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APPROXIMATION-SOLVABILITY OF A SEMILINEAR WAVE EQUATION

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We are concerned with the operator equation of the form

$$Lu = Su, \tag{1}$$

where L is a noninvertible linear operator and S a nonlinear perturbation over a real Hilbert space H. The kernel $N(L) = \ker L$ may be an infinite-dimensional subspace of H.

It is the purpose of this note to establish an existence theorem by using an appropriate A-proper operator technique [4]. This result is a simple example why the A-properness could be used to study evolution problems.

Let $(\cdot,\cdot)$ be the inner-product and $\|\cdot\|$ be the corresponding norm in H. $D(T)$ and $R(T)$ are domain and range, respectively, of a mapping T from H into itself.

Let us denote by $\mathcal{L}$ the class of linear operators $L: D(L) \subseteq H \to H$ which are closed, densely defined and such that $N(L) = (R(L))^{+}$. Therefore

$$H = N(L) + R(L),$$

as a topological direct sum. Notice that $\mathcal{L}$ contains all self-adjoint operators with closed range. Let P be the orthogonal projector onto $N(L)$. The restriction $L_{|D(L) \cap R(L)}$ being injective, we may define the right inverse of L associated with P

$$K = (L_{|D(L) \cap R(L)})^{-1}(I-P).$$

By the closed graph theorem, K is a continuous operator. Consider the subclass Λ of all operators in $\mathcal{L}$, whose right inverses are compact.

We fix an operator L in Λ. Let $\{X_n\}$ be a sequence of increasing finite-dimensional subspaces of $N(L)$ such that their union is dense in $N(L)$. Let P_n be the orthogonal operator onto X_n. By "$\to$" and "$\rightharpoonup$" we indicate strong and weak convergence, respectively.

Lemma 1. $P_n x \to x$ *as* $n \to \infty$ *for each* $x \in N(L)$.

Proof. Indeed, given $\delta > 0$, we can find, for each $x \in N(L)$, an element y in X_m for sufficiently large m such that $\|x-y\| \leq \delta$. Then, for $n \geq m$, u lies also in X_n and

$$\|P_n x - x\| \leq \|P_n y - x\| + \|P_n (x-y)\| \leq \delta + \delta = 2\delta. \quad []$$

Consider now the sequence $\{H_n\}$ of subspaces, finite-dimensional in the first component, in H,

$$H_n = X_n + R(L),$$

and the corresponding orthogonal projectors $Q_n = P_n + (I-P):H \to H_n$.

The double sequence $\Gamma = \{H_n, Q_n\}$ is a *complete projection scheme* for H, in the sense that $\{H_n\}$ is a sequence of monotonically increasing subspaces in H, whose union is dense in H, and $Q_n x \to x$ as $n \to \infty$ for each $x \in H$.

Let $\{u_n\}$ be a sequence in H. We say that $\{u_n\}$ is L-*convergent* to u and write $u_n \to u$ if $Pu_n \rightharpoonup Pu$ and $(I-P)u_n \to (I-P)u$. In particular, the 0-convergence is the weal convergence while, when L has a finite-dimensional kernel, the L-convergence is the strong convergence.

Given an operator $T:H \to H$ and an element $f \in H$, we associate with the equation

$$Tu = f, \tag{2}$$

a sequence of *approximant equations* with respect to the scheme Γ (w.r.t. Γ)

$$T_n u_n = Q_n f, \tag{3}$$

where $T_n = Q_n T$ and $u_n \in H_n$. Equation (2) is said to be L-*approximation solvable* w.r.t.Γ if there is some $n_f \in \mathbb{N}$ such that the n-th approximant equation (3) admits a solution u_n in H_n for every $n \geq n_f$ and the approximants $\{u_n\}$ L-converge in H to a solution u of the equation (2). L-approximation-proper operators correspond to this L-approximation solvability.

A mapping $T:H \to H$ is A_L-*proper* w.r.t. Γ if the restrictions $T_n:H_n \to H_n$ are continuous for every n, and if whenever $\{n_k\}$ is an increasing sequence of positive integers and $\{u_{n_k}\}$ is a bounded sequence such that $T_{n_k} u_{n_k} \to g$ for some g in H, it follows that $\{u_{n_k}\}$ L-converges to $u \in H$, at least on a subsequence, and $Tu = g$.

Although, A_L-properness depends on the chosen scheme, we refer, throughout this note, to the complete projection scheme Γ drawn above.

Theorem. Let $L \in \Lambda$ *and* $S:H \to H$ *be given. Suppose that there is an open bounded set* $\Omega \subset H$, *such that* $(I-P)S(\bar{\Omega})$ *is bounded and* $Lu = Su$ *is finite-dimensional solvable in* $\bar{\Omega}$. *If* $L - S$ *is* A_L*-proper, then the equation (1) is* L*-approximation solvable.*

Proof. By the finite-dimensional solvability, for every $n \geq n_0$, there exists a solution $u_n \in D(L) \cap (\overline{H_n \cap \Omega})$ of the equation

$$Lu_n = Su_n. \tag{4}$$

Since Ω is bounded, we may assume that $u_n \rightharpoonup u$ for some $u \in D(L) \cap \overline{\text{co}\ \Omega}$. Apply P_n and $(I-P)$ to (4) and obtain the equivalent system

$$P_n S u_n = 0$$

$$Lu_n = (I-P)Su_n.$$

The last one is equivalent with

$$(I-p)u_n = K(I-P)Su_n,$$

whence, by our hypotheses, $\{(I-P)u_n\}$ contains a strongly convergent subsequence to $(I-P)u$. So $Pu_n \rightharpoonup Pu$ and $(I-P)u_n \to (I-P)u$, hence $u_n \overset{L}{\to} u$. Because $(L-S)u_n \to 0$ and $L - S$ is A_L-proper, it follows that $Lu = Su$. []

Later on we point out a class of nonlinear perturbation S of monotone type with respect to $L \in \Lambda$ ([6]) such that $L - S$ is A_L-proper.

A mapping $S:\Omega \to H$ is said to be L-*pseudomonotone* if $(I-P)S(\bar{\Omega})$ is bounded and for every sequence $\{v_n\}$ such that $v_n \overset{L}{\to} v$ and $\lim(Sv_n, v_n - v) = 0$ it follows that $Sv_n \rightharpoonup Sv$. In particular, any bounded 0-pseudomonotone mapping is demicontinuous, i.e., $v_n \to v$ implies $Sv_n \rightharpoonup Sv$.

We denote by $\Phi(L)$ the set of finite-dimensional vector subspaces F of $\ker L$ and by $P_F:N(L) \to F$ the corresponding orthogonal projector. With each

$F \in \Phi(L)$ we associate the direct sum $H_F = F + R(L)$ and the projection $Q_F = P_F + I - P$. In this context, the finite-dimensional solvability implies the existence of a $F^* \in \Phi(L)$ such that the equation

$$(L_F - S_F)u_F = 0,$$

where $L_F = Q_F L$ and $S_F = Q_F S$, admits a solution $u_F \in D(L) \cap (\overline{H_F \cap \Omega})$ for every $F \in \Phi(L)$ with $F \supset F^*$.

Proposition. Let $L \in \Lambda$ and $S:\Omega \to H$ a L-pseudomonotone mapping such that $Lu = Su$ is finite-dimensional solvable. Then $L - S$ is A_L-proper.

Proof. For each $F_0 \in \Phi(L)$ with $F_0 \supset F^*$, we consider the set

$$V_{F_0} = \{u_F \in D(L) \cap (\overline{H_F \cap \Omega}) \,|\, (L_F - S_F)u_F = 0, \ \forall\, F \in \Phi(L),\ F \supset F_0\},$$

and denote by W_{F_0} the weak closure of V_{F_0}. Each W_{F_0} is weakly compact and the family $\{W_{F_0} | F_0 \in \Phi(L)\}$ has the finite intersection property. Then there exists

$$u \in \bigcup_{F^* \supset F_0 \in \Phi(L)} W_{F_0}.$$

Let $\tilde F$ be in $\Phi(L)$ such that $Pu \in \tilde F$ and $\tilde F \supset F^*$. Then $u \in H_{\tilde F} = \tilde F + R(L)$. As $u \in W_{\tilde F}$ and W_F is the weak closure of a bounded set in H, according to the Kaplanski theorem (see e.g., [3], p. 3), there exists a sequence $\{F_n\}$ with $F_n \supset \tilde F$ for all n, and $u_{F_n} = u_n \in D(L) \cap (\overline{H_n \cap \Omega})$, $H_{F_n} = H_n$, such that

$$L_{F_n} u_n = S_{F_n} u_n, \tag{5}$$

and $u_n \rightharpoonup u$. As we saw in the proof of the theorem, $Pu_n \rightharpoonup Pu$ and $(I-P)u_n \to (I-P)u$ and so $u_n \overset{L}{\to} u$. To prove that $\lim(Su_n, u_n - u) = 0$, we notice that $H_{\tilde F_n} \supset H_{\tilde F}$ and

$$u_n - u = (P_n + I - P)(u_n - u) \ \forall\, n \in \mathbb{N},$$

where $P_n = P_{F_n}$. Because $P_n Su_n = P_n S_n u_n = 0$, we derive

$$\begin{aligned}(Su_n, u_n - u) &= (Su_n, (P_n + I - P)(u_n - u)) = ((P_n + I - P)Su_n, u_n - u)\\ &= ((I-P)Su_n, u_n - u) = ((I-P)Su_n, (I-P)(u_n - u)) \to 0\end{aligned}$$

as $n \to \infty$. As S is L-pseudomonotone, it follows that $Su_n \rightharpoonup Su$.

Letting $n \to \infty$ in (5), we get

$$L_{F_n} u_n = Lu_n \rightharpoonup (I-P)Su.$$

Since L is a closed linear operator, it is also weakly closed, that is,

$$u \in D(L) \quad \text{and} \quad Lu = (I-P)Su.$$

On the other hand, for every $f \in F_n$,

$$(Su_n, f) = (P_n Su_n, f) = 0,$$

because $P_n Su_n = P_n S_n u_n = 0$. As $Su_n \rightharpoonup Su$, we have $(Su, f) = 0$ for every $f \in \ker L$, that is, $PSu = 0$. Hence

$$(L-S)u = 0,$$

which proves that $L - S$ is an A_L-proper mapping. []

To prove that periodic boundary conditions for the wave equation lead to an operator L in class Λ, we refer to some results due to J. Mawhin [2].

We consider the equation

$$\Box u = u_{tt} - u_{xx} = f(t,x,u), \tag{6}$$

in the domain $\Omega =]0,2\Pi[\times]0,\Pi[$. Suppose that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions and the growth restriction

$$|f(t,x,u)| \leq h(t,x) + c|u|, \quad \forall \quad (t,x) \in \Omega,$$

where $h \in L^2(\Omega)$ and $c > 0$. Then the Nemitskyi operator $S : L^2(\Omega) \to L^2(\Omega)$ which is defined by $Su = f(\cdot,\cdot,u)$, is bounded and continuous, (see e.g., [3], p. 166).

A function $u \in L^2(\Omega)$ is said to be *a generalized solution* of the periodic Dirichlet problem to the equation (6) provided that

$$\int_\Omega \{u \Box v + f(\cdot,\cdot,u)v\} = 0$$

for all $v \in C^2(\bar{\Omega})$ satisfying the boundary-value conditions

$$\left\{ \begin{array}{lr} v(0,x) - v(2\Pi,x) = v_t(0,x) - v_t(2\Pi,x) & x \in [0,\Pi], \\ v(t,0) = v(t,\Pi) = 0 & t \in [0,2\Pi]. \end{array} \right. \tag{7}$$

Let us investigate the spectrum of the operator $Lu = \Box u$ with the boundary conditions (7). As the Fourier series corresponding to the $u \in L^2(\Omega)$ has the form

$$u(t,x) = \sum_{(k,n)\in Z\times N} u_{kn} e^{ikt} \sin nx \quad \text{with} \quad u_{-kn} = \overline{u_{kn}},$$

we may define $L: D(L) \subset L^2(\Omega) \to L^2(\Omega)$ as follows

$$Lu = \sum_{(k,n)\in Z\times N} (n^2-k^2) u_{kn} e^{ikt} \sin nx$$

$$D(L) = \{u \in L^2(\Omega) \mid \sum_{(k,n)\in Z\times N} (n^2-k^2)^2 |u_{kn}|^2 < \infty\}.$$

In particular, for $g \in L^2(\Omega)$, $g(t,x) = \Sigma g_{kn} e^{ikt} \sin nx$, the generalized solution of the equation $Lv = g$ subjected to (7) has the Fourier coefficients

$$v_{kn} = \frac{g_{kn}}{k^2-n^2}.$$

We deduce that $\ker L = \{e^{\pm int} \sin nx \mid n \in \mathbb{N}\}$ and so $\dim \ker L = \infty$. Consequently,

$$\sigma(L) = \{\lambda = n^2-k^2 \mid n \in \mathbb{N},\ k \in Z\}.$$

Thus, every eigenvalue $\lambda \in \sigma(L)\setminus\{0\}$ has a finite multiplicity and the spectrum $\sigma(L)$ is unbounded from below and from above.

Furthermore, it is easy to check that $D(L)$ is dense in $L^2(\Omega)$, L is an auto-adjoint operator and $R(L) = (\ker L)^{\perp}$. Thereby, L possesses a right inverse K, which has a discrete spectrum $\sigma(K) = \{\frac{1}{\lambda} \mid \lambda \in \sigma(L)\setminus\{0\}\}$. In addition:

Lemma 2. K *is a compact operator.*

Proof. Let $\{v_{kn}\}$ be an orthonormal system of eigenfunctions of K in $L^2(\Omega)$. We write

$$Ku = \sum_{\substack{k,n \\ k\neq n}} \frac{(u,v_{kn})}{k^2-n^2} v_{kn}$$

and define the operators of finite rank

$$K_m u = \sum_{|k|+|n|\leq m} \frac{(u,v_{kn})}{k^2-n^2} v_{kn}.$$

Then

$$\|Ku-K_m u\|^2 = \Big\| \sum_{|k|+|n|>m} \frac{(u,v_{kn})}{k^2-n^2} v_{kn} \Big\|^2 \le \frac{1}{m^2} \sum_{k,n} (u,v_{kn})^2 = \frac{1}{m^2} \|u\|^2,$$

since $\frac{1}{|k-n|^2|k+n|^2} \le \frac{1}{m^2}$ for $|k| + |n| > m$, and so K is compact as uniform limit of operators $\{K_m\}$ of finite rank. []

This lemma completes the description of our example of operator $L \in \Lambda$.

We conclude with the following remark. The restrictions of L to H_F's are Fredholm operators of index xero, i.e., dim ker = codim R(L) < ∞, and so L_F are A-proper, in the usual sense. Moreover, $L_F + P_F$ are bijections of $D(L) \cap H_F$ onto H_F and S_F are L_F-compact. Thus, the finite-dimensional solvability conditions can be required in terms of the coincidence degree [1] or the Browder-Petryshyn degree [4].

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