta}_p. \tag{2.13}$$

Therefore we may say, that all globally defined (2p+1)-forms $\tilde{\Theta}_p$, are closed, ξ-conformal invariant and having exterior recurrent Lie derivative with respect to ξ.

Theorem. Let $\tilde{M}(U,\xi,\tilde{\eta},\tilde{\Omega},\tilde{g},\tilde{f})$ *be a c.c-manifold endowed with a pseudo-Sasakian structure and with the sixfold* $(U,\xi,\tilde{\eta},\tilde{\Omega},\tilde{g},\tilde{f})$ *as structure tensors.*

Any such manifold is foliated by *para-Kählerian* [1] hyperspheres M_0, orthogonal to ξ and if $\tilde{M}$ is a space form $\tilde{M}(c)$, then M_0 are *Einstein* submanifolds. In addition on $M(U,\xi,\tilde{\eta},\tilde{\Omega},\tilde{g},\tilde{f})$ one has the following properties:

(i) the scalar function $\tilde{f}$ is an eigenfunction of the harmonic operator;

(ii) the structure 2-form $\tilde{\Omega}$ is ξ-conformal invariant, and the Lie derivative of $\tilde{\Omega}$ with respect to ξ is exterior recurrent;

(iii) let $\tilde{X}$ be any vector field of the distribution $D_{\tilde{\eta}}$ annihilated by $\tilde{\eta}$. Then the necessary and sufficient condition that $\tilde{X}$ be an infinitesimal automorphism of the c.c-structure $(\tilde{\Omega},\tilde{\eta},\tilde{f})$ is that the 1-form $\mu(\tilde{X})$ (μ:fiber isomorphism defined by $\tilde{\Omega}$) be exterior recurrent, with the gradient $-2f\tilde{\eta}$ as recurrence form;

(iv) any globally defined (2p+1)-form $\tilde{\Theta}_p$ $(p < m)$ defined by $\tilde{\Theta}_p = 2L^p\tilde{u}$ $(\tilde{u} = (1-\tilde{f}^2)\tilde{\eta};\ L^p\tilde{u} = \tilde{u} \wedge (\Lambda\tilde{\Omega})^p)$ is: 1° closed; 2° ξ-conformal invariant; 3° and has exterior recurrent Lie derivative with respect to ξ.

3° ξ-VERTICAL CR SUBMANIFOLD OF $\tilde{M}(\tilde{U},\xi,\tilde{\eta},\tilde{\Omega},\tilde{g},\tilde{f})$

Let $x:M \to \tilde{M}$ be the proper immersion in $\tilde{M}$ of a submanifold M tangent to the structure vector field ξ, and let $T_p(M)$ (resp. $T_p^{\perp}(M)$) be the tangent space (resp. normal space) at each point p of M (we denote the lements induced by x, by suppressing $\sim$).

Referring to [4], [11], [12] we give the following:

Definition 1°. M is called a ξ-vertical CR submanifold of $\tilde{M}$ if there exists a differentiable distribution $D:p \to D_p \subset T_p(M)$ such that:

(i) D is invariant with respect to U, i.e., $UD_p \subset D_p$ for each $p \in M$, and

(ii) the complementary orthogonal distribution $D^{\perp}:x \to D_p^{\perp} \subset T_p(M)$ is *anti-invariant* with respect to U, i.e., $UD_p^{\perp} \subseteq T_p^{\perp}(M)$ for each $p \in M$, and $\xi \in D_p^{\perp}$.

D (resp. $D^{\perp}$) is called the *horizontal* distribution (resp. *vertical* (distribution).

Let $\ell < m$ be the codimension of M. Without loss of generality we assume that M is defined by

$$\omega^{r^*} = 0; \tag{3.1}$$

where $r^*, s^*, t^* = 2m-\ell+1,\dots,2m$, denote the normal indices. Consider then the $2(m-\ell)$-dimensional distribution $D_p = \{e_i, e_{i^*};\ i= ,\dots,m-\ell;\ i^*=i+m\}$ and its orthogonal complement $D_p^{\perp} = \{e_r, \xi;\ r=m-\ell+1,\dots,m\}$. Since $Ue_a = e_{a^*}$; $U^2 = +1$, one has $UD_p \in T_p(M)$ and $UD_p^{\perp} = T_p^{\perp}(M)$.

So we conclude that the submanifold M, defined by (3.1) is a ξ-*vertical CR submanifold* [12] of $\tilde{M}$.

Denote by ψ the *simple unit* from which sorresponds to D_p. Clearly by (1.10) and (3.1), the restriction $\Omega = \tilde{\Omega}/M$ of $\tilde{\Omega}$ on D_p is:

$$\Omega = \textstyle\sum_i \omega^i \wedge \omega^{i^*} \tag{3.2}$$

and we readily infer

$$\psi = \frac{(\wedge\Omega)^{m-\ell}}{(m-\ell)!}\ . \tag{3.3}$$

Taking account of (1.11), exterior derivative of (3.3), gives

$$d\psi = -2(m-\ell)f\ \eta \wedge \psi, \tag{3.4}$$

which shows that ψ is exterior recurrent. It follows that the ideal $I(D^{\perp}) = \{\psi \in \wedge(M)\ ,\ \psi$ annihilates $D^{\perp}$, and is completely integrable$\}$ is a *differentiable ideal* and we conclude that the distribution $D^{\perp}$ is always involutive (as in the case of CR submanifolds of para Kählerian manifolds [13] and of pseudo-Sasakian manifolds [14]). Further if X and Y are any vector fields tangent to M, one has the *Gauss formula*

$$\tilde{\nabla}_X Y = \nabla_X Y + B(X,Y) \tag{3.5}$$

where $B(X,Y)$ which is called the *second fundamental form* of M, is the normal part of (3.5).

Making use of (1.6) and (1.9) and referring to [11], [2] one finds

$$\begin{aligned} T_{r,g}B &= \sum_i (B(e_i,e_i)-B(Ue_i,Ue_i)) \\ &= \sum_i (B(e_i,e_i)-U^2B(e_i,e_i)) = 0. \end{aligned} \tag{3.6}$$

Hence, according to [4], the above formula proves that any ξ-vertical CR submanifold M of $\tilde{M}(U,\xi,\tilde{\eta},\tilde{\Omega},\tilde{g},\tilde{f})$ is D-*minimal*. Further, the *mean curvature vector* H of M is given by

$$H = \frac{1}{2m+1-\ell}\,[(\Sigma_i B(e_i,e_i)-B(Ue_i,Ue_i))+\Sigma_r B(e_r,e_r)] \tag{3.7}$$

and taking account of (3.6) one has $H = \frac{\Sigma_r B(e_r,e_r)}{2m+1-\ell}$. Consequently we may say that the necessary and sufficient condition that M be minimal is that it be $D^{\perp}$-*minimal*.

Let now

$$\phi = \omega^{m-\ell+1} \wedge \dots \omega^m \wedge \eta \tag{3.8}$$

be the simple unit form which corresponds to the vertical distribution $D_p^{\perp}$. Assume that ϕ is exterior recurrent, that is $d\phi$ satisfies:

$$d\phi = \gamma \wedge \phi;\ \gamma = \text{recurrence 1-form.} \tag{3.9}$$

Under this condition, and since ϕ annihilates D_p, it follows by an argument we have already used, that the distribution D is involutive. In this case we say that the submanifold M under consideration is *foliate* [4].

Now by (1.7) and (1.9) one finds that necessary and sufficient conditions that equation (3.9) be fulfilled are

$$\gamma_{ij}^r - \gamma_{ji}^r = 0,\ \gamma_{i^*j}^{r^*} + \gamma_{ji^*}^{r^*} = 0,$$

and γ is given by

$$\gamma = (\Sigma_r \gamma_{ip}^r)\omega^i - (\Sigma_r \gamma_{i^*r}^r)\omega^{i^*}.$$

Then if X, Y are any vector fields of D_p, one checks by a direct coputation, that $[X,Y] = 0$.

It is worthwhile to remark, that condition (3.9) is equivalent to:

$$B(X,UY) = B(UX,Y);\ \forall\ X,Y \in D_p.$$

In the following we denote by $M^{\top}$ (resp. $M^{\perp}$) the maximal integral manifold of D (resp. $D^{\perp}$). Clearly by (1.10) and (1.11), the 2-form Ω is closed and of maximal rank on $M^{\top}$. Hence M is a *symplectic* manifold. Let

$$dp = \omega^i \otimes e_i - \omega^{i*} \otimes e_{i*} \tag{3.10}$$

be the line element of $M^{\top}$. Then the *mean curvature form* [15] of $M^{\top}$ is the vectorial $(2(m-\ell)-1)$-form $\textcircled{H^{\top}} = *dp$. By virtue of (3.10) one finds

$$\textcircled{H^{\top}} = \Sigma_i(-1)^{i-1}\omega^1 \wedge \dots \wedge \hat{\omega}^i \wedge \dots \wedge \omega^m \wedge \omega^{1*} \wedge \dots \wedge \omega^{m*} \otimes e_i,$$

$$+ \Sigma_{i*}(-1)^{i*-m}\omega^1 \wedge \dots \wedge \omega^m \wedge \omega^{1*} \wedge \dots \wedge \hat{\omega}^{i*} \wedge \dots \wedge \omega^{m*} \otimes e_{i*},$$

($\wedge$: indicates the missing terms). Since $\textcircled{H^{\top}}$ is defined by (3.7) and $\omega^r = 0$, $\eta = 0$, taking exterior covariant derivative d^{∇} of $\textcircled{H^{\top}}$, gives with the help of (1.9) and (3.10), $d^{\nabla}\textcircled{H^{\top}} = 0$. Hence $\textcircled{H^{\top}}$ being closed it follows as is known that $M^{\top}$ is *minimal*. We shall now make some considerations regarding the symplectic structure $Sp(2(m-\ell),R)$ of the symplectic manifold $M^{\top}(\Omega,g^{\top})$ where $g^{\top}$ is the metric tensor of $M^{\top}$.

Since the metric $g^{\top}$ is *neutral* [15] we shall consider on $M^{\top}$ an adapted *Witt frame* $W = \{h_i,h_{i*}\}$ [15] derived from $\mathcal{O} = \{e_i,e_{i*}\}$. The real *null* basis of W is given by:

$$h_i = (e_1+Ue_1)/\sqrt{2}\ ;\ h_{i*} = (e_i-Ue_i)/\sqrt{2} \tag{3.11}$$

$(i = 1,\dots,m-\ell;\ i* = (m-\ell)^*;\ Uh_i = h_i;\ Uh_{i*} = -h_{i*})$ and $S_p = \{h_i\}$, $S^*_p = \{h_{i*}\}$ are two *self-orthogonal* distibutions [15] on M.

According to [13] we shall set $D^{*\perp}_p = C_{S*}T_p(M^{T}) \cap S^*_p$ and consider on M^{T} the two following distributions:

$$L_p = S_p \setminus D^{\perp}_p \subset D_p;\ L^*_p = S^*_p \setminus D^{*\perp}_p \subset D_p. \tag{3.12}$$

It is an easy matter to see that with respect to the symplectic structure of $M^{\top}(\Omega,g^{\top})$ one has

$$(3.13) \qquad \begin{cases} L_p = \mathrm{orthog}(L_p),\ L^*_p = \mathrm{orthog}(L^*_p) \\ \Omega/L_p = 0, \quad \Omega/L^*_p = 0, \end{cases}$$

and the above relations prove that L_p and L^*_p are two *Lagrangian distributions* [16] on M^T.

Let now X, Y (resp. X^*, Y^*) be any two vectir fields of L_p (resp. L^*_p). Making use of (3.10), (3.11) and (1.6) one finds:

$$(3.14) \qquad [X,Y] \subset L_p, \quad [X^*,Y^*] \subset L^*_p.$$

Hence we may say that L_p and L^*_p define two *polarizations* (i.e. Lagrangian foliations) on $M^T(\Omega, g^T)$. Referring to (3.12) and [15] these polarizations are defined as *principal*.

Theorem. Let $x:M \to \tilde{M}(U,\xi,\tilde{\eta},\tilde{\Omega},\tilde{g},\tilde{f})$ *be the proper immersion of a* ξ*-vertical CR submanifold* M *in the manifold* $\tilde{M}$ *discussed in section* 2°, *and let* $D^\perp$ *(resp.* D*) be the vertical (resp. horizontal) distribution on* M. *Then* $D^\perp$ *is always involutive and the necessary and sufficient condition that* M *be minimal is that it be* $D^\perp$*-minimal.*

Further the necessary and sufficient condition that M *be foliate (i.e.* D *be involutive) is that the simple unit form which corresponds to* $D^\perp$ *be exterior recurrent. If* M^T *is the maximal integral manifold of* D, *then* M^T *is minimal in* M^T *and is endowed with a symplectic structure* (M^T,Ω). *Moreover the two principal Lagrangian distributions on* (M^T,Ω) *define two polarizations.*

4° CO-ISOTROPIC HYPERSURFACE OF M

Let $x:M_c \to \tilde{M}$ be the improper immersion of a co-isotropic hypersurface M_c in the manifold $\tilde{M}(U,\xi,\tilde{\eta},\tilde{\Omega},\tilde{g},\tilde{f})$ defined in section 2°. Denote by e the normal vector field at each point of M_c. As is known [7], [2], c is a *null* vector field, tangent to M_c and in [17] we have called it the *characteristic* vector of M_c.

In the case under discussion, one may, without loss of generality, define M_c by

$$(4.1) \qquad \omega^m = \omega^{2m} = \omega.$$

By a simple argument it follows

(4.2) $$e = e_m - e_{2m} \Rightarrow \langle e,e\rangle = 0,$$

and so the line element dp of M_c is given by

(4.3) $$dp = \omega^i \otimes e_i - \omega^{i*} \otimes e_{i*} + \eta \otimes \xi + \omega \otimes e,$$

where the range of indices is $i = 1,\dots,m-1$; $i* = i + m$. Taking covariant derivative of (4.2) one finds by (1.6)

(4.4) $$\nabla e = \textstyle\sum_i (\omega_i^m + \omega_{i*}^m) \otimes (e_{i*} - e_i).$$

On the other hand, one derives from (4.1) by exterior differentiation

(4.5) $$\gamma_{im}^m + \gamma_{i*m}^m + \gamma_{im*}^m + \gamma_{i*m*}^m = 0,$$

With the help of the above equation one readily gets from (4.4)

(4.6) $$\nabla_e e = 0.$$

Since $\langle e,e\rangle = 0$ we may say that the null vector e_i is a *strict geodesic*.

Next by (1.6), (4.1) and (4.2) one finds

(4.7) $$\nabla\xi = -f(\omega^i \otimes e_i - \omega^{i*} \otimes e_{i*} + \omega \otimes e)$$

and from (4.4) and (4.7) we obtain

(4.8) $$[\xi,e] = L_\xi e = (f + \omega_m^{2m}(\xi))e.$$

The above equation proves that e defines an *infinitesimal transformation of generator* ξ.

Let now $\sigma = \omega^1 \wedge \dots \wedge \omega^{m-1} \wedge \omega^{1*} \wedge \dots \wedge \omega^{m*-1} \wedge \eta \wedge \omega$ be the volume element of M_c. By (4.5) one finds after some calculation

(4.9) $$L_e \sigma = 0 \Longrightarrow \operatorname{div} e = 0.$$

Since e is the characteristic vector of M_c, equation (4.9) proves according to [17] that the co-isotropic hypersurface M_c is *almost minimal*. This property is also in accordance with some more recent definitions [18].

Consider the two complementary differentiable distrbutions $D_p = \{e_i, e_{i*}\}$ and $D_p^* = \{e,\xi\}$. One readily finds $UD_p \subset T_p(M_c)$, $UD_p^* \subset T_p^*(M_c)$ and we conclude that M_c is a ξ-vertical CR submanifold of M. Let now X,Y be any vector fields of D_p^*. With the help of (4.7) one finds $[X,Y] \in D_p^*$, which that D_p^* is always involutive (as for proper immersion). Clearly the restriction Ω of $\tilde{\Omega}$ on M_c is given by

(4.10) $$\Omega = \sum_i \omega^i \wedge \omega^{i^*}$$

and obviously $\ker(\Omega) = D^*$. Then for any vector field $X \in D_p^*$, one readily finds by (1.11)

(4.11) $$L_X\Omega = -2f\eta(X)\Omega$$

since $X \in D_p^* = \{\xi, e\}$, in general $\eta(X) \neq 0$).

Definition. Let Π be a q-form and $\mathcal{D}$ a distribution annihilated by Π. If for any vector field W of $\mathcal{D}$ one has $L_W\Pi = \lambda\Pi$ (λ:scalar field) we say that Π is *$\mathcal{D}$-conformal invariant*. Hence in the case under consideration, Ω is *D^*-conformal invariant*.

Finally consider the second fundamental quadratic form $II = -\langle dp, \nabla e\rangle$ of M_c. By (4.3), (4.4) one quickly obtains:

(4.2) $$II = \sum(\omega_i^m + \omega_{i^*}^m) \otimes (\omega^{i^*} - \omega^i).$$

Then by (4.5) one finds that $\forall X, Y \in D_p^*$, one has $II(XY) = 0 \Leftrightarrow II(D^*, D^*) = 0$. Hence according to [4] we may say that the hypersurface M_c is *D^*-totally geodesic*.

Theorem. Let $x: M_c \to \tilde{M}(U, \xi, \tilde{\eta}, \tilde{\Omega}, \tilde{g}, \tilde{f})$ be the improper immersion of a co-isotropic hypersurface M_c in the manifold M defined in section 2° and let e be the characteristic vector of M_c. Such a hypersurface is a ξ-vertical CR submanifold of $\tilde{M}$, and the vertical distribution $D^* = \{e, \xi\}$ is always involutive.

In addition, one has the following properties:

(i) e defines an infinitesimal transformation of generation ξ;

(ii) M_c is almost minimal and D^*-totally geodesic;

(iii) the restriction Ω of the structure 2-form $\tilde{\Omega}$, on M is D^*-conformal invariant.

R E F E R E N C E S

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