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A COMBINATORIAL APPROACH TO THE THEORY OF THE BARGAINING SETS

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0. Introduction

The theory of the bargaining sets for cooperative n-person games with side payments, initiated by R.J. Aumann and M. Maschler in [2], has become a classical topic to be included in all comprehensive texts on game theory (see [5]). Surprisingly, no effort has been done to extend the results to bargaining sets defined by means of multicoalitional objections and counter objections. This is precisely the aim of the present paper, that tries also to exploit further the combinatorial character of such results. New terms as bargaining proposal, distribution, counter proposal, counter distribution, have been used, in order to emphasize the slight difference between these concepts and the extensions of the concepts of the bargaining sets theory. A particular attention on their algorithmic use has led us to the introduction of these concepts, however we tried to describe the meaning of the results in classical terms. The main result is Theorem 3.3, that characterizes the payoff configurations belonging to the bargaining set M_0, and extends the similar result by M. Maschler (see [4]). Note that the concept of core for cooperative games with coalition structures introduced by R.J. Aumann and J.H. Dreze in [1] has been used.

A cooperative n person game with side payments is a pair $G = (I,v)$, where I is a finite set, $|I| = n$, (the set of players), and v is a real-valued function on $P(I)$, (the power set of I), with $v(\phi) = 0$. A payoff vector of G is a real-valued function x on I. For $S \subseteq I$ we shall write $x(S) = \sum_{i \in S} x_i$, $(x(\phi) = 0)$.

A *constrained game* is a triplet $G_F = (I,v,F)$, where $G = (I,v)$ is a game and F is a subset of R^n defining the admissible (feasible) payoff vectors of G. A *coalition structure* $\mathcal{S}$ is a partition of I. For every coalition structure $\mathcal{S}$ the set of admissible payoff vectors is

(0.1) $$F_{\mathcal{S}} = \{x | x \in R^n, x(S) = v(S), \forall S \in \mathcal{S}\}.$$

A game with coalition structures is a constrained game $G_F = (I,v,F)$ in which F is the union of all $F_{\mathcal{S}}$. In the following the word game will designate only such games.

For $S \in P(I)$ and $x \in F$ the excess $e(x,S)$ is defined by

(0,2) $$e(x,S) = v(S) - x(S)$$

The core of G is

(0.3) $$C(G) = \{x | x \in F, e(x,S) \leq 0, \forall S \in P(I)\}.$$

Note the agreement of this concept of core with that used by R.I. Aumann, J.H. Dreze (see [1]). Notice that the above defined admissible payoff vectors are the x-parts of the payoff configurations used extensively in the theory of the bargaining sets.

The basic concepts of the theory of the bargaining sets are those of objections and counter objections. We shall define two similar concepts in a more general way than this is usually done, in order to emphasize their combinatorial character and therefore we shall use different terms. First, no restriction will be imposed upon the appartenence of the group of objecting or counter objecting players to the same coalition of a given coalition structure and upon the number of the new coalitions they intend to form. In fact, this approach has been suggested in the definition of the bargaining set M_0 (see [2], sec. 10). Secondly, no rationality principle, coalitional or individual, will be imposed. Finally, it has to be mentioned that an objection will be done against a given payoff associated to a given coalition structure and not against another group of players, as it is usually done. Similarly, a counter objection will be aimed to provide better opportunities for all players rather than to counter a given objection. It can be considered as a tentative of the winners of an objection to improve their outcomes rather than a reaction of the threatened players.

The first section will be devoted to the definition and some properties of the bargaining proposals and bargaining distributions, the analogues of the two parts of an objection. The bargaining counter proposals and bargaining counter distributions, the analogues of the two parts of a counter objection are considered in the second section, where the concept of double excess has been introduced and used. The third section introduces the concept of bargaining set and gives a combinatorial characterization of the admissible payoff vectors belonging to the bargaining set. This result (Theorem 3.3) is further interpreted in graph theoretical terms, on a weighted graph associated to the game. Finally, a basic algorithm stated at the end of the paper reduces the game problem to a graph problem on the set of all maximal independent sets of vertices of a weighted graph.

1. Bargaining proposals and bargaining distributions.

Consider any $x \in F_S$ for some coalition structure S. Denote a generic coalition structure by $T = (P_1,\dots,P_\pi;N_1,\dots,N_\nu;O_1,\dots,O_\omega)$, where P_i, $i=\overline{1,\pi}$ N_j, $j=\overline{1,\nu}$, O_k, $k=\overline{1,\omega}$, stand for the coalitions with positive, negative and zero excesses, respectively. Denote

(1.1) $$P = \bigcup_1^{\pi} P_i, \quad N = \bigcup_1^{\nu} N_j, \quad O = \bigcup_1^{\omega} O_k.$$

If $x \in C(G)$, then $P = \phi$ and if $P = \phi$ in all coalition structures, then $x \in C(G)$. If $x \notin C(G)$, then there exist coalition structures with $P \neq \phi$, even though $P = \phi$ for some coalition structures, for example $\mathcal{S}$ itself.

Any coalition structure $\mathcal{T}$ with $P \neq \phi$ will be called a bargaining proposal w.r.t. x and $\mathcal{S}$.

Lemma 1.1. If $\mathcal{T}$ is a bargaining proposal, then there exists $y \in F_{\mathcal{T}}$ such that we have $y_h > x_h$, $\forall\, h \in P$.

Proof. For each $i = \overline{1,\pi}$ consider

(1.2) $$y_h = x_h + e(x,P_i)/|P_i|, \quad \forall\, h \in P_i$$

and define y_h, $\forall\, h \notin P$ in an appropriate way to satisfy $y(S) = v(S)$, $\forall\, S \in \mathcal{T}-\{P_1,\ldots,P_\pi\}$. As $e(x,P_i) > 0$, $i = \overline{1,\pi}$ and $y(P_i) = v(P_i)$, $i=\overline{1,\pi}$ follow from (1.2) we have $y_h > x_h$, $\forall\, h \in P$ and $y \in F_{\mathcal{T}}$.

If $x \notin C(G)$, then the bargaining proposals are considered by any coalition $K \subseteq P$, because K could try to convince $P-K$ to work together, in order to obtain supplementary gains w.r.t. x. Of course, if some coalition $K \subseteq P$ initiates the bargaining, then this coalition could use the available excesses in its best interest, for example by offering to all players of $K-P$ only their shares in x. Note that a bargaining proposal is in fact the support of an objection against x of any coalition $K \subseteq P$.

For any bargaining proposal $\mathcal{T}$, any $y \in F_{\mathcal{T}}$ such that

(1.3) $$y_h \geq x_h, \quad \forall\ h \in P$$
$$y_h > x_h, \quad \text{for some } h \in P_i, \quad \forall\ i = \overline{1,\pi}$$

will be called a bargaining distribution of the gain provided by $\mathcal{T}$.

A bargaining distribution $y \in F_{\mathcal{T}}$ defines in each $P_i \in \mathcal{T}$ two subsets:

(1.4) $$K_i = \{h \mid h \in P_i,\ y_h > x_h\} \quad \text{and} \quad P_i - K_i$$

where $P_i - K_i$ could be empty. Each $h \in K_i$ might be called an active player and each $h \in P_i - K_i$, if any, might be called a passive player of P_i. Notice that $(y, \mathcal{T})$, where y is a bargaining distribution, is an objection against x of any coalition $K \subseteq P$ of active players. Lemma 1.1 says that: for any bargaining proposal (support of an objection) there exists an objection of any coalition $K \subseteq P$.

Lemma 1.2. If $\mathcal{T}$ has $P = \phi$, then there is no coalition $S \in \mathcal{T}$ such $y_h \geq x_h$, $\forall\ h \in S$ and $y_h > x_h$ for some $h \in S$, for some $y \in F_{\mathcal{T}}$.

Proof. From $y_h \geq x_h$, $\forall\ h \in S$ and $y_h > x_h$ for some $h \in S$ and $y(S) = v(S)$ we get $e(x,S) > 0$, which contradicts $P = \phi$.

Theorem 1.3. Consider any coalition structure $\mathcal{T}$. Then, there exists a bargaining distribution $y \in F_{\mathcal{T}}$, if and only if $\mathcal{T}$ is a bargaining proposal.

Proof. Follows from Lemmas 1.1 and 1.2.

As the players are looking for bargaining distributions, this result motivates the definition of the concept of bargaining proposal. Note that Theorem 1.3 is an extension of a result by M. Maschler ([4], Lemma 2.1), about the coalition structures that can support an one coalition objection.

Any bargaining proposal $\mathcal{T}$ with $N = \phi$ will be called a trivial (improper) bargaining proposal. The following result is obvious:

Lemma 1.4. If $\mathcal{T}$ is a trivial bargaining proposal, then there exists $y \in F_{\mathcal{T}}$ such that $y_h > x_h$, $\forall h \in P$ and $y_h = x_h$, $\forall h \in 0$.

Practically, this means that each coalition of a trivial bargaining proposal will agree to move to $\mathcal{T}$, hence any bargaining around $\mathcal{T}$ is useless, except perhaps for distributing the gain within each new coalition.

Lemma 1.5. If $\mathcal{T}$ is a nontrivial bargaining proposal, then there is no $y \in F_{\mathcal{T}}$ such that $y_h \geq x_h$, $\forall h \in N_j$, for some $j \in \{1,\ldots,\nu\}$.

Proof. From $y_h \geq x_h$, $\forall h \in N_j$ we get $e(y,N_j) \leq e(x,N_j) < 0$ and $y \notin F_{\mathcal{T}}$.

By Lemma 1.5 each coalition N_j, $j=\overline{1,\nu}$, will have at least one player threatened to lose his share from x. However, if no assumption has been done about the game, it might be possible that a partition of N into subsets with non negative excesses does exist.

In the following we shall assume that:

(A) There is no trivial bargaining proposal w.r.t. $x \in F$, $x \notin C(G)$ and $\mathcal{S}$.

Practically, this means that before starting the bargaining the trivial bargaining proposals have been searched and $x \in F$, $x \notin C(G)$ satisfying (A) has been discovered and provisionally accepted. Note that under this assumption, the set of players I could be covered but not partitioned by using only coalitions with non negative excesses. On the other hand, a reaction of some players of N, who are threatened by $\mathcal{T}$, can be expected. However, in the following we shall consider that an action against $\mathcal{T}$ could be initiated by the winners of $\mathcal{T}$, who would like to discover better opportunities than those offered by $\mathcal{T}$.

Now, we shall define some bargaining distributions $y(L)$ offering higher gains to a subset $L \neq \phi$ of players of P, in order to counter the attraction of these players for another bargaining proposal. Such distributions will be used in the next section. To prove some results we shall denote

(1.5) $$\alpha_h = y_h - x_h, \quad \forall\, h \in I$$

and we shall remark that: $y \in F_{\mathcal{T}}$ is a bargaining distribution associated to $\mathcal{T}$ if and only if

(1.6) $$\alpha_h \geq 0,\ \forall\, h \in P;\ \alpha_h > 0 \text{ for at least one } h \in P_i,\quad \forall\, i=\overline{1,\pi}$$

$$\sum_{h\in S} \alpha_h = e(x,S),\ \forall\, S \in \mathcal{T}$$

<u>Lemma 1.6</u>. If $\mathcal{T}$ is a bargaining proposal and $y \in F_{\mathcal{T}}$ is a bargaining distribution, then for any π-tuple of subsets $L_i \subseteq P_i$, $i=\overline{1,\pi}$, (some of them might be empty), we must have

(1.7) $$e(x,P_i) - [y(L_i) - x(L_i)] \geq 0, \quad i = \overline{1,\pi}$$

The equality sign could be met, only for $L_i = P_i$, or $L_i \neq P_i$, but $P_i - L_i$ will consist of passive players.

<u>Proof</u>. If $L_i = \phi$ for some i, then (1.7) follows from $e(x,P_i) > 0$. If $L_i \neq \phi$ and the inequality does not hold for some i, then from (1.5) and the last condition (1.6) we get $\sum_{h\in P_i - L_i} \alpha_h < 0$, that contradicts the first condition (1.6).

The equality sign will be met for some i, when $L_i \neq \phi$ and $\sum_{h\in P_i-L_i} \alpha_h = 0$ and this implies $L_i = P_i$ or $L_i \neq P_i$ but $\alpha_h = 0$, $\forall\, h \in P_i - L_i$.

Now, consider any $L \subseteq P$, $L \neq \phi$ and define

(1.8) $$L_i = L \cap P_i,\ i=\overline{1,\pi}, \quad I(L) = \{i \mid L_i \neq \phi\}.$$

Let $\varepsilon > 0$ be any number subject to

(1.9) $$0 < \varepsilon < \min e(x,P_i)$$

and $y(L,\varepsilon) \in E_T$ defined by (1.5) and $\alpha(L,\varepsilon)$ as follows:

a) for $i \in I(L)$

(1.10) $$\alpha_h(L,\varepsilon) = \frac{e(x,P_i)-\delta_i\varepsilon}{|L_i|}, \quad \forall h \in L_i$$

$$\alpha_h(L,\varepsilon) = \frac{\delta_i\varepsilon}{|P_i-L_i|}, \quad \forall h \in P_i-L_i, \text{ if } L_i \neq P_i$$

where $\delta_i = 1$, if $L_i \neq P_i$ and $\delta_i = 0$, if $L_i = P_i$;

b) for any $S \in T - \{P_i | i \in I(L)\}$

(1.11) $$\sum_{h \in S} \alpha_h(L,\varepsilon) = e(x,S)$$

Lemma 1.7. For any $L \subseteq P$ and ε subject to (1.9), $y(L,\varepsilon)$ defined by (1.5), (1.10), (1.11) is a bargaining distribution $y(L,\varepsilon) \in E_T$. Moreover, we have

(1.12) $$e(x,P_i) - [y(L_i)-x(L_i)] = \delta_i\varepsilon, \quad \forall i \in I(L)$$

Proof. From (1.10) we get $\sum_{h \in P_i} \alpha_h(L,\varepsilon) = e(x,P_i)$, $\forall i \in I(L)$ and (1.10), (1.11) show that the other conditions (1.6) are fulfilled, hence $y(L,\varepsilon)$ is a bargaining distribution. Obviously, (1.12) follows from (1.10).

In words, this result says that $y(L,\varepsilon)$ for small ε allocates almost the entire available excess of any P_i, $i \in I(L)$, to the players of L_i, while a small supplementary gain will be available for players of P_i-L_i to make the bargaining proposal still attractive for them.

2. Bargaining counter proposals and bargaining counter distributions.

Consider any $x \in F_{\mathcal{S}}$, for some coalition structure $\mathcal{S}$. Let $\mathcal{T}$ be any bargaining proposal and $y \in F_{\mathcal{T}}$ a bargaining distribution. The double excess of any coalition S w.r.t. x and $(y,\mathcal{T})$ is

(2.1) $$e(x,S;y,\mathcal{T}) = v(S) - x(S\cap(N\cup O)) - y(S\cap P).$$

Note the identity connecting the double excess to the excess

(2.2) $$e(x,S;y,\mathcal{T}) = e(x,S) - [y(S\cap P)-x(S\cap P)]$$

and showing the meaning of the double excess.

Notice that we have

(2.3) $$e(x,P_i;y,\mathcal{T}) = 0, \quad i=\overline{1,\pi}$$
$$e(x,S;y,\mathcal{T}) = e(x,S), \ \forall\, S \in \mathcal{T} - \{P_1,\ldots,P_\pi\}$$

and the fact that the last equality holds for any S with $S \cap P = \phi$, too.

Lemma 2.1. If $\mathcal{T}$ is any bargaining proposal and $y \in F_{\mathcal{T}}$ is any bargaining distribution, then for any coalition S we have

(2.4) $$e(x,S;y,\mathcal{T}) \leq e(x,S)$$

Proof. Follows from (2.2) and $y_h \geq x_h$, $\forall\, h \in P$.

Consider any other bargaining proposal $\mathcal{T}^* = (P_1^*,\ldots,P_{\pi^*}^*;N_1^*,\ldots,N_{\nu^*}^*;O_1^*,\ldots,O_{\omega^*}^*)$. If $P \cap P^* = \phi$, then $\mathcal{T}^*$ cannot play any role in the bargaining around $\mathcal{T}$. Therefore, from now on we shall assume

(B) $$P \cap P^* \neq \phi.$$

Under this assumption, we can suppose without loss of generality that there is an integer r, $0 \leq r \leq \pi^*$, such that

(2.5) $$P \cap P_{i^*}^* \neq \phi, \quad i^*=\overline{1,r}$$
$$P \cap P_{i^*}^* = \phi, \quad i^*=\overline{r+1,\pi^*}, \quad \text{if } r < \pi^*.$$

Remark that by Lemma 2.1 we have

$$(2.6)\qquad \begin{aligned} &e(x,N^*_{j^*};y,\mathcal{T}) < 0,\ j^*=\overline{1,\nu^*},\quad e(x,C^*_{k^*};y,\mathcal{T}) \leq 0,\quad k^* = \overline{1,\omega^*} \\ &e(x,P^*_{i^*};y,\mathcal{T}) = e(x,P^*_{i^*}) > 0,\quad i^* = r+1,\pi^*,\quad \text{if}\quad r < \pi^* \end{aligned}$$

so that for $r < \pi^*$ some of the double excesses $e(x,P^*_{i^*};y,\mathcal{T})$ are positive. For $r = \pi^*$ and some y's this might also be true, but in general no result can be obtained about the sign of the double excesses $e(x,P^*_{i^*};y,\mathcal{T})$, $i^*=\overline{1,r}$.

Any bargaining proposal $\mathcal{T}^*$ subject to (B) with

$$(2.7)\qquad e(x,P^*_{i^*};y,\mathcal{T}) \geq 0,\quad i^*=\overline{1,r}$$

for all bargaining distributions $y \in F_{\mathcal{T}}$ associated to a bargaining proposal $\mathcal{T}$, will be called a <u>bargaining counter proposal</u> w.r.t. x and $\mathcal{T}$.

Note that a bargaining counter proposal has been defined as a bargaining proposal having a certain property w.r.t. the set of all bargaining distributions associated to another bargaining proposal.

Consider a pair $(\mathcal{T},\mathcal{T}^*)$ consisting of two bargaining proposals subject to (B).

If $y \in F_{\mathcal{T}}$ is a bargaining distribution associated to $\mathcal{T}$, then any $z \in F_{\mathcal{T}^*}$ such that

$$(2.8)\qquad \begin{aligned} &z_h \geq y_h,\quad \forall\, h \in P \cap P^* \\ &z_h \geq x_h,\quad \forall\, h \in P^* - P \cap P^* \end{aligned}$$

will be called a <u>bargaining counter distribution</u> w.r.t. y.

Note that in this definition $\mathcal{T}^*$ could be a bargaining counter proposal or not.

Obviously, from (2.6), (2.8) we conclude that any bargaining counter distribution $z \in F_{T*}$ w.r.t. some bargaining distribution $y \in F_T$, is also a bargaining distrition w.r.t. x and $T*$. In general, this is a more profitable distribution than y for $P*$.

Lemma 2.2. Consider a pair $(T, T*)$ of bargaining proposals. If $T*$ is a bargaining counter proposal w.r.t. x and T, then for any bargaining distribution $y \in F_T$ there exists a bargaining counter distribution $z \in F_{T*}$.

Proof. For any bargaining distribution $y \in F_T$, consider for each $i*=\overline{1,\pi*}$

$$(2.9)\qquad \begin{aligned} z_h &= y_h + e(x,P^*_{i*};y,T)/|P^*_{i*}| \quad \text{if} \quad h \in P \cap P^*_{i*} \\ z_h &= x_h + e(x,P^*_{i*};y,T)/|P^*_{i*}| \quad \text{if} \quad h \in P^*_{i*} - P \cap P^*_{i*} \end{aligned}$$

and define z_h, $\forall\, h \notin P*$, in an appropriate way to satisfy $z(S) = v(S)$, $\forall\, S \in T* - \{P^*_1,\dots,P^*_{\pi*}\}$. From (2.6), (2.7) and $y(P^*_{i*}) = v(P^*_{i*})$, $i*=\overline{1,\pi*}$, we get (2.8) and $z \in F_{T*}$.

The bargaining counter proposals are considered by any coalition $L \subseteq P*$, because L would try to convince $P*-L$, (if $L \neq P*$), to work together in order to obtain a supplementary gain w.r.t. x and y. Of course, if some coalition L initiates the counter bargaining, then the available double excesses could be used by this coalition in its best interest and the offers for the players of $(P*-L) \cap P$ are depending on y, i.e.on the offers they get in T.

Lemma 2.3. Consider a pair $(T,T*)$ of bargaining proposals. If for some bargaining distribution $y \in F_T$ we have

$$(2.10)\qquad e(x,P^*_{i*};y,T) < 0 \quad \text{for some} \quad i* \in \{1,\dots,r\}$$

then there exists no bargaining counter proposal $z \in F_{T*}$ w.r.t. y.

Proof. If $z \in F_{\mathcal{T}^*}$ is a bargaining counter proposal w.r.t. y, then from (2.8) and $z(P^*_{i^*}) = v(P^*_{i^*})$, $i^*=\overline{1,\pi^*}$ we get $e(x,P^*_{i^*};y,\mathcal{T}) \geq 0$ $i^*=\overline{1,\pi^*}$, hence (2.10) does not hold.

From Lemmas 2.2 and 2.3 follows:

Theorem 2.4. Consider a pair $(\mathcal{T},\mathcal{T}^*)$ of bargaining proposals. Then, $\mathcal{T}^*$ is a bargaining counter proposal w.r.t. x and $\mathcal{T}$, if and only if for each bargaining distribution $y \in F_{\mathcal{T}}$ there exists a bargaining counter distribution $z \in F_{\mathcal{T}^*}$.

This result justifies the definition of the bargaining counter proposals. On the other hand, remark that a bargaining counter proposal $\mathcal{T}^*$ and a bargaining counter distribution $z \in F_{\mathcal{T}^*}$ form a counter objection $(z,\mathcal{T}^*)$ w.r.t. the objection $(y,\mathcal{T})$, (see [2]). So, the conditions (2.7) to be fulfilled for all objections represent a characterization of the coalition structures $\mathcal{T}$ for which each objection $(y,\mathcal{T})$ w.r.t. x has a counter objection $(z,\mathcal{T}^*)$. The case $\pi=1$, $\pi^*=1$ has been considered in [4]. Note also that (2.7) could give upper bounds for $y(P \cap P^*_{i^*}) - x(P \cap P^*_{i^*})$, $i^*=\overline{1,\pi^*}$, for those objections $(y,\mathcal{T})$ which have a counter objection $(z,\mathcal{T}^*)$ in the case where they are not fulfilled by all objections $(y,\mathcal{T})$.

Now, we intend to give a characterization of the bargaining counter proposals which does not depend explicitly on the set of all bargaining distributions to be countered. Some previous lemmas are needed.

Lemma 2.5. Consider a pair $(\mathcal{T},\mathcal{T}^*)$ of bargaining proposals. Then, $\mathcal{T}^*$ is a bargaining counter proposal w.r.t. x and $\mathcal{T}$ if and only if for all bargaining distributions $y \in F_{\mathcal{T}}$ we have

$$(2.11) \qquad e(x,P^*_{i^*}) \geq y(P \cap P^*_{i^*}) - x(P \cap P^*_{i^*}), \quad i^*=\overline{1,r}$$

Proof. The conditions (2.7) have been rewritten, by using (2.2).

Lemma 2.6. If $\mathcal{T}$ is a bargaining proposal and $\mathcal{T}^*$ is a bargaining counter proposal w.r.t. x and $\mathcal{T}$, then we have

$$\text{(C)} \qquad e(x,P^*_{i^*}) \geq \sum_{i \in I(P^*_{i^*})} e(x,P_i), \quad i^*=\overline{1,r}$$

Proof. Consider a fixed $i^* \in \{1,\ldots,r\}$ and $L = P \cap P^*_{i^*} \subseteq P$, $L \neq \phi$. We have $L_i = L \cap P_i = P_i \cap P^*_{i^*}$, $i=\overline{1,\pi}$, and by using Lemma 1.7, we conclude that there exists a bargaining distribution $y(L,\varepsilon)$ such that

$$\text{(2.12)} \qquad y(L_i) - x(L_i) = e(x,P_i) - \delta_i\varepsilon, \ \forall\, i \in I(P^*_{i^*})$$

If $\mathcal{T}^*$ is a bargaining proposal, then from (2.11), (2.12) we get (C) for $P^*_{i^*}$. Of course, L depends on i^*, but the reasoning is valid for all $i^*=\overline{1,r}$, because $\mathcal{T}^*$ is able to counter all bargaining distributions defined for all $i^*=\overline{1,r}$, hence (C) hold.

Theorem 2.7. Consider a pair $(\mathcal{T},\mathcal{T}^*)$ of bargaining proposals and let r, $1 \leq r \leq \pi^*$, be defined by (2.5). Then $\mathcal{T}^*$ is a bargaining counter proposal w.r.t. x and $\mathcal{T}$ if and only if (C) are satisfied.

Proof. By Lemma 2.6, (C) are necessary conditions, so that we should prove only the "if" part of the Theorem. We shall show that for all bargaining distributions $y \in F_{\mathcal{T}}$ the conditions (2.11) of Lemma 2.5 hold. Consider any bargaining distribution $y \in F_{\mathcal{T}}$. For any fixed $i^* \in \{1,\ldots,r\}$ define $L_i = P_i \cap P^*_{i^*}$, $i=\overline{1,\pi}$ and by Lemma 1.6 we have

$$\text{(2.13)} \qquad y(P_i \cap P^*_{i^*}) - x(P_i \cap P^*_{i^*}) \leq e(x,P_i), \quad i=\overline{1,\pi}$$

By summing up these inequalities for $i \in I(P^*_{i^*})$ we get

(2.14) $$y(P \cap P^*_{i*}) - x(P \cap P^*_{i*}) \leq \sum_{i \in I(P^*_{i*})} e(x,P_i)$$

where in the left hand side the null terms for $i \notin I(P^*_{i*})$ have been added. From (C) and (2.14) we will obtain (2.11) and Lemma 2.5 shows that $\mathcal{T}^*$ is a bargaining counter proposal.

If any coalition S with $e(x,S) > 0$ is called a profitable coalition then Theorem 2.7 says that: $\mathcal{T}^*$ is a bargaining counter proposal w.r.t. x and $\mathcal{T}$ if and only if the excess of any of its profitable coalitions say S, is at least equal to the sum of excesses of all profitable coalitions of $\mathcal{T}$ having common players with S. Note that such a characterization has been given in [4], for $\pi=1$, $\pi^*=1$, i.e. in the case of one-coalition objections and counter objections.

3. Bargaining sets and further combinatorial results.

Any $x \in F_{\mathcal{S}}$ for some coalition structure $\mathcal{S}$ is said to be stable, if either $x \in C(G)$ or for any bargaining proposal $\mathcal{T}$ w.r.t. x there exists a bargaining counter proposal $\mathcal{T}^*$ w.r.t. x and $\mathcal{T}$.

The set of all stable $x \in F_{\mathcal{S}}$, for all coalition structures $\mathcal{S}$ is called the bargaining set of G. It will be devoted by $M_0(G)$.

The combinatorial characterization of the elements of $M_0(G)$ given by Theorem 2.7 suggests a basic algorithm for determining whether or not a given $x \in F_{\mathcal{S}}$ belongs to $M_0(G)$. This algorithm could be refined by using other combinatorial results.

Consider the set of coalitions with positive excesses:

(3.1) $$X^+ = \{S \mid S \subseteq I,\ S \neq \phi,\ e(x,S) > 0\}$$

A subset of pairwise disjoint elements of X^+ will be called a positive partial partition of I. Obviously, if a partial partition of I is covering I, then it is a partition. In the following the word positive will be dropped out.

Note that if $\mathcal{T}$ is any bargaining proposal w.r.t. x, then $\{P_1,\ldots,P_\pi\}$ is a partial partition of $\mathcal{T}$. A similar remark is valid for any bargaining counter proposal.

A <u>maximal partial partition</u> is a partial partition that either covers I or it does not cover I, but no element of X^+ could be added to form a new partial partition. Of course, a bargaining proposal containing a partition of I is trivial. Under assumption (A) such partial partitions w.r.t. x do not exist.

In the basic algorithm to be described we are interested to confine our search for bargaining proposals and counter proposals to a subset of the set of partial partitions. This will be possible due to the following two results:

<u>Lemma 3.1</u>. If there exists a bargaining proposal $\mathcal{T}$, containing a partial partition $\{P_1,\ldots,P_\pi\}$, such that any other bargaining proposal is not a bargaining counter proposal w.r.t. x and $\mathcal{T}$, then there exists a bargaining proposal $\hat{\mathcal{T}}$, containing a maximal partial partition, and having the same property.

<u>Proof</u>. If $\{P_1,\ldots,P_\pi\}$ is not maximal, then $\{\hat{P}_1,\ldots,\hat{P}_\pi,\hat{P}_{\pi+1},\ldots,\hat{P}_{\pi+s}\}$ with $\hat{P}_i = P_i$, $i=\overline{1,\pi}$ and $s > 0$, a maximal partition containing the given one can be found. Obviously, $\hat{\mathcal{T}} = (\hat{P}_1,\ldots,\hat{P}_{\pi+s},I-\hat{P})$ with $\hat{P} = \bigcup_{1}^{\pi+s} \hat{P}_i$ and $I-\hat{P} \neq \phi$ is a bargaining proposal. If there is no bargaining counter proposal w.r.t. x and $\mathcal{T}$, then for each bargaining proposal $\mathcal{T}^*$ there exists an $i^* \in \{1,\ldots,r\}$ such that

$$e(x,P^*_{i^*}) < \sum_{i\in I(P^*_{i^*})} e(x,P_i) \tag{3.2}$$

Denote

$$(3.3) \qquad \hat{I}(P^*_{i*}) = \{ \hat{i} \,|\, P_{\hat{i}} \cap P^*_{i*} \neq \phi \}$$

and remark that $I(P^*_{i*}) \subseteq \hat{I}(P^*_{i*})$. Therefore, from (3.2) we get

$$(3.4) \qquad e(x,P^*_{i*}) < \sum_{i \in \hat{I}(P^*_{i*})} e(x,P_i)$$

hence by Theorem 2.7, T^* is not a bargaining counter proposal w.r.t. x and $\hat{T}$.

Note that a converse result is not true.

Now, if a bargaining proposal T containing a partial partition is available and we are looking for another bargaining proposal T^* able to counter T, then we shall also be interested to confine our search to a subset of partial partitions that could be included in T^*. Of course, any subset of a partial partition is also a partial partition, therefore a minimal partial partition consists of only one element of X^+. The following result shows that we can confine our search to the subset of one element partial partitions.

Lemma 3.2. If T is a bargaining proposal which can be countered by a bargaining counter proposal T^*, then there exists a bargaining counter proposal $\hat{T}$ containing only a one element partial partition.

Proof. Consider $\hat{T}^* = (P^*_{i*}; I-P^*_{i*})$ for a fixed $i^* \in \{1,\ldots,r\}$. Under assumption (A), $\hat{T}^*$ contains the one element partial partition $\{P^*_{i*}\}$. As $i^* \in \{1,\ldots,r\}$, $\hat{T}^*$ satisfies (B) and $\hat{I}(P^*_{i*}) = I(P^*_{i*})$. The bargaining proposal $\hat{T}^*$ is a counter proposal w.r.t. x and T, because the inequality (C) for this particular $i^* \in \{1,\ldots,r\}$ is among the inequalities (C) expressing the fact that T^* is a bargaining proposal. Hence by Theorem 2.7, $\hat{T}^*$ is a bargaining counter proposal w.r.t. x and T.

The above two lemmas can be summarized in the following result that describes in fact the basic algorithm for determining whether or not a given $x \notin C(G)$ belongs to $M_0(G)$.

Theorem 3.3. Any $x \notin C(G)$ belongs to $M_0(G)$ if and only if for every maximal positive partial partition $\{P_1,\dots,P_\pi\}$, there exists a one element positive partial partition P^* such that

$$(3.5) \qquad P^* \cap P \neq \phi, \quad P = \bigcup_{1}^{\pi} P_i$$
$$e(x,P^*) \geq \sum_{i \mid P^* \cap P_i \neq \phi} e(x,P_i)$$

Proof. Follows from Lemmas 3.1 and 3.2 and Theorem 2.7.

The algorithm based upon this theorem consists of the repeated application of two steps:

(1) Find a maximal positive partial partition of I. If all such partitions have already been considered, stop and conclude $x \in M_0(G)$. Otherwise, a bargaining proposal $T = (P_1,\dots,P_\pi;I-P)$ is available, go to (2).

(2) Find a minimal positive partial partition P^* of I subject to (3.5). If all minimal positive partial partitions have failed to pass one of the tests (3.5), stop and conclude $x \notin M_0(G)$. Otherwise, a new bargaining proposal must be considered, return to (1).

This algorithm can be described in graph theoretical terms.

Consider the graph $H = (X,E)$ where

$$(3.6) \qquad X = \{S \mid S \neq \phi,\ S \subset I\}$$
$$E = \{(S,S') \mid S \in X,\ S' \in X,\ S \cap S' \neq \phi\}.$$

Let $H^+ = (X^+,E^+)$ the subgraph generated by the set X^+ of proper coalitions of positive excesses. Any positive partial partition of I is an independent set of vertices of H^+ and conversely. Each vertex $S \in X^+$ is weighted by $w(S) = e(x,S)$, and the weight of a set of vertices is defined as the sum of the weights of its vertices. Then the steps of the above algorithm can be restated as follows:

(1) Find a maximal independent set of vertices in H^+. If all these sets have already been considered, stop and conclude $x \in M_0(G)$. Otherwise a maximal independent set $\{P_1,\ldots,P_\pi\}$ is available, go to (2).

(2) Find a vertex P^* of H^+ adjacent to $\{P_1,\ldots,P_\pi\}$, such that $w(P^*)$ is greater than or equal to $w(P^* \cap P)$. If all vertices of H^+ adjacent to $\{P_1,\ldots,P_\pi\}$ do not satisfy this condition, stop and conclude $x \notin M_0(G)$. Otherwise, a new maximal independent set must be considered, return to (1).

Of course, an algorithm for finding all maximal independent sets of a given graph has to be used as a subroutine.

All the above results can be extended to the so-called bargaining sets with thresholds (see [3]).

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