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EXTENDED CLASSES OF ANALYTIC FUNCTIONS OF SEVERAL VARIABLES

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1. INTRODUCTION. In this paper we consider an extended class of analytic functions of several complex variables and investigate several topological properties of the same including the Schwartz property. For the sake of brevity we confine our attention to the two variables case only and attempt to extend the class of analytic functions of two variables in terms of infinite matrices. This new class $\Lambda(P)$ of infinite matrices envelopes the space of analytic functions in the bi-cylinder considered in [6] as well as the space of entire functions of two variables initiated in [8]. Several necessary and sufficient conditions are laid down to guarantee the Schwartz property of $\Lambda(P)$ and hence of different classes of analytic functions of two complex variables. Characterizations of continuous linear functionals and bounded sets in a certain special case of $\Lambda(P)$ are also incorporated. In the last section we confine to a special class of matrices which, in particular, includes the class of entire functions of two complex variables having order point 0 and type point at most equal to 1. Specific attention is paid to characterize the dual of this class of matrices besides investigating certain unusual structural properties of the same.

2. NOTATION AND TERMINOLOGY. Let $\mathbb{K}$ denote the field of real or complex numbers. We follow [11] to recall a few unfamiliar definitions and notation. By Ω we mean the collection if all $\mathbb{K}$ valued infinite matrices (a_{mn}) with usual coordinatewise vector operations. Let (e^{mn}) denote the infinite matrix whose all entries are zeros except the meet of m-th row and n-th column which is one. Any subspace Λ of Ω with $\Lambda \supset \Phi$, the linear space generated by $\{e^{mn}: m + n \geq 0\}$, is called a *matrix space*. For any matrix space Λ, let us introduce the K-*dual* $\Lambda^{\#}$ of Λ as follows:

$$\Lambda^{\#} = \{b = (b_{mn}) \in \Omega: \sum\sum_{m+n\leq 0} |a_{mn} b_{mn}| < \infty, \forall\, a \in \Lambda\}.$$

Further for $a \in \Omega$, if

$$a^{(s)} = \sum\sum_{0\leq m\leq n\leq s} a_{mn} e^{mn},$$

then $a^{(s)}$ is called the s-th *plane section* of the matrix $a = (a_{mn})$.

Let us also recall the following three inifinite matrix spaces

$$\ell^{11} = \{x = (x_{mn}): \sum\sum_{m+n\geq 0} |x_{mn}| < \infty\};$$

$$c_{00} = \{x = (x_{mn}): x_{mn} \to 0 \quad \text{as} \quad m+n \to \infty\};$$

and

$$\ell^{\infty\infty} = \{x = (x_{mn}): \sup_{m,n} |x_{mn}| < \infty, \quad m,n \geq 0\},$$

Whenever there is a locally convex space (X,T) (T being the Hausdorff locally convex topology on a nontrivial vector space X over $\mathbb{K}$), we write B_X or B_T for the family of all balanced, convex and T-closed neighborhoods at the origin for the topology T, and D or D_T will stand for the family of Minkowski functionals p_u corresponding to u in B_T.

Let (X,T) be a locally convex space (l.c. TVS) and $u,v \in B_T$ with $v \prec u$ [u *absorbs* v]. Suppose X_u is the quotient of X relative to the kernel of p_u. As usual we write K_u^v for the canonical continuous surjection from X_v onto X_u and $\hat{K}_u^v$ for the extension of K_u^v from $\hat{X}_v$ onto $\hat{X}_u$.

In the sequel we shall also be using the concept of the Kolmogorov diameter δ_n of an arbitrary closed unit ball $U_u = \{x \in X: p_u(x) \leq 1\}$ corresponding to the Minkowski functional p_u, for u in B_T. In fact, if A and B are subsets of a vector space X with $A \prec B$, then the infinum $\delta_n(A,B)$ of all positive numbers δ such that there is a linear subspace F of X with dimension at most n for which

$$A \subset \delta B + F,$$

is called the n-th *Kolmogorov diameter of* A *with respect* to B. The relevant information concerning this matter can be found in [16] and also in [20]. We shall have occasions to use the following well known results from [16] and [20]:

Lemma 2.1. *Let* A *be a bounded subset of a normed space* $(x, ||\cdot||)$ *and* $U = \{x : x \in X, ||x|| \leq 1\}$. *Let* L *be an* $(n+1)th$ *dimensional subspace of* X. *Suppose* $\alpha \geq 0$ *satisfies the relation*

$$\alpha(U \cap L) \geq A,$$

then

$$\delta_n(A,u) \geq \alpha.$$

Lemma 2.2. *Let* X *be a vector space containing two sets* A *and* B *with* $A \prec B$. *Let* Y *be another vector space and* F *a linear operator from* X *to* Y. *Then*

$$\delta_n(F(A),F(B)) \leq \delta_n(A,B)$$

Proposition 2.3. A bounded subset B of an ℓ.c.TVS(X,T) is precompact iff for each $u \in B_T$, $\delta_n(B,u) \to 0$ as $n \to \infty$.

Proposition 2.4. $\delta_n(K_u(v), K_u(u)) = \delta_n(v,u)$, K_u being the canonical injection from X onto X_u.

3. KOTHE MATRIX SPACES. We begin with the following:

Definition 3.1. A set $P \subset Q$ is called a K-*box* if the following three conditions hold:

(i) For each x in P, $x_{mn} \geq 0$, $\forall$ $m,n \geq 0$.

(ii) For x,y in P, there exist $z \in P$ such that

$$x_{mn} \leq z_{mn}, \; y_{mn} \leq z_{mn}, \; \forall \; m,n \geq 0.$$

(iii) For each $(m,n) \in N_0 \times N_0$, there exist $x \in P$ with $x_{mn} > 0$.

Corresponding to a K-box P, let

$$\Lambda(P) = \{x \in Q : p_a(x) \equiv \sum_{m+n \geq 0}\sum |x_{mn}| a_{mn} < \infty, \; \forall \; a \in P\}.$$

The family $\{p_a : a \in P\}$ generates a Hausdorff locally convex topology on $\Lambda(P)$ and we shall denote it by T_p.

The space $\delta^{\alpha,\beta}$: We consider a particular type of the K-box $P_{\alpha,\beta}$ given by

$$P = P_{\alpha,\beta} = \{R_1^{\alpha_m} R_2^{\beta_n} : 0 < R_1 < \infty,\ i = 1,2\},$$

where $\alpha, \beta \in \omega$ and satisfy the condition

$$0 = \alpha_0 \leq \alpha_1 \leq \dots \leq \alpha_m \to \infty;\ 0 = \beta_0 \leq \beta_1 \leq \dots \leq \beta_n \to \infty.$$

The space $(P_{\alpha,\beta})$ envelopes in particular the space of all entire functions of two complex variables. Indeed we have,

Proposition 3.2. Let $\alpha, \beta \in \omega$ be allowed to satisfy the *E-condition*, namely,

(3.3) $$\sum_{m+n \geq 0}\sum R_1^{-\alpha_m} R_2^{-\beta_n} < \infty,$$

for some $R_1, R_2 > 1$. Then

$$\Lambda(P) = \delta^{\alpha,\beta} = \{x \in Q : \lim_{m+n\to\infty} |x_{mn}|^{\frac{1}{\alpha_m+\beta_n}} = 0\}.$$

Proof. Let $x \in \Lambda(P)$ and suppose

$$|x_{mn}|^{\frac{1}{\alpha_m+\beta_n}} \nrightarrow 0 \quad \text{as} \quad m+n \to \infty.$$

Hence for some $\varepsilon > 0$, we can find increasing sequences $\{m_i\}$, $\{n_j\}$ such that

$$|x_{m_i n_j}| > \varepsilon^{\alpha_{m_i}+\beta_{n_j}}, \quad i+j \geq 0.$$

Choose R_1, R_2 so that εR_1, $\varepsilon R_2 > 1$; then we have

$$\sum_{i+j\geq 0}\sum |x_{m_i n_j}| R_1^{\alpha_{m_i}} R_2^{\beta_{n_j}} \geq \sum_{i+j\geq 0}\sum (\varepsilon R_1)^{\alpha_{m_i}} (\varepsilon R_2)^{\beta_{n_j}} = \infty,$$

which is a contradiction to the fact that $x \in \Lambda(P)$. Thus $\Lambda(P) \subset \delta^{\alpha,\beta}$.

Let now $x \in \delta^{\alpha,\beta}$. Then we get

$$\lim_{m+n\to\infty} |x_{mn}|^{\frac{1}{\alpha_m+\beta_n}} = 0.$$

The condition (3.3) is equivalent to the following, $\sum_{m+n\geq 0}\sum u_1^{\alpha_m} u_2^{\beta_n} < \infty$, for some

u_1, u_2 with $0 < u_i < 1$, $i = 1,2$. Then for any R_1, R_2 with $0 < R_i < \infty$, choose $\varepsilon > 0$ so that $\varepsilon R_1 < u_1$, $\varepsilon R_2 < u_2$. By (+) there exist integer N such that

$$|x_{mn}| < \varepsilon^{\alpha_m+\beta_n} \quad \text{for} \quad m+n > N.$$

Hence we have

$$\sum\sum_{m+n>N} |x_{mn}| R_1^{\alpha_m} R_2^{\beta_n} \leq \sum\sum_{m+n>N} (\varepsilon R_1)^{\alpha_m} (\varepsilon R_2)^{\beta_n} \leq \sum\sum_{m+n>N} u_1^m u_2^n < \infty.$$

Therefore, $x \in \Lambda(P)$; that is, $\delta^{\alpha,\beta} = \Lambda(P)$.

Corollary 3.4. *If* $\alpha_m = m$, $\beta_n = n$, *then the corresponding space* $\delta^{\alpha,\beta}$ *is the space* δ *of all entire functions of two complex variables considered in* [6].

The Topology on $\delta^{\alpha,\beta}$. For $x \in \delta^{\alpha,\beta}$, let

$$(3.5) \qquad ||x||_{\alpha,\beta} = \sup\{|x_{00}|, |x_{mn}|^{\frac{1}{\alpha_m+\beta_n}} \ \forall\ m,n \geq 1\},$$

and denote by $T_{\alpha,\beta}$, the topology on $\delta^{\alpha,\beta}$ generated by the F-norm $||\cdot||_{\alpha,\beta}$.

We will have occations to make use of the following *Mendelbrojt restriction* as well on $\alpha;\beta$, namely,

$$(3.6) \qquad \varliminf_{m\to\infty} (\alpha_m - \alpha_{m-1}) = 2h; \quad \varliminf_{n\to\infty} (\beta_n - \beta_{n-1}) = 2k,$$

where $h,k > 0$ are finite numbers.

Lemma 3.7. *If* α *and* β *satisfy* (3.6), *then the topologies* T_p *and* $T_{\alpha,\beta}$ *on* $\delta^{\alpha,\beta}$ *are equivalent.*

Proof. Let $x^p \to x$ in $T_{\alpha,\beta}$. Take $R_1,R_2 > 0$ and $\varepsilon > 0$ arbitrarily. One can find an integer N such that

$$\alpha_n - \alpha_{n-1} \geq h; \quad \beta_n - \beta_{n-1} \geq k, \ \forall\ n \geq N.$$

Choose $\eta > 0$ so small that ηR_1, $\eta R_2 \leq 1$ and

$$\eta + \frac{(\eta R_1)^{\alpha_1}(\eta R_2)^{\beta_1}}{[1-(\eta R_1)^h][1-(\eta R_2)^k]} [N(L-(\eta R_1)^h)+(\eta R_1)^h][N(1-\eta R_2)^k)+(\eta R_2)^k] < \varepsilon$$

Now for $p \geq Q = Q(\eta)$,

$$|x^p_{00}-x_{00}| < \eta, \quad |x^p_{mn}-x_{mn}| < \eta^{\alpha_m+\beta_n}, \ \forall\, m,n \geq 0.$$

Hence for $p \geq Q$, we have

$$p_{R_1,R_2}(x^p-x) = \sum\sum_{m+n\geq 0} |x^p_{mn}-x_{mn}| R_1^{\alpha_m} R_2^{\beta_n}$$

$$< \eta + \sum\sum_{m+n\geq 1} (\eta R_1)^{\alpha_m}(\eta R_2)^{\beta_n}$$

$$\leq \eta + \left(\sum_{i=1}^{N-1} (\eta R_1)^{\alpha_i} + \frac{(\eta R_1)^{\alpha_N}}{1-(\eta R_1)^h}\right)\left(\sum_{i=1}^{N-1} (\eta R_2)^{\beta_j} + \frac{(\eta R_2)^{\beta_N}}{1-(\eta R_2)^k}\right)$$

$$\leq \eta + \frac{(\eta R_1)^{\alpha_1}(\eta R_2)^{\beta}}{(1-(\eta R_1)^h(1-(\eta R_2)^k)} (N(1-(\eta R_1)^h)+(\eta R_1)^h)(N(1-(\eta R_2)^k) + (\eta R_2)^k) < \varepsilon.$$

Thus $T_p \subset T_{\alpha,\beta}$.

Suppose now that $x^p \to x$ in T_p, then in particular $x^p_{00} \to x_{oo}$; suppose $\varepsilon > 0$ and choose R_1, R_2 such that $R_1, R_2 \geq \frac{1}{\varepsilon}$, then there exists $Q = Q(\varepsilon,R_1,R_2)$ such that

$$|x^p_{00}-x_{00}| < \varepsilon; \quad p_{R_1,R_2}(x^p-x) < 1,$$

for $p \geq Q$. Hence for all m,n with $m+n \geq 1$ and $p \geq Q$,

$$|x^p_{mn}-x_{mn}|^{\frac{1}{\alpha_m+\beta_n}} \leq \frac{1}{R_1^{\frac{\alpha_m}{\alpha_m+\beta_n}} R_2^{\frac{\beta_n}{\alpha_m+\beta_n}}} \leq \frac{1}{R} \leq \varepsilon,$$

where $R = \min\{R_1,R_2\} \geq \varepsilon^{-1}$, giving $x^p \to x$ in $T_{\alpha,\beta}$. Hence $T_{\alpha,\beta} \subset T_p$. Therefore $T_{\alpha,\beta} \approx T_p$.

Proposition 3.8. The space $(\delta^{\alpha,\beta}, \|\cdot\|_{\alpha,\beta})$ is complete.

Proof. It is sufficient to show that for each Cauchy sequence $\{x^p\}$ in $\delta^{\alpha,\beta}$, there corresponds a unique $x \in \delta^{\alpha,\beta}$ such that $\|x^p - x\| \to 0$. For $\varepsilon > 0$, we find $Q = Q(\varepsilon)$ such that

$$(*) \qquad |x^p_{00} - x^q_{00}|, \; |x^p_{mn} - x^q_{mn}|^{\frac{1}{\alpha_m+\beta_n}} < \varepsilon, \; \forall \; pq \geq Q; \; m,n \geq 0;$$

and so $\{x^p_{mn}\}$ is Cauchy in $\mathbb{K}$ for all $m,n \geq 0$. So let $x^p_{mn} \to x_{mn}$, say, as $p \to \infty$ for each pair $m,n \geq 0$. But we have

$$|x_{mn}|^{\frac{1}{\alpha_m+\beta_n}} \leq |x^p_{mn} - x_{mn}|^{\frac{1}{\alpha_m+\beta_n}} + |x^p_{mn}|^{\frac{1}{\alpha_m+\beta_n}}$$

So for any fixed p (say, $p = Q$) we get

$$|x^p_{mn}|^{\frac{1}{\alpha_m+\beta_n}} \to 0 \quad \text{as} \quad m+n \to \infty$$

$$\Rightarrow \lim_{m+n\to\infty} |x_{mn}|^{\frac{1}{\alpha_m+\beta_n}} = 0.$$

Hence $x \in \delta^{\alpha,\beta}$. Using (*) once again we find that the result is proved.

Lemma 3.9. The space $(\delta^{\alpha,\beta}, T_{\alpha,\beta})$ is nonnormable.

Proof. Consider G to be an arbitrary 0-neighborhood in $(\delta^{\alpha,\beta}, T_{\alpha,\beta})$. Then for some $\varepsilon > 0$ we get

$$\{x : p_{R_1,R_2}(x) < \varepsilon\} \subset G.$$

Define $x^p \in \delta^{\alpha,\beta}$ by

$$x^p = \frac{\varepsilon}{2} \left(\frac{1}{R_1}\right)^{\alpha_p} \left(\frac{1}{R_2}\right)^{\beta_p} e_{pp}.$$

Then $x^p \in G$ for $p \geq 1$. If $\varepsilon_p = 2^{-(\alpha_p+\beta_p)}$, then

$$p_{2R_1,2R_2}(\varepsilon_p x^p) = \frac{\varepsilon}{2} > \frac{\varepsilon}{4},$$

and so

$$\varepsilon_p x^p \notin \{x \in \delta^{\alpha,\beta} : p_{2R_1,2R_2}(x) < \frac{\varepsilon}{4}\}$$

and hence $\varepsilon_0 x^p \not\to 0$. Therefore, no neighborhood G of 0 is bounded with respect to T_p and hence with respect to $T_{\alpha,\beta}$ by Lemma 3.7.

Theorem 3.10. If α and β satisfy (3.6), then $(\delta^{\alpha,\beta}, T_{\alpha,\beta})$ is a nonnormable Fréchet space.

Proof. This follows from Lemma 2.7, Propositions 2.8 and 3.9.

Remark 3.11. In particular, Theorem 3.10 includes the Theorem 2.1 of [6].

4. ESTIMATION OF KOLMOGOROV'S DIAMETERS

Consider $\Lambda(P)$ and $u,v \in P$ with $u_{mn} \leq v_{mn}$ for $m,n \geq 0$. Define $\{\alpha_{mn}\} \in Q$ by

$$\alpha_{mn} = \begin{cases} \dfrac{u_{mn}}{v_{mn}}, & \text{when } v_{mn} \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Then we have the following estimation:

Proposition 4.2. Let $u_{mn} > 0$ for all m,n, then we have

$$\inf_{m+n\geq s} \alpha_{mn} \leq \delta_{\frac{(s)(s+3)}{2}}(U_v, U_u) \leq \delta_{\frac{(s+1)(s+2)}{2}}(U_v, U_u) - \sup_{m+n\geq s} \alpha_{mn}$$

where U_u is the closed unit ball relative to p_u and a similar expression for U_v.

Proof. Let

$$L_s = sp\{e^{mn} : 0 \leq m+n \leq s-1\}.$$

For $x \in \Lambda(P)$ such that $x \in U_v$, we have

$$p_u(x-x^{(s-1)}) = \sum\sum_{m+n\geq s} |x_{mn}| u_{mn}$$

$$= \sum\sum_{m+n\geq s} |x_{mn}| \frac{u_{mn}}{v_{mn}} v_{mn}$$

$$\leq \sup_{m+n\geq s} \alpha_{mn} \sum\sum_{m+n\geq s} |x_{mn}| v_{mn}$$

$$\leq \sup_{m+n\geq s} \alpha_{mn} \sum\sum_{m+n\geq 0} |x_{mn}| v_{mn}$$

$$= \sup_{m+n\geq s} \alpha_{mn} p_v(x) \leq \sup_{m+n\geq s} \alpha_{mn}.$$

Hence

$$x-x^{(s-1)} \in (\sup_{m+n\geq s} \alpha_{mn}) U_u \Rightarrow x \in L_s + (\sup_{m+n\geq s} \alpha_{mn}) U_u$$

(*)

$$\Rightarrow \frac{\delta_{(s+1)(s+2)}(U_v, U_u)}{2} \leq {}_{m+n\geq s}\, \alpha_{mn},$$

Next, let $x \in \inf_{m+n\leq s} \alpha_{mn}[U_u \cap L_{s+1}]$. Then for $x \in L_{s+1}$, we have

$$p_v(x) = \sum\sum_{m+n\geq 0} |x_{mn}| v_{mn}$$

$$= \sum\sum_{0\leq m+n\leq s} |x_{mn}| v_{mn}$$

$$\leq (\sup_{m+n\leq s} \alpha_{mn}^{-1}) \sum\sum_{0\leq m+n\leq s} |x_{mn}| u_{mn}$$

$$= (\sup_{m+n\leq s} \alpha_{mn}^{-1}) \inf_{m+n\leq s} \alpha_{mn} \sum\sum_{0\leq m+n\leq s} \frac{|x_{mn}|\, u_{mn}}{\inf_{m+n\leq s} \alpha_{mn}}$$

$$\leq (\sup_{m+n\leq s} \alpha_{mn}^{-1}) \cdot \inf_{m+n\leq s} \alpha_{mn}$$

$$= 1$$

since $x \in (\inf_{m+n\leq s} \alpha_{mn})U_u$. Thereforem for $x \in \inf_{m+n\leq s} \alpha_{mn}[U_u \cap L_{s+1}]$ we get $p_v(x) \leq 1$. Consequently we have

$$(\inf_{m+n\leq s} \alpha_{mn})[U_u \cap L_{s+1}] \subset U_v$$

which leads to

$$(**) \qquad \frac{\delta_{s(s+3)}(U_v, U_u)}{2} \geq \inf_{m+n\leq s} \alpha_{mn},$$

by Lemma 2.1. The required result follows from (*) and (**).

Lemma 4.3. If a bounded subset B *of* ℓ^{11} *relatively compact, then we have*

$$(*) \qquad \lim_{m+n\to\infty} \sup_{x\in B} \sum\sum_{i+j\geq m+n} |x_{ij}| = 0.$$

Proof. Suppose (*) is not true, then there exists increasing sequence $\{m_k\}$, $\{n_k\}$ and $\varepsilon > 0$ such that

$$A_{1_k} \geq 2\varepsilon > 0; \forall\, k \geq 1$$

where

$$A_1 = \sup_{x\in B} \sum\sum_{i+j\geq 1_k} |x_{ij}| \quad \text{with} \quad 1_k = m_k + n_k.$$

Hence B contains a sequence $\{x^{1_k}\}$ such that

$$\sum\sum_{i+j\geq 1_k} |x_{ij}^{1_k}| \geq \varepsilon, \forall\, k \geq 1.$$

Let $p_1 = 1_1$, then we can find $p_2 = 1_{k_0}$ for some k_0 such that

$$(+) \qquad \sum\sum_{i+j\geq p_2} |x_{ij}^{p_2}| \geq \varepsilon \quad \text{and} \quad \sum\sum_{i+j\geq p_2} |x_{ij}^{p_1}| < \frac{\varepsilon}{2}.$$

Proceeding in this way we can find an increasing subsequence $\{p_k\}$ of $\{1_k\}$ such that

$$\sum\sum_{i+j\geq p_k} |x_{ij}^{p_k}| \geq \varepsilon; \quad \sum\sum_{i+j\geq p_{k+1}} |x_{ij}^{p_k}| < \frac{\varepsilon}{2}, \forall\, k \geq 1.$$

Therefore

$$\sum_{i+j\geq 1}\sum |x_{ij}^{p_k}-x_{ij}^{p_t}| > \frac{\varepsilon}{2}, \ \forall\, k,t \geq 1, \quad k \neq t.$$

For otherwise, say, for $k > t$ we have

$$\begin{aligned}\sum_{i+j\geq p_k}\sum |x_{ij}^{p_k}| &\leq \sum_{i+j\geq p_k}\sum |x_{ij}^{p_t}| + \frac{\varepsilon}{2}\\ &\leq \sum_{i+j\geq p_{t+1}}\sum |x_{ij}^{p_t}| + \frac{\varepsilon}{2} < \varepsilon,\end{aligned}$$

by using (+) and this gives a contradiction. Therefore $\{x^{p_k}\}$ which belongs to B cannot have any convergent subsequence, and this again contradicts the fact that B is relatively compact.

Lemma 4.4. Let X *be an* l.c.TVS *and* B_X *the neighborhood system at origin consisting of all balanced, closed and convex sets. Suppose* $A \subset X$ *is bounded. Then* A *is precompact if and only if* $\lim_{m+n\to\infty} \delta_{m+n}(A,u) = 0$ *for* $u \in B_X$.

Proof. Let $\lim_{m+n\to\infty} \delta_{m+n}(A,u) = 0$. Take $\varepsilon > 0$. One can find a larger integer N such that $\delta_N(A,u) \leq \frac{\varepsilon}{4}$. Hence correspondingly there exists a finite set $\{y_1,\ldots,y_N\}$ of X and a positive number α such that (cf. Proposition 1.3 in [20])

$$\delta_N(A,u) + \frac{\varepsilon}{4} ; \quad A \subset \alpha u + \Gamma\{y_1,\ldots,y_N\}$$

where Γ represents the balanced convex hull. Hence

$$A \subset \frac{\varepsilon}{2} u + \Gamma\{y_1,\ldots,y_N\}, \quad \text{for } u \text{ is balanced.}$$

But $\Gamma\{y_1,\ldots,y_N\}$ is precompact, because let us define $A_i = \{y_i\}$; $1 \leq i \leq N$, then A_i's are compact and convex. Let

$$L = \{(\alpha_1,\ldots,\alpha_N) : \alpha_i \in K \text{ and } \sum_{i=1}^{N} |\alpha_i| \leq 1\},$$

then L is a compact subset of $\mathbb{K}^N$. Let

$$S = L \times X_1 \times \dots \times A_N, \quad \text{then } S \text{ is compact in } \mathbb{K}^N \times X^N.$$

Define a function $f: \mathbb{K}^N \times X^N \to X$ by

$$f(\alpha_1, \dots, \alpha_N, x_1, \dots, x_N) = \sum_{i=1}^{N} \alpha_1 x_1.$$

As f is continuous, $f(S)$ is compact. But $f(S)$ is nothing but balanced convex hull of $\{y_1, \dots, y_N\}$. So $\Gamma\{y_1, \quad , y_N\}$ is precompact being compact. Thus

$$\Gamma\{y_1, \dots, y_N\} \subset \bigcup_{i=1}^{M} \{x_i + \frac{\varepsilon}{2} u\}.$$

So

$$A \subset \frac{\varepsilon}{2} u + \bigcup_{i=1}^{M} \{x_i + \frac{\varepsilon}{2} u\} \subset \bigcup_{i=1}^{M} \{x_i + \varepsilon u\}$$

Therefore A is precompact.

Conversely let A be precompact. So by Lemma 2.3 we have $\delta_n(A,u) \to 0$. But we know $\delta_{m+n} \le \delta_n$ for each m,n. Hence the result follows.

Proposition 4.5. Let $u,v \in P$ with $u_{mn} \le v_{mn}$, then

$$\delta_{m+n}(U_v, U_u) \to 0 \iff u_{mn}/v_{mn} \in c_{00},$$

where u_{mn}/v_{mn} is regarded to be zero whenever $v_{mn} = 0$.

Proof. Let $\{u_{mn}/v_{mn}\} \in c_{00}$. By Proposition 4.2 we have

$$\delta_{\frac{(s+1)(s+2)}{2}}(U_v, U_u) \le \sup_{m+n \ge s} \alpha_{mn}$$

$$\delta_{\frac{(s+1)(s+2)}{2}}(U_v, U_u) \to 0 \text{ as } s \to \infty .$$

Choose m and n so large that $m+n \ge (s+1)(s+2)/2 \ge s$, $s > 1$ and this proves the sufficiency part.

Conversely let now $\delta_{m+n}(U_v, U_u) \to 0$. Define a map $T:\Lambda(P) \to \ell^{11}$ by

$$Tx = xu$$

where $u \in P$. Then

$$\|T_x\|_{\ell^{11}} = \sum_{m+n>0}\sum |x_{mn}| u_{mn} = p_u(x).$$

Obviously

$$T[U_u] = \{\{x_{mn}u_{mn}\}: x \in \Lambda(P), \sum\sum_{m+n\geq 0} |x_{mn}|u_{mn} \leq 1\}$$

$$\Rightarrow T[U_u] \subset u_{\ell^{11}}$$

where $U_{\ell^{11}}$ is the closed unit ball in ℓ^{11}. Therefore

$$\delta_{m+n}(T(u_v), u_{\ell^{11}}) \leq \delta_{m+n}(T(U_v), T(U_u)) \leq \delta_{m+n}(U_v, U_u) \to 0$$

by Lemma 2.2. So $T(U_v)$ is precompact by Lemma 4.4 and hence relatively compact subset of ℓ^{11}. Define now

$$\theta^{st} = \alpha_{st} e^{st}; \quad s,t \geq 0,$$

where α_{st} stands as in (4.1). Observe that $\theta^{st} \in U_{\ell^{11}}$. If $v_{st} = 0$, then $\theta^{st} = 0$ and so $\theta^{st} \in U_v u$. If $U_{st} \neq 0$ then $\theta^{st} = \frac{U_{st}}{v_{st}} e^{st}$. But $e^{st}/v_{st} \in U_v$, and hence $\theta^{st} \in U_v, u$. Thus $\theta^{st} \in T[U_v]$ for all $s+t \geq 0$. So

$$|\alpha_{st}| \leq \sup_{x\in T[U_v]} \sum\sum_{m+n\geq s+t} |x_{mn}|.$$

But $T(U_v)$ is relatively compact, so by Lemma 4.3 we get

$$\lim_{s+t\to\infty} \sup_{x\in T(U_v)} \sum\sum_{m+n\geq s+t} |x_{mn}| = 0$$

$$\Rightarrow \{\alpha_{st}\} \in c_{00}$$

and hence $\{u_{mn}/v_{mn}\} \in c_{00}$.

<u>Proposition 4.6.</u> $K_u^v : X_v \to X_u$ is precompact if and only if

(*) $$\delta_{m+n}(v,n) \to 0 \quad \text{as} \quad m+n \to \infty$$

<u>Proof.</u> Let K_u^v be precompact. The closed unit bell in X_v looks like $\{V+N_v : v \in V\}$. So it follows that

$$K_u^v[v+N_v] = V+N_u \text{ is precompact on } X_u$$

$$\Longleftrightarrow \delta_{m+n}(v+N_u, U+N_u) \quad 0, \text{ by Lemma 4.4.}$$

$$\Longleftrightarrow \delta_{m+n}(v,u) \to 0, \text{ by Proposition 2.4.}$$

Proposition 4.7. The space $\Lambda(P)$ is Schwartz if and only if for each $u \in P$, there exist $v \in P$ with $u \leq v$ and such that $\{u_{mn}/v_{mn}\} \in c_{00}$.

Proof. This follows from Propositions 4.5 and 4.6.

Definition 4.8. If a power set P satisfies the conditions:

(i) Each $u \in P$ is non-decreasing, that is to say, $u_{mn} \leq u_{st}$ whenever $m+n \leq s+t$.

(ii) For each $a \in P$, there exists $b \in P$ such that $a_{mn} \leq b_{mn}^2$.

Then P is called a set of *infinite type* and the corresponding K-matrix space is called a $G_{\infty\infty}$*-space*.

Definition 4.9. If P satisfies the conditions:

(i) Each $a \in P$ is non-increasing, that is to say, $a_{mn} \leq a_{st}$ whenever $m+n \geq s+t$.

(ii) For each $a \in P$, there exists $b \in P$ such that $a_{mn} \leq b_{mn}$, $\forall\, m,n$.

Then P is said to be of *finite type* and the corresponding $\Lambda(P)$ is called a G_{11}*-space*.

Proposition 4.10. A $G_{\infty\infty}$-space $(\Lambda(P),T)$, is Schwartz if and only if there exists u in P such that $\{1/u_{mn}\} \in c_{00}$.

Proof. If the given space is Schwartz then by Proposition 4.7, to every $a \in P$ there corresponds $u \in P$ such that $a \leq u$ and $\{a_{mn}/u_{mn}\} \in c_{00}$; but due to the non-decreasing character of $\{a_{mn}\}$

$$\frac{a_{00}}{u_{mn}} \leq \frac{a_{mn}}{u_{mn}}$$

$$\Rightarrow \left\{ \frac{a_{00}}{u_{mn}} \right\} \in c_{00}$$

$$\Rightarrow \left\{ \frac{1}{u_{mn}} \right\} \in c_{00}$$

Conversely, let there be an u in P with $\{1/u_{mn}\} \in c_{00}$. Choose an arbitrary a in P. Then one can find b and d in P such that

$$b_{mn} \geq \max\{a_{mn}, u_{mn}\}, \quad \text{and} \quad a_{mn}^2 \leq d_{mn}.$$

Let there be g in P with $g_{mn} \geq \max\{b_{mn}, d_{mn}\}$. Hence

$$\frac{a_{mn}}{g_{mn}} \leq \frac{\sqrt{d_{mn}}}{g_{mn}} \leq \frac{1}{\sqrt{g_{mn}}} \leq \frac{1}{\sqrt{b_{mn}}} \leq \frac{1}{\sqrt{u_{mn}}} \to 0 \quad \text{as} \quad m+n \to \infty$$

$$\Rightarrow \left\{ \frac{a_{mn}}{g_{mn}} \right\} \in c_{00}.$$

So by Proposition 4.7, $\Lambda(P)$ is Schwartz.

<u>Proposition 4.11</u>. A G_{11}-space $(\Lambda(P),T)$ is Schwartz if and only if $P \subset c_{00}$.

<u>Proof</u>. Let $P \subset c_{00}$. Take u in P, so by the G_{11}-character, we can find a $v \in P$ with $u_{mn} \leq v_{mn}^2$, for $m,n \geq 0$. But $v \in P \subset c_{00}$ and so

$$\frac{u_{mn}}{v_{mn}} \leq v_{mn} \to 0 \quad \text{as} \quad m+n \to \infty$$

Therefore, $\{u_{mn}/v_{mn}\} \in c_{00}$ and hence $\Lambda(P)$ is Schwartz by Propsition 4.7.

Conversely let $\Lambda(P)$ be Schwartz and assume $u \in P$. Then we can find some $v \in P$ such that

$$\{u_{mn}/v_{mn}\} \in c_{00}.$$

But members of P are non-increasing

$$\Rightarrow \frac{u_{mn}}{v_{mn}} \geq \frac{u_{mn}}{v_{00}}$$

$$\Rightarrow \{u_{mn}/v_{00}\} \in c_{00}$$

$$\Rightarrow \{u_{mn}\} \in c_{00}$$

$$\Rightarrow P \subset c_{00},$$

and the result is proved.

Corollary 4.12. If a G_{11}-space $(\Lambda(P),T)$ is not Schwartz, then $\Lambda(P) = \ell^{11}$.

Proof. Suppose $\Lambda(P) \not\subseteq \ell^{11}$, then there exists on $x \in \ell^{11}$ such that $x \notin \Lambda(P)$, and so

$$\sum\sum_{m+n\geq 0} |x_{mn}| < \infty;$$

also there exists an $u \in P$ for which the series

$$\sum\sum_{m+n\geq 0} |x_{mn}| u_{mn} \qquad (+)$$

diverges. Hence for $N > 0$ we can determine integers R_N and S_N so that

$$\sum_{m=0}^{R_N} \sum_{n=0}^{S_N} |x_{mn}| u_{mn} > N.$$

Hence we get

$$N < \sum_{m=0}^{R_N} \sum_{n=0}^{S_N} |x_{mn}| u_{mn}$$

$$\leq u_{R_N S_N} \sum_{m=0}^{R_N} \sum_{n=0}^{S_N} |x_{mn}|$$

$$\leq u_{R_N S_N} \sum\sum_{m+n\geq 0} |x_{mn}|$$

So by (+) we have

$$\frac{1}{u_{R_N S_N}} \leq \frac{1}{N} \left(\sum\sum_{m+n\geq 0} |x_{mn}| \right) \to 0$$

as $N \to \infty$. Since $\{u_{mn}\}$ is non-decreasing, $\{1/u_{mn}\} \in c_{00}$. But this contradicts the fact that $\Lambda(P)$ is not Schwartz. Hence the result follows.

5. We begin with the following Definition:

Definition 5.1. Corresponding to an l.c.TVS (X,T), let $\Delta_{B_X}(X) = \{\alpha=\{\alpha_{mn}\}$: to every $u \in B$ there exist $v \in B_X$ such that $v \prec u$ and $\lim_{m+n\to\infty} |\alpha_{mn}|\delta_{m+n}(v,u)=0\}$. Then $\Delta_{B_X}(X)$ is called *bi-dimetrical dimension* of X.

Proposition 5.2. If B'_X is another neighbourhood system at origin besides B_X, then

$$\Delta_{B_X}(X) = \Delta_{B'_X}(X).$$

Proof. Let $x \in \Delta_{B_X}(X)$. Since neighborhood systems are equivalent, if $u' \in B'_X$, then there exists $u \in B_X$ with $u \subset u'$. So then for $v \in B_X$ with $v < u$ we get

$$\lim_{m+n\to\infty} |x_{mn}|\delta_{m+n}(v,u) = 0.$$

Given $v \in B_X$ then exists $v' \in B'_X$ such that $v' \subset v$. Thus

$$v' \subset v < u \subset u'$$

$$\Longrightarrow \delta_{m+n}(v',u') \leq \delta_{m+n}(v',u) \leq \delta_{m+n}(v,u)$$

$$\Longrightarrow |x_{mn}|\delta_{m+n}(v',u') \leq |x_{mn}|\delta_{m+n}(v,u) \to 0 \quad \text{as} \quad m+n \to \infty.$$

So $x \in \Delta_{B'_X}(x)$, and hence $\Delta_{B_X}(X) \subset \Delta_{B'_X}(X)$.

Now let $x \in \Delta_{B'_X}(X)$. Take $u \in B_X$, then there exists $u' \in B'_X$ such that $u' \subset u$. So then for $v' \in B'_X$ with $v' < u'$ such that

$$\lim_{m+n\to\infty} |x_{mn}|\delta_{m+n}(v',u') = 0.$$

Given $v' \in B'_X$, there exist $v \in B_X$ such that $v \not\subset v'$. So

$$v \subset v' < u' \subset u$$
$$\Rightarrow \delta_{m+n}(v,u) \le \delta_{m+n}(v,u') \le \delta_{m+n}(v',u')$$
$$\Rightarrow |x_{mn}|\delta_{m+n}(v,u) \le |x_{mn}|\delta_{m+n}(v',u') \to 0 \quad \text{as} \quad m+n \to \infty.$$

So $x \in \Delta_{B_X}(X)$, and hence $\Delta_{B'_X}(X) = \Delta_{B_X}(X)$.

Remark 5.3. From above, we find that $\Delta_{B_X}(X)$ is independent of the choice of B_X, and so we would prefer to use the notation $\Delta(X)$ for $\Delta_{B_X}(X)$.

Proposition 5.4. If two locally convex spaces $(X,\mathcal{F})$ and $(Y,\mathcal{Y})$ are topologically isomerphic by the map T, then we get

$$\Delta(X) = \Delta(Y).$$

Proof. By Lemma 2.2, we have,

$$\delta_{m+n}(v,u) \ge \delta_{m+n}(Tv,Tu) \ge \delta_{m+n}(T^{-1}(Tv),T^{-1}(Tu)) = \delta_{m+n}(v,u)$$

where $u,v \in B_x$ with $v < u$. The above inequality implies that

(*) $$\delta_{m+n}(v,u) = \delta_{m+n}(Tv,Tu)$$

Let

$$B^* = \{Tu : u \in B_x\}$$

From (*) we find that $\Delta(X) = \Delta_B(Y)$. But by 5.3 $\Delta_B(Y) = \Delta(Y)$. Hence $\Delta(Y) = \Delta(X)$.

Corollary 5.5. We have $c_{00} \subset \Delta(X)$.

Proof. Let $x = \{x_{mn}\} \in c_{00}$. Hence for u in B_X, there exists a large integer N Such that $x_{mn} \in u$ for all $m+n \ge N$, then it is enough to show that if $v \in B_X$ with $v < u$, then

$$\lim_{m+n\to\infty} |x_{mn}|\delta_{m+n}(v,u) = 0$$

Since $\delta_0(v,u) \leq \rho$, where ρ comes from $v \subset \rho u$ and

$$\delta_0(v,u) \geq \delta_1(v,u) \geq \delta_2(v,u) \geq \dots$$

we find

$$\delta_{m+m}(v,u) \leq \rho, \ \forall\ m,n, \quad m+n \geq N$$

$$\Rightarrow \lim_{m+n\to\infty} |x_{mn}|\delta_{m+n}(v,u) = 0,$$

and hence the result is proved.

Proposition 5.6. A closed subspace M of an l.c. TVS has codimension less than n, that is to say $\dim(X/M) \leq n$, if and only if there exist $f_1,\dots,f_n \in X^*$ such that

$$M = \{x \in X: f_i(x) = 0,\ 1 \leq i \leq n\} .$$

Definition 5.7. A closed subspace M is said to be *complemented* in space X, if there exists a closed subspace N of X such that

$$X = M \oplus N$$

We would need the following well known result in the sequel (cf. [18]).

Lemma 5.8. Let M *be a closed subspace of* X *such that*

$$\dim(X/M) < \omega$$

Then M *is complemented in* X.

Theorem 5.9. Let $\langle X,Y\rangle$ *be a dual system. Then*

$$\Delta(X,\sigma(X,Y)) = \Omega$$

Proof. Clearly $\Delta(X,\sigma(X,Y)) \subset \Omega$. So let u be a neighbourhood of origin in X with respect to $\sigma(X,Y)$. Then there exists $\varepsilon > 0$ such that $\varepsilon u_s \subset u$, where for $y_1,\dots,y_s \in Y$,

$$u_s = \{x \in X: |\langle x,y_i\rangle| \leq i,\ 1 \leq i \leq s\}.$$

Let

$$N_s = \{x \in X: \langle x,y_i\rangle = 0,\ 1 \leq i \leq s\}.$$

So N_s is a $\sigma(X,Y)$-closed subspace of X. Then by Definition 5.6, we have $\dim(X/N_s) \leq s$. There exists a closed subspace L_s such that

$$X = N_s \oplus L_s$$

Now observe that

$$\dim(L_s) = \dim(X/N_s) \leq s.$$

If $\rho < 1$, then

$$u_s \subset N_s + L_s \subset \rho u_s + L_s$$

$$\Rightarrow \delta_s(u_s,u_s) \leq \rho.$$

Since ρ is arbitrary, allow $\rho \to 0$. Then we get

$$\delta_s(u_s,u_s) = 0$$

$$\Rightarrow \delta_{m+n}(u_s,u_s) = 0, \ \forall\ m,n \text{ with } m+n \geq s.$$

Because $\varepsilon u_s \subset u$, we have the inequality

$$\delta_{m+n}(u_s,u) \leq \delta_{m+n}(u_s,u_s)$$

$$\delta_{m+n}(u_s,u) = 0, \ \forall\ m+n \geq s.$$

Now take any $x \in \Omega$. Then we have

$$|x_{mn}|\delta_{m+n}(u_s,u) = 0, \ \forall\ m+n \geq s$$

$$\Rightarrow \lim_{m+n\to\infty} |x_{mn}|\delta_{m+n}(u_s,u) = 0,$$

and so $x \in \Delta(X,\sigma(X,Y))$, thereby proving the required result.

Definition 5.10. Let X,Y be two l.c.TVS. A linear map $T:X \to Y$ is called *almost open* if

$$\overline{Tu} \in B_y, \ \forall\ u \in B_X.$$

Theorem 5.11. (Permanence Theorem). Let X,Y *be two* l.c.TVS. *Suppose* $T:X \to Y$ *is linear continuous and almost open, then we have*

$$\Delta(X) \subset \Delta(Y).$$

Proof. Let $u^* \in B_Y$. So $T^{-1}(u^*) \supset u$, for some u in B_X by our assumption of continuity on the linear map T. Let $x \in \Delta(X)$, so for $u \in B_X$ there exists $v \in B_X$ with $v \prec u$ and such that

$$\lim_{m+n\to\infty} |x_{mn}| \delta_{m+n}(v,u) = 0$$

We now proceed to find v^* in B_Y such that $v^* \prec u^*$ with

$$\lim_{m+n\to\infty} |x_{mn}| \delta_{m+n}(v^*,u^*) = 0.$$

Let us define $v^* = \overline{Tv}$. So $v^* \in B_Y$ for T is almost open. Then we have

$$\lim_{m+n\to\infty} |x_{mn}| \delta_{m+n}(v^*,u^*) \leq \lim_{m+n\to\infty} |x_{mn}| \delta_{m+n}(Tv,Tu)$$
$$\leq \lim_{m+n\to\infty} |x_{mn}| \delta_{m+n}(v,u)$$

and hence $x = \{x_{mn}\} \in \Delta(Y)$, consequently $\Delta(X) \subset \Delta(Y)$.

Corollary 5.12. For an l.c.TVS X, *the following statements are equivalent:*

(i) X *is Schwartz*

(ii) $\ell^{\infty\infty} \subset \Delta(X)$.

(iii) $c_{00} \subsetneq \Delta(X)$.

Proof. (i) ⇒ (ii) Let X be Schwartz and assume $x \in \ell^{\infty\infty}$. So to each $u \in B_X$, there exists $v \in B_X$, $v \prec u$ such that

$$\delta_{m+n}(v,u) \to 0 \text{ as } m+n \to \infty;$$

adn also there exists $k > 0$ such that $|x_{mn}| \leq k$, $\forall\, m,n \geq 0$.
Thus

$$\lim_{m+n\to\infty} |x_{mn}| \delta_{m+n}(v,u) = 0,$$

which gives $x \in \Delta(X)$ and hence $\ell^{\infty\infty} \subset \Delta(X)$.

(ii) ⇒ (iii). Assume that $\ell^{\infty\infty} \subset \Delta(X)$, but we know that $c_{00} \subsetneq \ell^{\infty\infty}$, and hence $c_{00} \subsetneq \Delta(X)$.

(iii) ⇒ (i). Let $c_{00} \subsetneq \Delta(X)$, then there exists $x \in \Delta(X)$ but $x_{mn} \not\to 0$ as $m+n \to \infty$. So there exist sequences of integers $\{m_k\}$ and $\{n_k\}$ such that

$$\inf |x_{m_k n_k}| = \eta > 0.$$

Take $u \in B_X$, then there exists $v \in B_X$ with $v \prec u$ and such that

(*) $$\lim_{m+n\to\infty} |x_{mn}|\delta_{m+n}(v,u) = 0.$$

But we know

$$\eta\delta_{m_k+n_k}(v,u) \leq |x_{m_k+n_k}|\delta_{m_k+n_k}(v,u)$$

$$\Rightarrow \eta\delta_{m_k+n_k}(v,u) \to 0$$

by (*). So we get $\delta_{m+n}(v,u) \to 0$, and hence X is Schwartz by Lemma 4.5 and Proposition 4.7.

Corollary 5.13. Let $T:X \to Y$ *be a continuous linear and almost open. Suppose* X *is Schwartz, then* Y *is Schwartz.*

Proof. Under the given hypothesis we have

$$\Delta(X) \subset \Delta(Y)$$

by Theorem 5.11. But X is Schwartz, so we get

$$\ell^{\infty\infty} \subset \Delta(X)$$

$$\Rightarrow \ell^{\infty\infty} \subset \Delta(Y).$$

Therefore Y is Schwartz by Proposition 5.12.

6. DUALITY OF $\delta^{\alpha,\beta}$:

We begin with the following; define

(6.1) $$d^{\alpha,\beta} = \{x\in\Omega: \sup\{|x_{00}|, |x_{mn}|^{\frac{1}{\alpha_m+\beta_n}} \ \forall\ m+n \geq 1\} < \infty\}.$$

Then we have

Proposition 6.2. $(\delta^{\alpha,\beta})^{\times} = d^{\alpha,\beta}$ and $(d^{\alpha,\beta})^{\times} = \delta^{\alpha,\beta}$.

Proof. We follow [12] for the proof. Let $x \in d^{\alpha,\beta}$. Then there exist $M > 0$ with

$$|x_{ij}| < M^{\alpha_i+\beta_j}, \ i+j > 0.$$

One may choose $\varepsilon > 0$ such that $\varepsilon M < 1$. If $y \in \delta^{\alpha,\beta}$ then we get

$$|y_{ij}| \leq \varepsilon^{\alpha_i+\beta_j}, \quad \text{for} \quad i+j \geq i_0 = i_0(\varepsilon)$$

therefore

$$\sum\sum_{i+j\geq 0} |x_{ij}y_{ij}| = \sum\sum_{0\leq i+j<i_0} |x_{ij}y_{ij}| + \sum\sum_{i+j\geq i_0} |x_{ij}y_{ij}|$$

$$\leq \sum\sum_{0\leq i+j<i_0} |x_{ij}y_{ij}| + \sum\sum_{i+j\geq i_0} (\varepsilon M)^{\alpha_i+\beta_j} < \infty$$

$$\Rightarrow d^{\alpha,\beta} \subset (\delta^{\alpha,\beta})^{\times}.$$

On the other hand, let $x \in (\delta^{\alpha,\beta})^{\times}$ but $x \notin d^{\alpha,\beta}$. Hence there exist increasing sequences $\{m_k\}$ and $\{n_k\}$ such that

$$|x_{m_k n_k}| > k^{2(\alpha_{m_k}+\beta_{n_k})}, \quad k \geq 1.$$

If

$$y_{mn} = \begin{cases} k^{-(\alpha_{m_k}+\beta_{n_k})}, & m = m_k,\ n = n_k; \\ 0, & \text{elsewhere.} \end{cases}$$

Then $y = y_{mn} \in \delta^{\alpha,\beta}$, but $\sum\sum_{i+j\geq 0} |x_{ij}y_{ij}| = \infty$. Thus $x \notin (\delta^{\alpha,\beta})^{\times}$ which is a contradiction. Therefore $(\delta^{\alpha,\beta})^{\times} \subset d^{\alpha,\beta}$. So we got $(\delta^{\alpha,\beta})^{\times} = d^{\alpha,\beta}$. Similarly we can show $(d^{\alpha,\beta})^{\times} = \delta^{\alpha,\beta}$.

<u>Proposition 6.3</u>. A subset B of $\delta^{\alpha,\beta}$ is $\sigma(\delta^{\alpha,\beta}, d^{\alpha,\beta})$ bounded if and only if the following hold:

(i) B is $\sigma(\delta^{\alpha,\beta}, \Phi)$ bounded.

(ii) For each $\varepsilon > 0$, there exists $N_0 = N_0(\varepsilon)$ such that

$$|x_{mn}| \leq \varepsilon^{\alpha_m+\beta_n}, \ \forall\ m+n \geq N_0 \ \text{ and } \forall\ x \in B.$$

<u>Proof</u>. Let (i) and (ii) be true. Choose $y \in d^{\alpha,\beta}$, so for a suitable $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that

$$|y_{mn}| \leq (2\varepsilon)^{-(\alpha_m+\beta_n)}, \ \forall\ m+n \geq N.$$

Let $N^* = \max\{N_0, N\}$. Then we have

$$\sum\sum_{i+j\geq N^*} |x_{ij}y_{ij}| \sum\sum_{i+j\geq N^*} \frac{1}{2^{\alpha_i+\beta_j}} \leq \sum\sum_{i+j\geq 1} \frac{1}{2^{\alpha_i+\beta_j}}, \ \forall\, x \in B.$$

Because of (i) exists a constant $M = M(N^*) > 0$ such that

$$\sum\sum_{0\leq i+j\leq N^*} |x_{ij}y_{ij}| \underline{\Phi} M$$

$$\Rightarrow \sum\sum_{i+j\geq 0} |x_{ij}y_{ij}| \leq M + \sum\sum_{i+j\geq 0} \frac{1}{2^{\alpha_i+\beta_j}} < \infty.$$

Conversely, let B be a $\sigma(\delta^{\alpha,\beta},d^{\alpha,\beta})$ bounded then it is enough to show that (ii) holds. Suppose (ii) is not true, then for some ε 0 there exist increasing sequences $\{m_k\}$ and $\{n_k\}$ and a sequence $x^{(k)}$ in B such that

$$|x^{(k)}_{m_k n_k}| > \varepsilon^{\alpha_{m_k}+\beta_{n_k}}, \quad k \geq 1.$$

Define $y \in \Omega$ by

$$y_{mn} = \begin{cases} \alpha_{m_k}^{1+\frac{\beta n_k}{\alpha_{m_k}}} \cdot \beta_{n_k}^{1+\frac{\beta_{m_k}}{\alpha_{m_k}}}, & m = m_k,\ n = n_k; \\ 0, & \text{elsewhere.} \end{cases}$$

Clearly $y \in d^{\alpha,\beta}$. Consider now

$$p_y(x^{(k)}) \geq |x_{m_k n_k}|\ |y_{m_k n_k}| = \alpha_{m_k}^{1+\frac{\beta_{n_k}}{\alpha_{m_k}}} \cdot \beta_{n_k}^{1+\frac{\alpha_{m_k}}{\beta_{n_k}}}$$

So B is not $\eta(\delta^{\alpha,\beta},d^{\alpha,\beta})$ bounded and hence not $\sigma(\delta^{\alpha,\beta},d^{\alpha,\beta})$ bounded, because $\delta^{\alpha,\beta}$ is perfect. This is a contradiction, so (ii) must hold.

Lemma 6.4. Consider the sequence $\{a_{mn}:m,n\geq 0\}$, *satisfying*

$$\lim_{m+n\to\infty} |a_{mn}|^{\frac{1}{\alpha_m+\beta_n}} = 0$$

Then $\sum\sum_{m+n\geq 0} C_{mn}a_{mn}$ *converges if and only if*

$$\{|c_{00}|; |c_{mn}|^{\frac{1}{\alpha_m+\beta_n}}; \ m+n \neq 0\}$$

is bounded .

Proof. Assume first of all that, there exists a positive constant M, such that

$$|c_{00}| \leq M, \ |c_{mn}| \leq M^{\alpha_m+\beta_n}, \ m,n \geq 0, \ m+n \neq 0.$$

By hypothesis we can get an integer N such that

$$|a_{mn}| \leq \left(\frac{1}{2M}\right)^{\alpha_m+\beta_n}, \ m+n \geq N,$$

and hence

$$|a_{mn}c_{mn}| \leq 2^{-(\alpha_m+\beta_n)}, \ m+n \geq N.$$

Therefore for $m+n \geq N$

$$\sum\sum_{m+n\geq 0} |a_{mn}c_{mn}| \leq \sum\sum_{0\leq m+n\leq N-1} |a_{mn}c_{mn}| + \sum\sum_{m+n\geq N} \frac{1}{2^{\alpha_m+\beta_n}}$$

$$\leq \sum\sum_{0\leq m+n\leq N-1} |a_{mn}c_{mn}| + \sum\sum_{m+n\geq 0} \frac{1}{2^{\alpha_m+\beta_n}} < \infty.$$

To show the necessity assume that the series in question is convergent but $\{|c_{00}|; |c_{mn}|^{\frac{1}{\alpha_m+\beta_n}}, \ m,n \geq 0\}$ $m+n \neq 0$ is unbounded. Thus in general there exist increasing sequences $\{m_k\}$ and $\{n_k\}$ such that

$$|c_{m_k n_k}| \geq k^{\alpha_{m_k}+\beta_{n_k}}, \ k \geq 1.$$

Now define a new sequence $a = \{a_{mn}\}$ by

$$a_{mn} = \begin{cases} k^{-(\alpha_{m_k}+\beta_{n_k})} & m = m_k, \ n = n_k; \\ 0, \ \text{elsewhere} & \end{cases}$$

$$\Rightarrow |a_{m_k n_k} C_{m_k n_k}| \geq 1, \ k \geq 1.$$

Hence $\sum\sum_{m+n\geq 0} |a_{mn} C_{mn}|$ does not converge though $\lim_{m+n\to\infty} |a_{mn}|^{\frac{1}{\alpha_m+\beta_n}} = 0.$

Theorem 6.5. Consider $\delta^{\alpha,\beta}$ *equipped with either* T_p *or* $T_{\alpha,\beta}$. *Then every continuous linear functional* Ψ *on* $\delta^{\alpha,\beta}$ *is of the form*

(6.6) $$\Psi(x) = \sum\sum_{m+n\ 0} x_{mn} C_{mn},$$

where

(6.7) $$\{|C_{00}|; |C_{mn}|^{\frac{1}{\alpha_m+\beta_n}}, \ m+n \neq 0, \ m,n \geq 0\}$$

is bounded.

Moreover for any double sequence $\{C_{mn}\}$, *satisfying (6.7), the map* $\Psi : \delta^{\alpha,\beta} \to \mathbb{C}$, *whose value at* x *is given by (6.6), represents a continuous linear functional.*

Proof. Let $\Psi \in (\delta^{\alpha,\beta})^*$. Now define

$$C_{mn} = \Psi(e^{mn})$$

For any element x in $\delta^{\alpha,\beta}$ we can write $x = \sum\sum_{m+n\geq 0} x_{mn} e^{mn}$ where s-th plane section of x is given by

$$x^{(s)} = \sum\sum_{0\leq m+n\leq s} x_{mn} e^{mn}$$

where we note that $x^{(s)} \to x$ in $T_p[T_{\alpha,\beta}]$. Hence we get $\Psi(x^{(s)}) \to \Psi(x)$. Now we observe

$$\Psi(x) = \lim_{s\to\infty} (x^{(s)})$$

$$= \lim_{s\to\infty} \sum\sum_{0\leq m+n\leq s} x_{mn} \Psi(e^{mn})$$

$$= \sum\sum_{m+m\geq 0} x_{mn} C_{mn}.$$

But

$$|x_{mn}|^{\frac{1}{\alpha_m+\beta_n}} \to 0 \quad \text{as} \quad m+n \to \infty$$

and so

$$\Rightarrow \{|c_{00}|, |c_{mn}|^{\frac{1}{\alpha_m+\beta_n}}; m+n \neq 0\}$$

is bounded by Lemma 6.4.

Conversely let Ψ be mentioned as in the hypothesis. It is enough to show that Ψ is continuous, linearity being clear from the definition. So let $x^P \to 0$ in T_P. Let $\varepsilon > 0$ be chosen such that

$$\varepsilon M < 1, \quad \text{where } M = \sup\{|c_{00}|, |c_{mn}|^{\frac{1}{\alpha_m+\beta_n}}; m+n \neq 0\}.$$

Then there exists $Q = Q(\varepsilon)$, so that for $p \geq Q$ we get

$$|x^P_{00}|, |x^P_{mn}|^{\frac{1}{\alpha_m+\beta_n}} < \varepsilon; \; m,n \geq 0, \; m+n \neq 0.$$

Then the continuity of Ψ follows from the following inequality by (3.6):

$$\Psi(x^P) \leq |x^P_{00}c_{00}| + \sum\sum_{m+n\geq 1} |x^P_{mn}c_{mn}| < \varepsilon M + \sum\sum_{m+n\geq 1} (\varepsilon M)^{\alpha_m+\beta_n}$$

$$\leq \varepsilon M + \sum_{m\geq 0} (\varepsilon M)^{\alpha_m} \sum_{n\geq 0} (\varepsilon M)^{\beta_n}.$$

7. CERTAIN SUBSPACES OF A FRÉCHET SPACE. We consider a special type of matrix space which in particular envelopes the class of entire functions of two variables having finite order and type points. The purpose of this section is to study the form of continuous linear functionals on this class of matrices.

Let $\{p_m\}$ and $\{q_n\}$ be two bounded sequences of real numbers such that $p_m > 0$ and $q_n > 0$ and p_m^{-1} and q_n^{-1} satisfy *Mandelbrojt condition*, namely if $\mu_m = \frac{1}{p_m}$ and $\lambda_n = \frac{1}{q_n}$, then

(7.1) $$\underline{\lim}\, (\mu_m - \mu_{m-1}) = h > 0; \; \underline{\lim}\, (\lambda_n - \lambda_{n-1}) = k > 0.$$

With this restriction on $\{p_m\}$ and $\{q_n\}$, let us introduce the following two spaces:

$$\Gamma_1(p,q) = \{a \in \Omega : \overline{\lim}\,(m+n)|a_{mn}|^{\frac{p_m+q_n}{p_m+q_n}} < \infty\}$$

$$\Gamma_2(p,q) = \{a \in \Omega : |(m+n)!a_{mn}|^{\frac{p_m q_n}{p_m+q_n}} \to 0 \quad \text{as} \quad m+n \to \infty\}$$

Clearly Γ_2 is a subspace of Γ_1. We observe that the space Γ_1 contains not only entire functions but analytic functions also, For example, if $p_m = 1 + \frac{1}{m}$ and $q_n = 1 + \frac{1}{n}$, then Γ_1 contains those functions which are analytic in the disc $|z| \leq r$, $1 \leq r < \infty$. We see also that if $p_m = \frac{1}{m^k}$ and $q_n = \frac{1}{n^k}$ for fixed $k \geq 1$, then the space Γ_1 contains entire functions only. At the same time, clearly $\Gamma_2(p,q)$ contains entire functions only because of the fact that

$$|a_{mn}|^{\frac{p_m q_n}{p_m+q_n}} \leq |(m+n)!a_{mn}|^{\frac{p_m q_n}{p_m+q_n}} \to 0 \quad \text{as} \quad m+n \to \infty .$$

Now for $a \in \Gamma_1(p,q)$ define

$$\phi(a) = \sup\{(m+n)|a_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_m^{-1}}} : m+n \neq 0\}$$

and for $a \in \Gamma_2(p,q)$ define

$$\Psi(a) = \sup\{|(m+n)!a_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} : m+n \neq 0\},$$

where $M = \max\{1, \sup_m p_m\}$ and $N = \max\{1, \sup_n q_n\}$. Then ϕ and Ψ are paranorms on $\Gamma_1(p,q)$ and $\Gamma_2(p,q)$ respectively making conplete matrix spaces, under the metric topology induced by them.

Regarding the inclusion relation between $_1(p,q)$ and $_2(p,q)$ we have

Theorem 7.2. If $\{p_m\}$ and $\{q_n\}$ are chosen so that

$$K = \overline{\lim}[(m+n-1)!]^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} < \infty$$

then $\Gamma_2(p,q)$ is a closed subspace of $\Gamma_1(p,q)$.

<u>Proof</u>. Let $a \in \Gamma_1(p,q)$ and $a^i \in \Gamma_2(p,q)$ with $a^i \to a$ in $\Gamma_1(p,q)$. Then given $\varepsilon > 0$, there exists $i_0 \equiv i_0(\varepsilon)$ such that

$$|(m+n)!\,a^i_{mn}-a_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} < \varepsilon,\ i \geq i_0.$$

Consequently we have

$$|(m+n)!\,a_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} \leq |(m+n)!\,a^i_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}}$$

$$+\ [(m+n-1)!]^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} \cdot (m+n)\,|a^i_{mn}-a_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}}$$

$$\leq |(m+n)!\,a^i_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} + K\varepsilon,$$

which gives the required result for $\{a^i_{mn}\} \in \Gamma_2$.

Remark 7.3. This above theorem includes the class like the case when p_m and q_n are of the form

$$p_m = \frac{1}{\alpha m^k}\ ;\quad q_n = \frac{1}{cn^k}\ ,\quad k \geq 1,\quad \alpha, c \in \mathbb{K}\,.$$

Lemma 7.4. $\Gamma_2(p,q) \subset \Gamma_2(r,s)$ *if and only if*

(7.5) $$\underline{\lim}\ \frac{r_m}{p_m} > 0 \quad\text{and}\quad \underline{\lim}\ \frac{s_n}{q_n} > 0.$$

Proof. Suppose (7.5) holds, then there exists λ and μ such that $r_m > \lambda p_m$; $s_n > \mu q_n$ for $m \geq N_1$ and $n \geq N_2$ respectively. Take $a \in \Gamma_2(p,q)$ then there exists a large integer I such that

$$|(m+n)!\,a_{mn}|^{\frac{p_m q_n}{p_m+q_n}} \leq 1,\quad m+n \geq I$$

$$\Rightarrow\ |(m+n)!\,a_{mn}| \leq 1,\quad m+n \geq I.$$

Now

$$\frac{1}{r_m} + \frac{1}{s_n} \leq \frac{1}{\lambda p_m} + \frac{1}{\mu q_n} \leq \frac{p_m+q_n}{\theta\, p_m q_n}\ ,\quad m \geq N_1,\quad n \geq N_2$$

where $\theta = \min\{\lambda,\mu\}$. So then we have

$$\frac{r_m s_n}{r_m+s_n} = \frac{1}{r_m^{-1}+s_n^{-1}} \geq \frac{\theta p_m q_n}{p_m+q_n}, \quad m \geq N_1, \quad n \geq N_2$$

Then for $m+n \geq \max\{N_1+N_2, I\}$, inview of (*) we get

$$|(m+n)!a_{mn}|^{\frac{r_m s_n}{r_m+s_n}} \leq [|(m+n)!a_{mn}|^{\frac{p_m q_n}{p_m+q_n}}]^{\theta}$$

and hence $a \in \Gamma_2(r,s)$.

Conversely suppose (5.5) is not true. Then there exists increasing sequence $\{m_i\}$ and $\{n_j\}$ such that

$$r_{m_i} < \frac{1}{i} p_{m_i} ; \quad s_{n_j} < \frac{1}{j} q_{n_j} .$$

Now we construct $a \in \Omega$ as follows.

$$(m+n)!a_{mn} = \begin{cases} [(i+j)^{-1}]^{\frac{p_m+q_n}{p_m q_n}}, & m = m_i,\ n = n_j; \\ 0, & \text{elsewhere.} \end{cases}$$

Then clearly $a \in \Gamma_2(p,q)$ and hence $a \in \Gamma_2(r,s)$. But

$$|(m_i+n_j)!a_{m_i n_j}|^{\frac{r_{m_i} s_{n_j}}{r_{m_i}+s_{n_j}}} = [(i+j)^{-1}]^{p_{m_i}^{-1}+q_{n_j}^{-1} / r_{m_i}^{-1}+s_{n_j}^{-1}}$$

$$\geq [(i+j)^{-1}]^{i^{-1}+j^{-1}} .$$

Since

$$\lim_{\substack{i+j\to\infty \\ i=j}} [(i+j)^{-1}]^{i^{-1}+j^{-1}} = 1,$$

we find a subsequence of $\{|(m_i+n_j)!a_{m_i n_j}|^{\frac{r_{m_i} s_{n_j}}{r_{m_i}+s_{n_j}}}\}$ which does not tend to zero.

Hence $a \notin \Gamma_2(r,s)$, a contradiction.

We next consider another totoplogy on $\Gamma_2(p,q)$ and proceed to establish its equivalence to the original topology. So for $a \in \Gamma_2(p,q)$ and $\delta > 0$,

$$||A,\delta|| = \sup[\frac{(m+n)!\,|a_{mn}|}{\delta^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}}}$$

Lemma 7.6. Let T* *be the topology generated by* $\{||a,\delta||, \delta>0\}$ *and* T *be the paranorm topology. Then* T *and* T* *are equivalent.*

Proof. Let $a^i \to 0$ in T. Let $\delta > 0$ and $0 < \varepsilon < 1$ be given. Choose $\lambda > 0$ such that $\lambda < \delta\varepsilon$, then we have

$$\left(\frac{\lambda}{\delta}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} < \varepsilon^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} \leq \varepsilon, \ \forall\ m+n \geq 0.$$

Since $a^i \to 0$, so for above choice of λ, there exists $I \equiv I(\lambda)$ such that

$$|(m+n)!\,a^i_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} < \varepsilon, \ \forall\ m+n \geq 0,\ i \geq I,$$

and hence for $i \geq I$,

$$\frac{|(m+n)!\,a^i_{mn}|}{\delta^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}}} < \left(\frac{\lambda}{\delta}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}} < \varepsilon, \ \forall\ m+n \geq 0$$

So $||a^i,\delta|| < \delta$, $i \geq I$, that is to say, $a^i \to 0$ in T^*.

Conversely let $a^i \to 0$ in T*. So we get $||a^i,\delta|| \to 0$ for each $\delta > 0$. Then for each $\varepsilon > 0$, choose $\lambda, \delta > 0$ with $\delta < \varepsilon$; $\lambda < \frac{\varepsilon}{\delta}$. Then

$$\lambda < \frac{\varepsilon}{\delta} \leq \left(\frac{\varepsilon}{\delta}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}}, \ \forall\ m+n \geq 0.$$

$$\Rightarrow \delta\lambda^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} \leq \varepsilon, \ \forall\ m+n \geq 0.$$

Now for the above choice of λ there exists $I \equiv I(\lambda)$ such that for $i \geq I$

$$\frac{|(m+n)!a^{i}_{mn}|}{\delta^{\frac{p_m^{-1}+q_m^{-1}}{M^{-1}+N^{-1}}}} < \lambda, \ \forall\ m+n \geq 0.$$

$$\Rightarrow |(m+n)!a^{i}_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} < \delta\lambda^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}} \leq \varepsilon, \ i \geq I, \ \forall\ m+n \geq 0$$

$$\Rightarrow a^{i}_{mn} \to 0 \ \text{ in } T,$$

and hence T and T^* are equivalent.

In the next main result we will need.

Lemma 7.7. For every $a \in \Gamma_2(p,q)$, *the series*

$$(+) \qquad \sum_{m+n\geq 0}\sum (m+n)!a_{mn}c_{mn}$$

converges if and only if

$$\{|c_{00}|;\ |c_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}};\ m+n \neq 0\}$$

is bounded.

Proof. Assume first of all that there exists a positive constant L, such that

$$|c_{00}| \leq L, \ |c_{mn}| \leq L^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}}, \ m+n \neq 0.$$

For each $a \in \Gamma_2(p,q)$ we find L^* such that

$$|(m+n)!a_{mn}| \leq \left(\frac{1}{2L}\right)^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}}, \ m+n \geq L^*$$

Hence for $m+n \geq L^*$, we have

$$|(m+n)!a_{mn}c_{mn}| \leq 2^{-\left(\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}\right)}.$$

So then we have

$$\sum_{m+n\geq 0}\sum |(m+n)!a_{mn}c_{mn}| = \sum_{0\leq m+n\leq L^*-1}\sum |(m+n)!a_{mn}c_{mn}|$$

$$\sum\sum_{m+n\geq L} |(m+n)!a_{mn}c_{mn}|$$

$$\leq \sum\sum_{0\leq m+n\leq L^*-1} |(m+n)!a_{mn}c_{mn}| + \sum\sum_{m+n\leq L^*} 2^{-\left(\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}\right)} < \infty.$$

Conversely assume that the series (+) is convergent for each a $\Gamma_2(p,q)$ but the sequence (*) is unbounded. Then there exists increasing sequences $\{m_k\}$ and $\{n_k\}$ such that

$$|c_{m_k n_k}| \geq k^{\frac{p_{m_k}^{-1}+q_{n_k}^{-1}}{M^{-1}+N^{-1}}}, \quad k \geq 1.$$

Now construct $a \in \Omega$ as follows:

$$a_{mn} = \begin{cases} k^{-\left(\frac{p_{m_k}^{-1}+q_{n_k}^{-1}}{M^{-1}+N^{-1}}\right)}; & m = m_k,\ n = n_k; \\ 0, & \text{otherwise.} \end{cases}$$

Let us note that

$$|(m+n)!a_{mn}|^{\frac{p_m q_n}{p_m+q_n}} = [k^{-1}]^{\frac{1}{M^{-1}+N^{-1}}} \to 0.$$

However

$$|(m_k+n_k)!a_{m_k n_k}c_{m_k n_k}| \geq 1,$$

and this contradicts the convergence in (+).

Theorem 7.8. Consider $\Gamma_2(p,q)$ *with either of the two topologies. Then every continuous linear functional* Ψ *on* $\Gamma_2(p,q)$ *is given by*

$$\Psi(x) = \sum\sum_{m+n\geq 0} (m+n)!x_{mn}c_{mn}, \quad x \in \Gamma_2(p,q)$$

where $\{c_{mn}\}$ is such that

$$\{|c_{00}|; |c_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}}, m+n \neq 0\}$$

is bounded.

Conversely, for $c \in \Omega$ satisfying (+), the function defined by (*) is continuous and linear.

Proof. Let $\Psi \in [\Gamma_2(p,q)]^*$. Suppose $\Psi(e^{mn}) = (m+n)!c_{mn}$, then for any element $x \in \Gamma_2(p,q)$, we can write

$$x = \sum\sum_{m+n\geq 0} x_{mn} e^{mn}$$

Consider the s-th plane section $x^{(s)} = \sum\sum_{0\leq m+n\leq s} x_{mn} e^{mn}$. Hence we get $\Psi(x^{(s)}) \to \Psi(x)$, and so

$$\Psi(x) = \lim_{\to\infty} \sum\sum_{0\leq m+n\leq s} x_{mn} \Psi(e^{mn})$$

$$= \sum\sum_{m+n\geq 0} (m+n)!c_{mn}x_{mn}, \ \forall\ x \in \Gamma_2(p,q).$$

So, by the previous lemma the sequence

$$\left\{|c_{00}|, |c_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}}, m+n \neq 0\right\}$$

is bounded.

Conversely let Ψ be mentioned as above in (*). We need show only the continuity of Ψ. Let $x^p \to 0$ in either of the two topologies, and put

$$k = \sup\left\{|c_{00}|, |c_{mn}|^{\frac{M^{-1}+N^{-1}}{p_m^{-1}+q_n^{-1}}}, m+n \neq 0\right\}.$$

Choose $\varepsilon > 0$ so that $\varepsilon K < 1$, then there exists $Q \equiv Q(\varepsilon)$ such that for $p \geq Q$ we get

$$|x^p_{00}| < \varepsilon; \ |(m+n);x^p_{mn}| < \varepsilon^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}}, m+n \neq 0.$$

The continuity of ψ follows from the following inequality

$$|\psi(x^p)| \leq |c_{00}x^p_{00}| + \sum\sum_{m+n\geq 1} |(m+n)!x^p_{mn}c_{mn}|$$

$$\leq \varepsilon K + \sum_{m+n\geq 1} [\varepsilon K]^{\frac{p_m^{-1}+q_n^{-1}}{M^{-1}+N^{-1}}}, \ p \geq Q.$$

Lemma 7.9. If for each $a \in \Gamma_1$ *the series*

$$(*) \qquad \sum_{m+n\geq 0}\sum (m+n)^{\frac{p_m+q_n}{p_m q_n}} x_{mn} a_{mn}, \ \forall\ a \in \Gamma_1$$

converges, then

$$(+) \qquad \{|x_{00}|; |x_{mn}|^{\frac{p_m q_n}{p_m+q_n}}, \ m+n \neq 0\}$$

is bounded.

Proof. Suppose (+) is not true, then there exists increasing sequences $\{m_k\}$ and $\{n_k\}$ such that

$$|x_{m_k n_k}| \leq k^{\frac{p_{m_k}+q_{n_k}}{p_{m_k} q_{n_k}}}, \ k \geq 1$$

Construct $a \in \Omega$ as follows

$$(m+n)^{\frac{p_m+q_n}{p_m q_n}} a_{mn} = \begin{cases} k^{-\frac{p_{m_k}+q_{n_k}}{p_{m_k} q_{n_k}}} ; & m = m_k, n = n_k, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $a \in \Gamma_1$, for when $m = m_k$ and $n = n_k$, then for $k \geq 1$,

$$(m+n)|a_{mn}|^{\frac{p_m q_n}{p_m+q_n}} = k^{-1},$$

and the left hand side is zero otherwise. But at the same time we have

$$|(m_k+n_k)^{\frac{p_{m_k}+q_{n_k}}{p_{m_k} q_{n_k}}} a_{m_k n_k} x_{m_k n_k}| \geq 1$$

and this is a contradiction to (*).

Theorem 7.10. Every continuous linear functional Ψ *on* Γ_1 *is of the form*

$$(+)\qquad \Psi(x) = \sum\sum_{m+n\geq 0} (m+n)^{\frac{p_m+q_n}{p_mq_n}} x_{mn}c_{mn},$$

where

$$(*)\qquad \{|c_{00}|; |c_{mn}|^{\frac{p_mq_n}{p_m+q_n}};\ m+n \neq 0\}$$

is bounded and conversely suppose for a sequence $\{c_{mn}\}$ *satisfying* (*), *the functional* Ψ *defined as in* (+), *is a continuous linear functional.*

Proof. Let $\Psi \in \Gamma_1^*$ and define $\{c_{mn}\} \in \Omega$ by

$$\Psi(e^{mn}) = (m+n)^{\frac{p_m+q_n}{p_mq_n}} c_{mn}.$$

We know that

$$x^{(s)} = \sum\sum_{0\leq m+n\leq s} x_{mn}e^{mn}$$

$$\Rightarrow x^{(s)} \to x, \text{ for any } x = \sum\sum_{m+n\geq 0} x_{mn}e^{mn}$$

$$\Rightarrow \Psi(x^{(s)}) \to \Psi(x).$$

So we have

$$\Psi(x) = \lim_{s\to\infty} \Psi(x^{(s)}) = \lim_{s\to\infty} \sum\sum_{0\leq m+n\leq s} x_{mn}\Psi(e^{mn}).$$

Consequently

$$\Psi(x) = \sum\sum_{m+n\geq 0} (m+n)^{\frac{p_m+q_n}{p_mq_n}} x_{mn}c_{mn}.$$

But we know that $\{x_{mn}\} \in \Gamma_1$ is arbitrary and so by the previous lemms, we find that

$$\{|c_{00}|; |c_{mn}|^{\frac{p_mq_n}{p_m+q_n}};\ m+n \neq 0\}$$

is bounded.

Conversely, let Ψ be the one mentioned in (+). So we need show only the continuity of Ψ as linearity of Ψ is already clear. Take $x^p \to 0$ in Γ_1, then our aim is to show that $\Psi(x^p) \to 0$. Set

$$M = \sup\{|c_{00}|; |c_{mn}|^{\frac{p_m q_n}{p_m+q_n}}, m+n \neq 0\}$$

and choose $\varepsilon > 0$ so that $\varepsilon M < 1$. Then there exists $Q \equiv Q(\varepsilon)$ such that for $p \geq Q$

$$|x^p_{mn}| < \left(\frac{\varepsilon}{m+n}\right)^{\frac{p_m+q_n}{p_m q_n}}, \ \forall\ m+n \geq 0.$$

Also we have

$$|c_{mn}| \leq M^{\frac{p_m+q_n}{p_m q_n}}, \ \forall\ m+n \geq 0.$$

So for $p \geq Q$, we have

$$|\Psi(x^p)| \leq \sum\sum_{m+n\geq 0} |x^p_{mn} c_{mn}| (m+n)^{\frac{p_m+q_n}{p_m q_n}}$$

$$\leq \sum\sum_{m+n\geq 0} (\varepsilon M)^{\frac{p_m+q_n}{p_m q_n}} .$$

Recalling the Mandelbrojt condition on p and q, it follows that

$$|\Psi(x^p)| \leq \sum_{m\geq 0} (\varepsilon M)^{\frac{1}{p_m}} \sum_{n\geq 0} (\varepsilon M)^{\frac{1}{q_n}} .$$

7.11. Remark. In particular, Theorems 5.8 and 5.10 contain results developed by Maddox Iyer [4]. Kamthan [7], etc.

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